Renormalization of fermion fields in cosmological spacetimes

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Francisco Torrentí

Instituto de Fisica Teorica UAM/CSIC & U. Autonoma de Madrid

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Contents

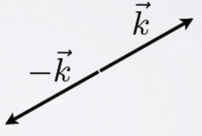
Introduction: Gravitational particle creation

We put a (free) **quantum field** in an **expanding** (FLRW) universe:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$

Consequences

- Particles of **all kinds** are **created** from the vacuum. (exception: conformal fields)



- They produce **new UV divergences** absent in Minkowski spacetime.

$$\langle \phi^2 \rangle \ , \langle \bar{\psi}\psi \rangle \ , \langle T_{\mu\nu} \rangle \quad \longrightarrow \quad \langle \phi^2 \rangle_{ren} \ , \langle \bar{\psi}\psi \rangle_{ren} \ , \langle T_{\mu\nu} \rangle_{ren}$$

- We need a regularization/renormalization procedure in curved spacetime $\rightarrow$ **adiabatic regularization**.

Adiabatic regularization for scalar fields

- **Example:** Variance of a scalar field.

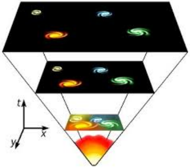
$$\phi(\vec{x},t) = \int \frac{d^3\vec{k}}{\sqrt{2}(2\pi a(t))^{3/2}} e^{i\vec{k}\vec{x}} (A_{\vec{k}} f_k(t) + A^\dagger_{-\vec{k}} f^*_{-k}(t))$$

$$\downarrow$$

$$\langle 0|\phi^2(\vec{x},t)|0\rangle = \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 |f_k(t)|^2 = \infty$$

This quantity contains **quadratic** and **logarithmic** ultraviolet divergences in momenta:

$$|f_k(t)|^2 \sim \frac{a}{k} - \frac{a^3}{2k^3}\left[m^2 + (-1+6\xi)\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)\right] + \frac{a^5}{8k^5}[\ldots] + \ldots$$



- $\langle T_{\mu\nu}\rangle$ contains also **quartic** divergences.

Adiabatic regularization for scalar fields

Adiabatic regularization: 1. Identification of the divergent terms and 2. Subtraction from the momentum integrand (Parker, Fulling (1974))

1. Divergent terms are identified through a **WKB adiabatic expansion** of $f_k(t)$ ($\omega^{(n)} \rightarrow$ **nth adiabatic order** $\rightarrow$ contains n derivatives of $a(t)$)

$$f_k(t) = \frac{1}{\sqrt{\omega_k(t) + \omega^{(2)} + \omega^{(4)} + \dots}} e^{-i\int^t (\omega_k(t') + \omega^{(2)}(t') + \dots)dt'} \qquad \omega_k(t) = \sqrt{\frac{k^2}{a^2(t)} + m^2}$$

$$\downarrow$$

$$|f_k(t)|^2 = (|f_k(t)|^2)^{(0)} + (|f_k(t)|^2)^{(2)} + (|f_k(t)|^2)^{(4)} \dots$$

2. Subtracting $(|f_k(t)|^2)^{(0)}$ removes **quadratic** divergence. Subtracting $(|f_k(t)|^2)^{(2)}$ removes **logarithmic** divergence:

$$\langle \phi^2(\vec{x}, t)\rangle_{ren} = \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 \Big(|f_k(t)|^2 \underbrace{- \frac{1}{\omega}}_{-(|f_k(t)|^2)^{(0)}} \underbrace{+ \frac{\dot{a}^2}{2a^2\omega^3}\left[1 + \frac{m^2}{\omega^2} - \frac{5m^4}{4\omega^4}\right] + \frac{\ddot{a}}{2a\omega^3}\left[1 - \frac{m^2}{2\omega^2}\right]}_{-(|f_k(t)|^2)^{(2)}} \Big)$$

Adiabatic regularization for spin-1/2 fields

Spin-1/2 field:

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi a(t))^{3/2}} e^{i\vec{k}\vec{x}} \sum_\lambda \left[B_{\vec{k}\lambda} u_{\vec{k}\lambda}(x) + D^\dagger_{-\vec{k}\lambda} v_{-\vec{k}\lambda}(x) \right] \ , \ u_{\vec{k}\lambda}(x) = \begin{pmatrix} h^I_k(t)\xi_\lambda(\vec{k}) \\ h^{II}_k(t)\frac{\vec{\sigma}\cdot\vec{k}}{k}\xi_\lambda(\vec{k}) \end{pmatrix}$$

$$\downarrow$$

- **Example:** Two-point function of a fermion field

$$\langle\bar{\psi}\psi\rangle = \frac{1}{\pi^2 a^3(t)} \int dk k^2 (|h^{II}_k(t)|^2 - |h^I_k(t)|^2) \ = \infty$$

This also contains extra **quadratic and logarithmic UV divergences**.

There is **not** a self-consistent adiabatic WKB-expansion for spin-1/2 fields. **How to identify the divergent terms?**

Adiabatic regularization for spin-1/2 fields

We use a more generic adiabatic expansion for fermion field modes:

$$h_k^I(t) = \sqrt{\frac{\omega_k + m}{2\omega_k}} e^{-i\int^{t'}(\omega_k(t')+\omega^{(1)}(t')+\omega^{(2)}(t')+\dots)dt'}(1 + F^{(1)} + F^{(2)} + \dots)$$

$$h_k^{II}(t) = \sqrt{\frac{\omega_k - m}{2\omega_k}} e^{-i\int^{t'}(\omega_k(t')+\omega^{(1)}(t')+\omega^{(2)}(t')+\dots)dt'}(1 + G^{(1)} + G^{(2)} + \dots)$$

(not WKB!) Landete, Navarro-Salas, F. T., **arXiv: 1305.7374**

$$\downarrow$$

$$|h_k^{II}|^2 - |h_k^I|^2 \sim -\frac{m}{2\omega} + \frac{(\omega - m)}{2\omega}(2G^{(2)} - (G^{(1)})^2 - \frac{(\omega + m)}{2\omega}(2F^{(2)} - (F^{(1)})^2) + \dots$$

$$\downarrow$$

$$\langle\bar{\psi}\psi\rangle_{ren} = \frac{1}{\pi^2 a^3(t)}\int dkk^2\left[|h_k^{II}|^2 - |h_k^I|^2 + \frac{m}{2\omega} - \frac{m\dot{a}^2}{4a^2\omega^3}\left(-1 + \frac{7m^2}{2\omega^2} - \frac{5m^4}{2\omega^4}\right) - \frac{m\ddot{a}}{4a\omega^3}\left(1 - \frac{m^3}{\omega^3}\right)\right]$$

Checks

- We recover the same expressions for anomalies obtained by other renormalization procedures:
  - Trace anomaly $\langle T^{\mu}_{\mu} \rangle_r$
  - Axial anomaly $\langle \nabla_\mu J^{\mu}_{A} \rangle_r = i2m \langle \bar{\psi} \gamma^5 \psi \rangle_r$

- For scalar fields, fermion-like expansion recovers WKB:

$$f_k(t) = \frac{1}{\sqrt{\omega}} e^{-i \int^t dt' (\omega(t') + \omega^{(1)} + \dots)} (1 + H^{(1)} + H^{(2)} + \dots) \rightarrow \textbf{WKB}$$

- $\langle T^{\mu\nu} \rangle_{ren}$ is **conserved**: $\nabla_\mu \langle T^{\mu\nu} \rangle_{ren} = 0$

Renormalized stress-energy tensor

$$\boxed{G^{\mu\nu} = 8\pi\langle T^{\mu\nu}\rangle_{ren}}$$ **(fermions)**

Del Rio, Navarro-Salas, F. T., **arXiv:1407.5058**

- Energy density:

$$\langle T^{00}\rangle_{ren} = \frac{1}{2\pi^2 a^3(t)}\int_0^\infty dkk^2\left[i\left(h_k^I\frac{\partial h_k^{I*}}{\partial t} + h_k^{II}\frac{\partial h_k^{II*}}{\partial t} - c.c\right) + 2\omega + \frac{m^4\dot{a}^2}{4\omega^5 a^2} - \frac{m^2\dot{a}^2}{4\omega^3 a^2}\right] - \langle T^{00}\rangle^{(4)}$$

