Boundary theories of two-dimensional tensor network states near the AKLT point

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Introduction

- The boundaries of materials with a finite volume are known to contain information about the phase of matter in the bulk. The localized edge modes in topologically ordered systems are a paradigmatic example of this *bulk-boundary correspondence.*
- Tensor network states are wavefunction Ansätze for quantum many-body systems. The wavefunction is parametrized with local tensors at each site, and eg. expectation values and correlation functions can be calculated by multiplying moderate-size matrices.
- The Affleck-Kennedy-Lieb-Tasaki (AKLT) [1] model is a special point of the integer-spin quantum Heisenberg model. Its ground state is a valence-bond state with energy $E_0 = 0$, and can be represented with a tensor network of bond dimension 2.

Bulk-boundary correspondence in tensor network states

- Projected Entangled-Pair States (PEPS) [2] generalize Matrix Product States in two dimensions. The wavefunction is written

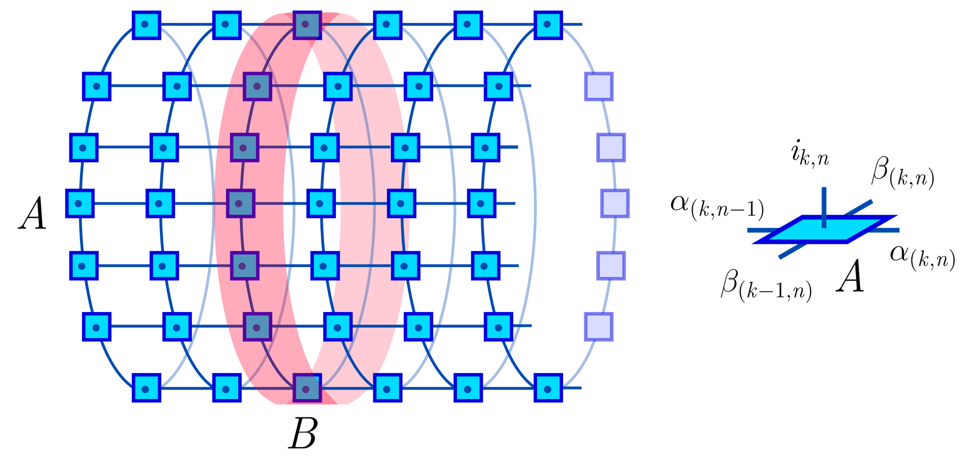
$$\left|\Psi\right\rangle = \sum_I c_I \left|I_1, I_2, \ldots, I_{N_h}\right\rangle$$

$$c_I = \mathrm{Tr}\left[X B^{I_1} B^{I_2} \ldots B^{I_{N_h}}\right]$$

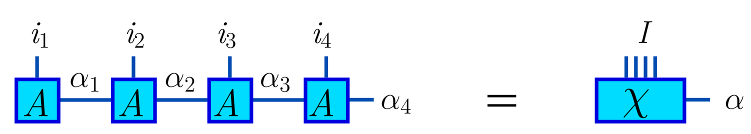
where $I_n = \{i_{1,n}, i_{2,n}, \ldots, i_{N_v,n}\}$ denotes all physical indices for column n, the matrix X encodes the boundary conditions and the column matrices B^{I_n} are written as

$$(B^{I_n})_{\Lambda_{n-1},\Lambda_n} = \mathrm{Tr}\left[A^{i_{1,n}}_{\alpha_{(1,n-1)},\alpha_{(1,n)}} \cdots A^{i_{N_v,n}}_{\alpha_{(N_v,n-1)},\alpha_{(N_v,n)}}\right]$$

with shorthand notation $\Lambda_n = \{\alpha_{(1,n)}, \alpha_{(2,n)}, \ldots, \alpha_{(N_v,n)}\}$ for all virtual indices in column n.



- PEPS admits a natural way to map states between the bulk and the boundary [3, 4].



$$\left|\phi_\alpha\right\rangle = \chi|\alpha)$$

The map χ does not preserve orthogonality. To achieve this, take the polar decomposition $\chi = UP$, where U is an isometry and $P = \sqrt{\chi^\dagger \chi}$. Now the states $\left|\Phi_\alpha\right\rangle = U|\alpha)$ form an orthogonal set, $\left\langle \Phi_{\alpha'} \middle| \Phi_\alpha \right\rangle = \delta_{\alpha',\alpha}$, and each physical bulk state $\left|\Phi_\alpha\right\rangle$ corresponds to a distinct virtual boundary mode α.

- The boundary theory of a Hamiltonian H_{bulk} is described by a Hamiltonian H acting on the virtual degrees of freedom at the boundary:

$$H = U^\dagger H_{\text{bulk}} U$$

In the first order of perturbation theory, the state can be written as $|\Psi\rangle = \sum_\alpha c_\alpha |\Phi_\alpha\rangle$. The bulk-boundary mapping can now be written as $U = \chi\left(\sqrt{\chi^\dagger \chi}\right)^{-1}$, where χ is the mapping for the ground state of H_{AKLT}.

