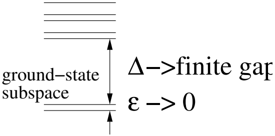
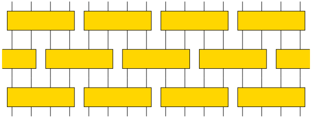
Highly entangled quantum states of matter – a general theory of gapped quantum liquids

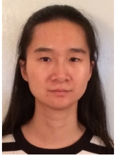
Xiao-Gang Wen, June 24, 2014

Gapped quantum phases and local unitary transformations



- An equivalence relation: $|\Psi(1)\rangle$ and $|\Psi(0)\rangle$ are equivalent iff the two states are related by a local unitary transformation

$\{|\Psi_\alpha(1)\rangle\} =$  $\{|\Psi_\alpha(0)\rangle\}$

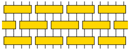


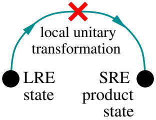
- Two kinds of equivalence class: Zeng-Wen 14
  - stable class → **emergence of unitarity**
  $|\Psi_\alpha\rangle$ remain to be orthogonal under a small local non-unitary transformation.
  A local non-unitary evolution induces a local unitary evolution in the ground state subspace.
  - un-stable class: example $\{|\uparrow\uparrow\dots\rangle, |\downarrow\downarrow\dots\rangle\}$.

Conjecture: **a gapped quantum liquid phase is a stable equivalence class of local unitary transformations**.

Long-range entanglement and topological order

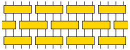
- There are **long range entangled (LRE) states**
- There are **short range entangled (SRE) states**

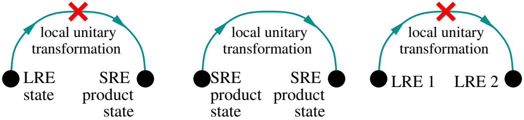
$|\text{LRE}\rangle \neq$  $|\uparrow\uparrow\cdots\uparrow\rangle = |\text{SRE}\rangle$

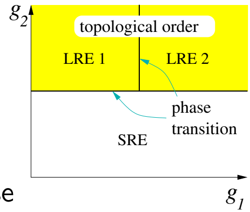


Long-range entanglement and topological order

- There are **long range entangled (LRE) states** $\to$ many phases
- There are **short range entangled (SRE) states** $\to$ one phase

$|\mathrm{LRE}\rangle \neq$  $|\uparrow\uparrow\cdots\uparrow\rangle = |\mathrm{SRE}\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to different phases

$=$ different **long-range entanglements**

$=$ different **topological orders**

Chen-Gu-Wen 10

LRE $\leftrightarrow$ **topological order** $\leftrightarrow$ **emergence of unitarity**

$\to$ fault tolerant topological quantum computation

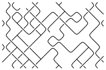
How to make long range entanglements (topo. orders)

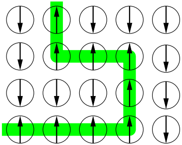
To make topological order, we need to sum over many different product states, but we should not sum over everything.

$\sum_{\text{all spin configurations}} |\uparrow\downarrow\downarrow\uparrow ...\rangle = |\rightarrow\rightarrow\rightarrow\rightarrow ...\rangle$

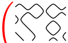
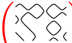
- Sum over a subset of spin configurations: sum over all the "string states", where the up-spins form strings:

$|\Phi_{\text{closed string}}\rangle =$ sum over loops $\Big|$  $\Big\rangle$

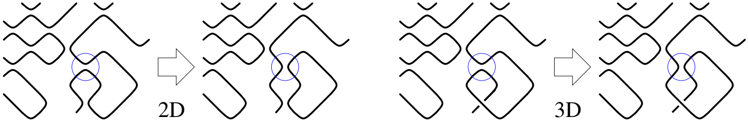
$|\Phi_{\text{string-net}}\rangle =$ sum over string-nets $\Big|$  $\Big\rangle$

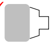
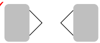


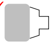
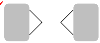
$\rightarrow$ string-net condensation Levin-Wen 05 (string-net liquid).

- Different amplitudes of string wave function $\Phi\Big($  $\Big)$ give rise to different topological orders.
- How to determine the string wave function $\Phi\Big($  $\Big)$?

A quantitative description of long-range entanglement Local dancing rule → global dancing pattern



- Local dancing rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=\Phi_{\text{str}}$ (), Φ_{str} () $=\Phi_{\text{str}}$ ()
  → Global dancing pattern Φ_{str} () $=1$ (Z_2 loop liquid)

- Local dancing rules of another string liquid (exist only in 2+1D):
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=\Phi_{\text{str}}$ (), Φ_{str} () $=-\Phi_{\text{str}}$ ()
  → Global dancing pattern Φ_{str} () $=(-)^{\#\text{ of loops}}$

- