Fractional Quantum Hall States :
From trial wavefunctions to Matrix Product States

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Numerical and analytical methods
for strongly correlated systems

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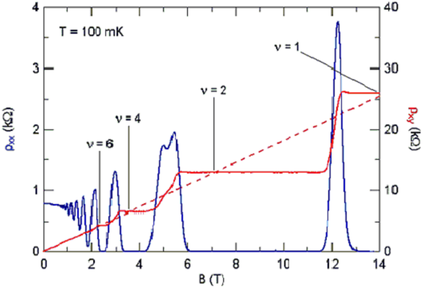
Integer quantum Hall effect



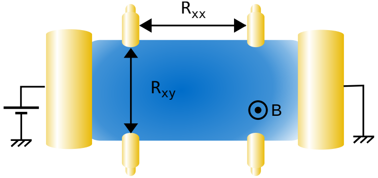
Landau levels

Classical Hall effect

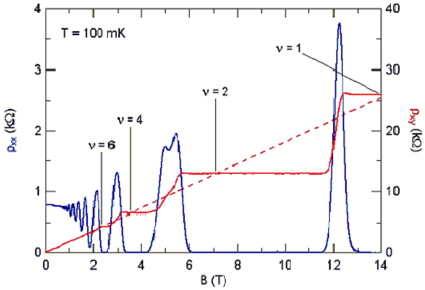
Hall effect : a 2D electron gas in a perpendicular magnetic field.

$\Rightarrow$ **current $\perp$ voltage**

$$R_{xy} \propto B$$



Integer Quantum Hall effect (IQHE)



IQHE : von Klitzing (1980)

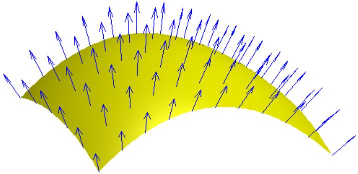
Quantized Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

ν is an integer up to $O(10^{-9})$
Used in metrology

Setup of the quantum Hall effect

Particles (fermions or bosons) confined in 2D, in a strong perpendicular magnetic field, and low temperature



$$H = \frac{1}{2m} \sum_i \left(\vec{p}_i - q\vec{A} \right)^2 + \sum_{i<j} V(\vec{x}_i - \vec{x}_j)$$

- Integer QHE : essentially a one-body problem
- Fractional QHE : strongly correlated quantum system

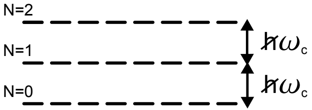
Landau levels on the plane

Two scales for the one-body problem :

$$\text{energy scale} \quad \omega_c = \frac{|qB|}{m}, \qquad \text{length scale} \quad l_B = \sqrt{\frac{\hbar}{|qB|}}$$

Hamiltonian in the plane

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right)$$



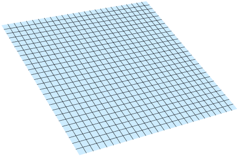
Discrete spectrum, huge degeneracy

Lowest Landau level (LLL) wavefunctions

$$\Psi_n(z, \bar{z}) = z^n \, e^{-z\bar{z}/4l_B^2} \qquad n = 0, 1, \cdots, \infty$$

$\Rightarrow$ **chirality :** $(x, y) \to z = (x + iy)$

Magnetic translations and dimensional reduction

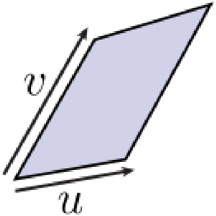


- Landau levels are degenerate, due to translation invariance
- translations do not commute ! ! ! !
  in symmetric gauge $\vec{A} = \frac{B}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$

Magnetic translations $R_{\vec{u}} = e^{iq\vec{u}.\vec{A}} e^{\vec{u}.\vec{\nabla}}$

Aharonov-Bohm effect :

$$R_{\vec{u}} R_{\vec{v}} = e^{i\frac{qB}{\hbar}\vec{u}\wedge\vec{v}} R_{\vec{v}} R_{\vec{u}}$$



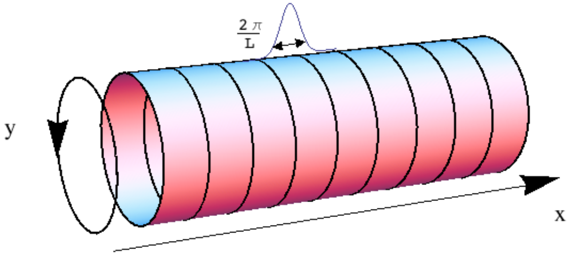
What is the meaning of the magnetic length ?