Bulk Hamiltonian

- The 2D AKLT model is a spin-2 Hamiltonian defined via nearest-neighbour interactions: $H_{\text{AKLT}} = \sum_{\{i,j\}} P^{(S=4)}_{i,j}$. Nontrivial boundary dynamics appear by introducing a perturbation:

$$H_{\text{bulk}} = H_{\text{AKLT}} + V$$

We are interested in two kinds of perturbations: one-body terms $V_1 = \sum_i S^i_z$ and two-body terms $V_2 = \sum_{\{i,j\}} \sum_{S=0}^{3} P^S_{i,j}$

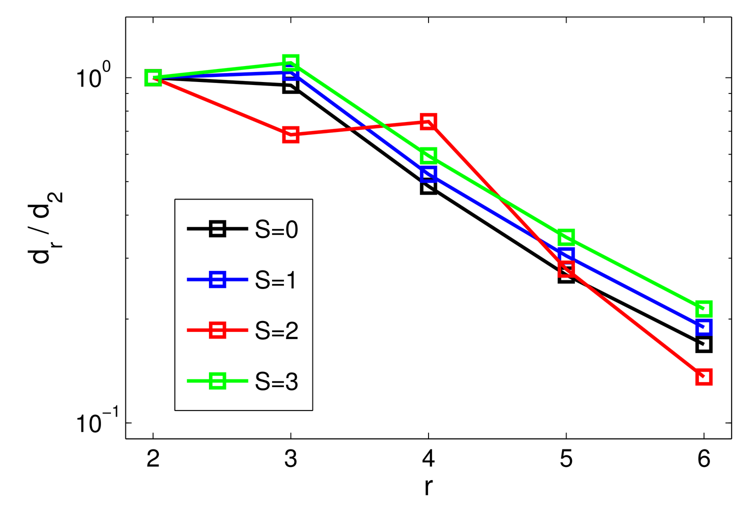
Interaction length of the boundary Hamiltonian

- If the boundary Hamiltonian can be written as a sum of quasilocal terms, the corresponding tensor network can be contracted efficiently [5].
- To study the locality of interactions, define the weight

$$d_r = \mathrm{Tr}(H_r^2)$$

(st. normalization) which quantifies the relative strength of terms with interaction length r. The r-body part of the Hamiltonian is given by $H_r = \mathrm{Tr}(H h_r)$, where h_r is the sum of all r-body Hamiltonians.

- The terms of the Hamiltonian vanish exponentially as a function of r ($N_v = 12$) [6]:

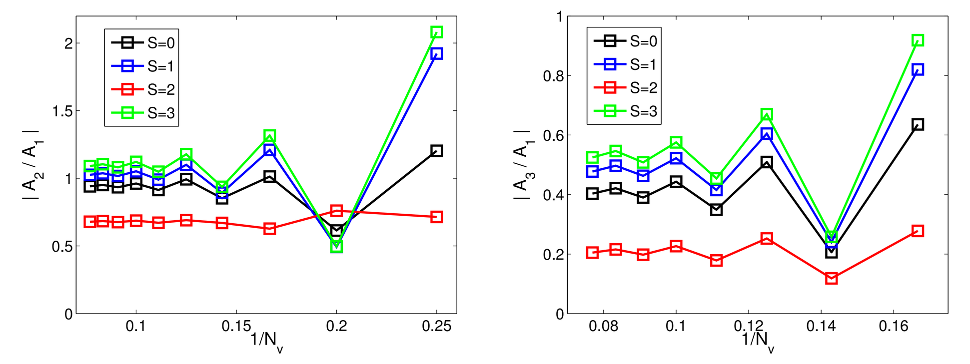


Structure of the boundary Hamiltonian

- On a more detailed level, the individual terms of the Hamiltonian can be found by defining the weight

$$A_r = \mathrm{Tr}\Big(H \sum_k \sigma^z_k \sigma^z_{k+r}\Big)$$

- As the diameter of the cylinder increases, the boundary Hamiltonian converges to a certain Hamiltonian:



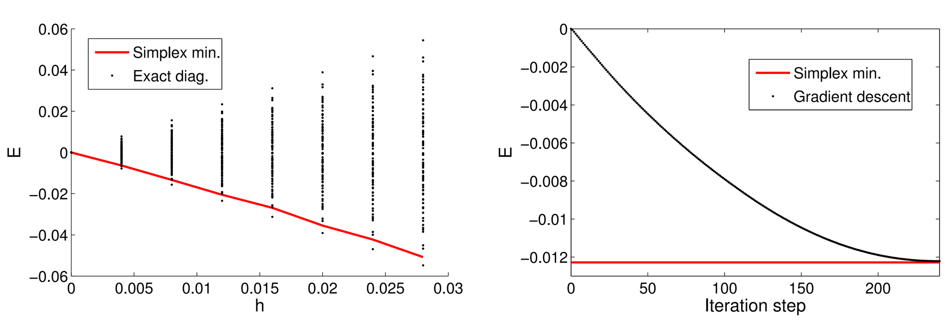
- A local magnetic field in the bulk induces a local field at the boundary as well. Chiral terms, such as $\mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3)$, induce chiral terms at the boundary. Owing to the structure of the PEPS, any local symmetry in the bulk is inherited by the boundary Hamiltonian.
- The leading terms of the boundary Hamiltonian can be written

$$H = \sum_{l \geq 1} \eta_l \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+l}$$

with $\eta_1 \approx 2.298$ and $\eta_2 \approx -2.394$.

Nonperturbative regime

- Find the ground state using simplex optimization or gradient descent methods:



Conclusions and outlook

- The bulk-boundary correspondence in two-dimensional integer spin systems was studied numerically with tensor network methods. The leading contribution to the boundary Hamiltonian consists of nearest and next-nearest neighbour terms.
- Locality of the boundary Hamiltonian implies that quantities such as correlation functions can be computed efficiently.
- Outlook: possible connections between boundary interactions and the entanglement spectrum [7].

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