- Pressure:

$$\langle T^{ii}\rangle_{ren} = \frac{-1}{2\pi^2 a^3(t)}\int_0^\infty dkk^2\left[\frac{2k}{3a}(h_k^I h_k^{II*} + c.c) - \frac{2}{3}\omega + \frac{2m^2}{3\omega} - \frac{m^2\dot{a}^2}{12\omega^3 a^2} - \frac{m^2\ddot{a}}{6\omega^3 a}\right.$$
$$\left. + \frac{m^4\ddot{a}}{6\omega^5 a} + \frac{m^4\dot{a}^2}{2\omega^5 a^2} - \frac{5m^6\dot{a}^2}{12\omega^7 a^2} + \frac{m^2\ddot{a}}{6\omega^3 a}\left(1 + \frac{m^2}{\omega^2}\right)\right] - \langle T^{ii}\rangle^{(4)}$$

where the fourth-order subtraction term is **finite** and **covariant**:

$$\langle T^{\mu\nu}\rangle^{(4)} = \frac{-2}{2880\pi^2}\left[-\frac{1}{2}\,{}^{(1)}H^{\mu\nu} + \frac{11}{2}\,{}^{(3)}H^{\mu\nu}\right]$$

$${}^{(1)}H_{\mu\nu} = R_{;\mu\nu} - 2\Box R g_{\mu\nu} + 2RR_{\mu\nu} - \frac{1}{2}R^2 g_{\mu\nu}\ , \quad {}^{(3)}H_{\mu\nu} = R^\rho_\mu R_{\rho\nu} - \frac{2}{3}RR_{\mu\nu} - \frac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} + \frac{1}{4}R^2 g_{\mu\nu}$$

Renormalized stress-energy tensor

$$\boxed{a(t) = e^{Ht}}$$ **de Sitter spacetime** **(fermions)**

$$\downarrow$$

$$h_k^I(t) = \frac{i}{2}\sqrt{\frac{\pi k}{H}}\, e^{\frac{\pi m}{2H} - \frac{Ht}{2}} H^{(1)}_{\frac{1}{2} - i\frac{m}{H}}\left(\frac{k}{H}e^{-Ht}\right)$$

$$\downarrow$$

$\langle T^{\mu\nu}\rangle$ in de Sitter spacetime

The stress-energy tensor is

$$\langle T^{\mu\nu}\rangle_{ren} = \frac{1}{960\pi^2} g^{\mu\nu}(11H^4 + 130H^2m^2 + 120m^2(H^2+m^2))\left(\log\left(\frac{m}{H}\right) - \mathfrak{Re}\left[\psi\left(-1 + i\frac{m}{H}\right)\right]\right)$$

Renormalized stress-energy tensor

$\boxed{a(t) = a_0 t^{1/2}}$ **Radiation-dominated universe** **(fermions)**

$$\downarrow$$

$$h_k^I = E_k \left(N \frac{W_{\kappa,\frac{1}{4}}(i2mt)}{\sqrt{a(t)}} \right) + F_k \left(N \frac{k}{2ma(t)^{3/2}} \left[W_{\kappa,\frac{1}{4}}(i2mt) + \left(\kappa - \frac{3}{4}\right) W_{\kappa-1,\frac{1}{4}}(i2mt) \right] \right)^*$$

$$\kappa = \frac{1}{4} - ix^2 \ , \quad N = \frac{a_0^{1/2}}{(2m)^{1/4}} e^{-\frac{\pi}{2}x^2} \ , \quad x^2 = k^2/(a_0^2 2m)$$

$$\downarrow$$

- Early times ($t << m^{-1}$): Created particles behave as **radiation**.

$$\rho(t) \sim \frac{\rho_{0r}}{a^4(t)} \ , \quad p(t) \sim \frac{\rho}{3} \quad , \quad \rho_{0r} = f(E_k, F_K)$$

- Late times ($t >> m^{-1}$): Created particles behave as **matter**.

$$\rho(t) \sim \frac{\rho_{0m}}{a^3(t)} \ , \quad p(t) \sim 0 \ , \quad \rho_{0m} = f(E_k, F_k)$$

Summary

- We have developed an **extension of the adiabatic regularization method** to free fermion fields in FLRW metrics.
- We have obtained $\langle T_{\mu\nu} \rangle_{ren}$ and applied it to two examples.
- This formalism **can be extended** to more generic situations, such as:
  - FLRW spacetimes with spatial curvature ($k = +1, -1$),
  - anisotropic metrics,

$$ds^2 = dt^2 - a_x^2(t)dx^2 + a_y^2(t)dy^2 + a_z^2(t)dz^2$$

  - Fields with (effective) time-dependent mass...

Thank you for your attention!