Projection to the LLL : x and y no longer commute $[\hat{x}, \hat{y}] = i\, l_B^2$

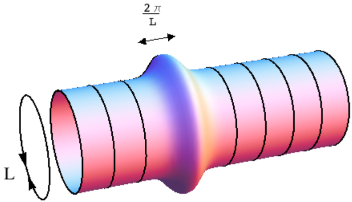
$$\sigma_x \sigma_y \geq l_B^2/2 \qquad \text{and} \qquad x \sim l_B^2 k_y$$

LLL Wavefunctions on the cylinder ($l_B = 1$)

$$\Psi_n(x,y) = e^{iyk} e^{-\frac{(x-k)^2}{2}}, \qquad k = \frac{2\pi n}{L}$$

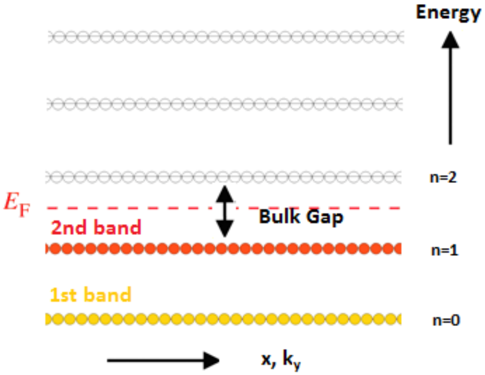


- the orbital with angular momentum $k = \frac{2\pi n}{L}$ is localized at $x = k$
- the interorbital distance is $2\pi/L$



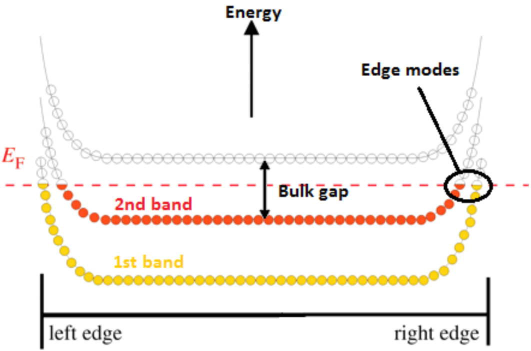
The IQHE : gapped bulk

Filling a Landau level is like a band insulator.



$\Rightarrow$ **Bulk insulator.**

The IQHE : conducting edges



$\Rightarrow$ **Conducting edges**
each channel contributes e^2/h to the Hall conductance

$$\sigma_{xy} = \frac{1}{\nu}\frac{e^2}{h}$$

Chiral (and therefore protected) massless edges

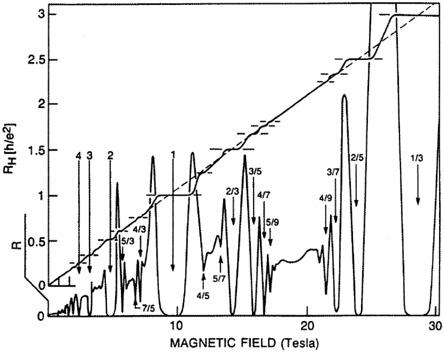
Topological insulator

This quantization is insensitive to disorder or strong periodic potential :

topological invariant : the Chern number

Fractional filling

the many-body problem

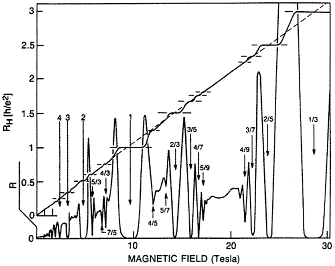


FQHE trial wavefunctions

Fractional filling : N particles in N_Φ orbital/states

N body wavefunctions in the LLL : (anti)symmetric polynomials

filling fraction $\nu = N/N_\phi < 1 \Rightarrow$ huge degeneracy (**no gap !**)



Electrons interactions

- Interactions can lift the degeneracy $\Rightarrow$ incompressible/gapped state
- What are the low energy properties ? Gapped bulk, Massless edge

Strongly correlated systems, emergence of exotic phases : fractional charges, non-abelian braiding.

What can we do ? Exact diagonalization, effective field theories, trial wavefunctions

The Laughlin state

Let's consider N particles in the LLL, at positions $\{z_1, \cdots, z_N\}$.

- $\nu = 1$ (filled band) :

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & & z_N \\ \vdots & & \ddots & \\ z_1^{N-1} & z_2^{N-1} & & z_N^{N-1} \end{vmatrix} = \prod_{i<j}(z_i - z_j)$$

- $\nu = 1/3$ Laughlin ground-state :

$$\Psi_L(z_1, \cdots, z_N) = \prod_{i<j}(z_i - z_j)^3$$

Unique ground state of a model interaction (short range part of Coulomb).

$\rightarrow$ **extremely high overlap with exact diagonalization !**
$\rightarrow$ **predicts excitations with fractional charge** $e/3$**.**

The Laughlin wave function : excitations

Excitations with fractional charge $\frac{\pm e}{3}$ and fractional statistics

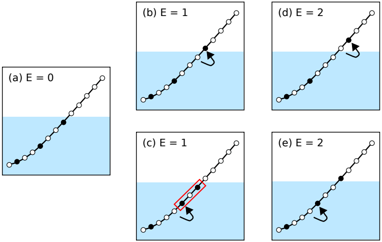
Bulk excitations

Add one flux quantum at w_0 = one quasi-hole

$$\Psi_{qh} = \prod_i (w_0 - z_i) \quad \Psi_L(z_1, \ldots z_N)$$

- For given number of particles and flux quanta, there is a specific number of qh states
- These numbers are a fingerprint of the phase (related to the statistics of the excitations).

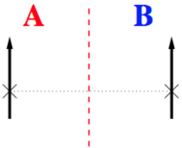
Edge excitations



- A chiral $U(1)$ boson linear dispersion relation
- The degeneracy of each energy level is given by the sequence $1, 1, 2, 3, 5, 7, \cdots$

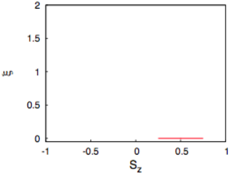
Entanglement

Example : system made of two spins 1/2

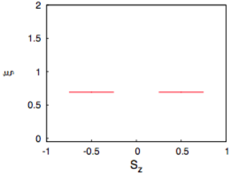


$\rho = |\Psi\rangle \langle\Psi|$, reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$
Entanglement spectrum : $\xi = -log(\lambda)$ (λ eigenvalues of ρ_A) as a function of S_z

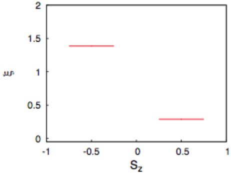
$|\uparrow\uparrow\rangle$



$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$



$\frac{1}{\sqrt{4}}|\uparrow\downarrow\rangle + \sqrt{\frac{3}{4}}|\downarrow\uparrow\rangle$



The counting (i.e the number of non zero eigenvalue) also provides informations about the entanglement

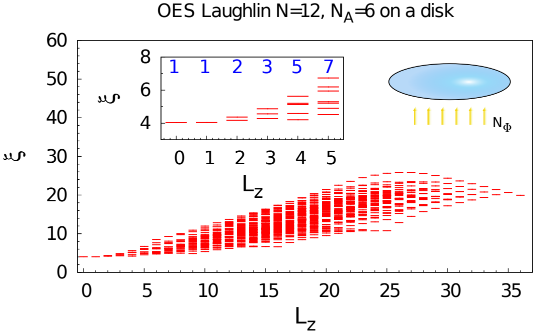
Entanglement entropy and spectrum

- Divide your system into two parts A and B (cut of length L)
- Compute the reduced density matrix or the Schmidt decomposition $|\Psi\rangle = \sum_\alpha \exp(-\xi_\alpha/2)\,|A,\alpha\rangle \otimes |B,\alpha\rangle$
- **Entanglement entropy** (Kitaev and Preskill 2005)

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A) = \alpha L - \gamma + \cdots$$

Entanglement spectrum
Li and Haldane (2008) :
spectrum of $\xi = -\log \rho_A$
(plot ξ vs momentum)

$\Rightarrow$ **Same counting** ...



Conformal field theories (CFT)

(2d) Conformal field theory : massless field theory (scale invariant)

- massless quantum systems in $1+1$ dimensions (for instance a QH edge)
- critical phenomena in 2 dimensions

The free boson

$$S = \int \mathrm{d}^2 z\, \partial\varphi\bar{\partial}\varphi$$

The free fermion

$$S = \int \mathrm{d}^2 z \, \left(\Psi\bar{\partial}\Psi + \bar{\Psi}\partial\bar{\Psi}\right)$$

Conformal symmetry $\Rightarrow$ classification of possible $2d$ CFTs
CFTs are exactly solvable !

The free boson a.k.a. U(1) CFT

The mode decomposition of the **chiral** free boson is

$$\varphi(z) = \varphi_0 - i a_0 \log(z) + i \sum_{n \neq 0} \frac{1}{n} a_n z^{-n}, \quad \text{with} \quad [a_n, a_m] = n \delta_{n+m,0}$$

Primary states/ vacua $|Q\rangle$ are defined by their U(1) charge Q

$$a_0 |Q\rangle = Q|Q\rangle, \qquad a_n|Q\rangle = 0 \text{ for } n > 0$$

The Hilbert space is simply a Fock space

Descendants are obtained with the lowering operators $a_n^\dagger = a_{-n},\ n > 0$

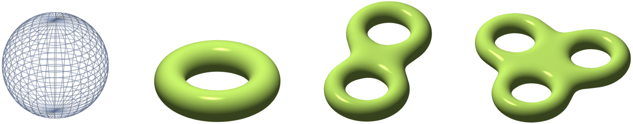
$$|Q, \mu\rangle = \prod_{i=1}^{n} a_{-\mu_i} |Q\rangle, \qquad a_0|Q,\mu\rangle = Q|Q,\mu\rangle$$

with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0$

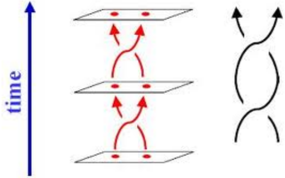
Topological phases

A system is in a topological phase if, at low energy, all observables are invariant under smooth deformation of the underlying space-time manifold, i.e. when its low energy effective field theory is a TQFT (with a gap).

- Ground state degeneracy depends on the genus



- Excitations ("quasi-holes") with fractional charges, possibly non-abelian anyons (non trivial action of the braid group)



Link between $2+1$ TQFT and $1+1$ CFT

Quasi-hole wavefunctions are conformal blocks.

- degeneracy = number of conformal blocks
- braiding = monodromies

FQH trial wave-function from CFT

Moore and Read (1990) proposed to write
FQH Trial wavefunctions as **CFT correlators**

$$\Psi(z_1, \cdots, z_N) = \langle u | V(z_1) \cdots V(z_N) | v \rangle$$

- **Operators or fields** $V(z) = \sum_n z^n V_n$
- **Infinite dimensional Hilbert space (graded by momentum/conformal dimension)**

Why is this ansatz sensible ?

- correct entanglement behavior (area law and counting)
- yields a consistent TQFT (pentagon and hexagon equations)
- Laughlin state is of this form

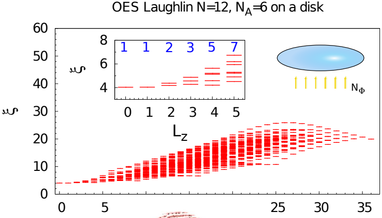
Laughlin state = free boson

$$\prod_{i<j}(z_i - z_j)^3 = \langle 0|V(z_1)\cdots V(z_N)|0\rangle, \qquad V(z) =: e^{i\sqrt{3}\varphi(z)}:$$

$$\Psi = \langle 0|V(z_1)\cdots\left(\sum_\alpha |\alpha\rangle\langle\alpha|\right)\cdots V(z_N)|0\rangle = \sum_\alpha \Psi_\alpha^A \Psi_\alpha^B$$

States per momentum sector :

- $1 : |Q\rangle$
- $1 : a_{-1}|Q\rangle$
- $2 : a_{-1}^2|Q\rangle,\ a_{-2}|Q\rangle$
- $3 : a_{-1}^3|Q\rangle,\ a_{-2}a_{-1}|Q\rangle,\ a_{-3}|Q\rangle$
- $5 : a_{-1}^4|Q\rangle,\ a_{-2}a_{-1}^2|Q\rangle,\ a_{-2}^2|Q\rangle,$ $a_{-3}a_{-1}|Q\rangle,\ a_{-4}|Q\rangle$
- $7 : \cdots$



$\Rightarrow$ **Same counting** ...

Excitations of the Laughlin state

Bulk excitations

Wavefunction for p quasiholes

$$\langle V_{\text{qh}}(w_1)\cdots V_{\text{qh}}(w_p)V(z_1)\cdots V(z_N)\rangle$$

- For the Laughlin state

$$V_{\text{qh}}(w) =: \exp\left(\frac{i}{\sqrt{3}}\varphi(w)\right):$$

- we recover

$$\prod_{a<b}(w_a - w_b)^{\frac{1}{3}}\prod_{a,i}(w_a - z_i)\Psi_{\text{ground-state}}$$

Edge excitations

$$\Psi_u = \langle u|V(z_1)\cdots V(z_N)|0\rangle$$

- edge mode = CFT descendant
- for instance $|u\rangle = a_{-n}|0\rangle$ gives

$$\Psi_u = \left(\sum_i z_i^n\right)\Psi_{\text{ground-state}}$$

- we recover $1, 1, 2, 3, 5, 7, \cdots$

Beyond Laughlin (for bosons)

- $U(1)$ **Laughlin state** $V(z) =: e^{i\sqrt{r}\varphi(z)}:$

$$\Psi_{\text{ground-state}} = \prod_{i<j}(z_i - z_j)^r$$

- $\mathrm{SU}(2)_2$ **Moore-Read state** $V(z) = \Psi(z) \otimes : e^{i\varphi(z)}:$

$$\Psi_{\text{ground-state}} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)$$

- $\mathrm{SU}(2)_k$ **Read-Rezayi state**

$$V(z) = J^+(z) = \Psi_1(z) \otimes : e^{i\sqrt{2/k}\varphi(z)}:$$

What about quasi-hole operators ?

$$V_{qh}(w) = \sigma_1(w) \otimes e^{i\sqrt{1/2k}\varphi(w)} :\Rightarrow \text{non-Abelian anyons}$$

Generic structure of the CFT Hilbert space

L_n (modes of the stress-energy tensor) obey the **Virasoro algebra**

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

Primary fields $|\Delta\rangle$ are annihilated by the positive modes

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_n|\Delta\rangle = 0 \qquad n > 0$$

Descendant states : lowering operators $L_n^\dagger = L_{-n}$, $n > 0$

$$|\Delta, \lambda\rangle = L_{-\lambda_1} L_{-\lambda_2} \cdots L_{-\lambda_n}|\Delta\rangle$$

Two issues :

- these states are not orthogonal
- they might not even be independant !

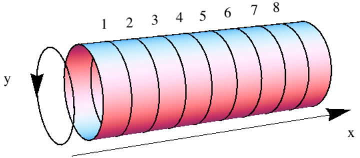
$\Rightarrow$ **No closed formula, has to be implemented numerically.**

Matrix Product State (MPS)

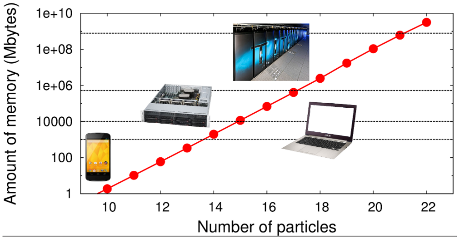
Limitations of exact diagonalizations and trial wf

$\longrightarrow$ decomposition of a state $|\Psi\rangle$ on a convenient occupation basis

$$|\Psi\rangle = \sum_{\{m_i\}} c_{\{m_i\}} \, |m_1, \ldots, m_{N_\Phi}\rangle$$



What is the amount of memory needed to store the Laughlin state ?



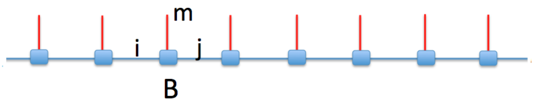
The most powerful computer in the world can't store more than 21 particles !

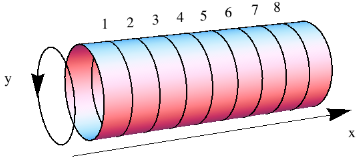
Matrix Product State : more compact and computationally friendly

Matrix Product States

Any state $|\Psi\rangle = \sum_{\{m_i\}} c_{\{m_i\}} |m_0, ..., m_{N_\Phi}\rangle$ can be written as

$$|\Psi\rangle = \sum_{\{m_i\}} \left(\langle u| B^{[m_1]} \ldots B^{[m_{N_{\mathrm{orb}}}]} |v\rangle \right) |m_1, ..., m_{N_{\mathrm{orb}}}\rangle$$





The $B^{[m]}_{\alpha,\beta}$ matrices have two types of indices

- $[m]$ is the physical index : occupied $(m = 1)$ or empty $(m = 0)$ orbital
- (α, β) are the bond indices (**auxiliary space**)
- **Bond dimension** χ (dimension of the auxiliary space)

Area law $\Rightarrow$ efficient MPS $(\chi \sim \exp \alpha L)$

Matrix Product State for FQH trial wave-function

$\Psi(z_1, \cdots, z_n) = \langle u | V(z_1) \cdots V(z_n) | v \rangle$ is a MPS in disguise !

Dubail, Read, Rezayi (2012)

$$|\Psi\rangle = \sum_{\{m_i\}} \left(\langle u | \, B^{m_1} B^{m_2} \cdots B^{m_n} \, | v \rangle \right) | m_1 \cdots m_n \rangle$$

Zaletel, Mong (2012)

- the matrices B^m are operators in the underlying CFT
- the auxiliary space is the (infinite dimensional) CFT Hilbert space ...
- ... which can be truncated with arbitrary large precision

Why is this formalism interesting ?

Many quantities (correlation functions, entanglement spectrum, ...) can be computed in the (relatively small) auxiliary space.

Where does this MPS structure come from ?

Start with a trial wavefunction given by a CFT correlator

$$\Psi(z_1, \cdots, z_N) = \langle u | V(z_1) \cdots V(z_N) | v \rangle = \sum_{m_i = 0,1} c_{(m_1, \cdots, m_n)} \, |m_1, \cdots, m_n\rangle$$

with electron operator $V(z)$ in some chiral $1+1$ CFT .

- Insert a complete basis of states

$$\sum_{\alpha_1, \cdots, \alpha_{N-1}} \langle u | V(z_1) | \alpha_1 \rangle \langle \alpha_1 | V(z_2) | \alpha_2 \rangle \cdots \langle \alpha_{N-1} | V(z_N) | v \rangle$$

- Project to $|m_1, \cdots, m_n\rangle$

One gets an infinite MPS (on any genus 0 geometry)

$$c_{(m_1, \cdots, m_n)} = \langle u | \, B^{m_1}[1] B^{m_2}[2] \cdots B^{m_n}[n] \, | v \rangle$$

Site/orbital dependent matrices :

$$\langle \alpha' | B^0[j] | \alpha \rangle = \delta_{\alpha', \alpha}, \qquad \langle \alpha' | B^1[j] | \alpha \rangle \propto \delta_{\Delta_{\alpha'}, \Delta_\alpha + h + j} \langle \alpha' | V(1) | \alpha \rangle$$

Translation invariant MPS on the cylinder

Uniform background charge $\Rightarrow$ site independant MPS

$$B^0 = e^{-\frac{i}{\sqrt{q}}\varphi_0}, \qquad B^1 = V_0\, e^{-\frac{i}{\sqrt{q}}\varphi_0}$$

where

- φ_0 is the bosonic zero mode (B_0 shifts the electric charge by $1/q$)
- V_0 is the matrix of the electron operator

$$\langle \alpha' | V_0 | \alpha \rangle = \delta_{\Delta_{\alpha'}, \Delta_\alpha + h} \langle \alpha' | V(1) | \alpha \rangle$$

What is required for a numerical implementation ?

- build the basis $|\alpha\rangle$ (auxiliary space) + truncation scheme
- compute the matrix elements $\langle \alpha' | B^m | \alpha \rangle$

Summary

FQHE : hard many-body problem

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Exact diagonalization

Trial wavefunctions
effective TQFTs

$\downarrow$

2d Conformal Field Theories

$\downarrow$

Matrix Product States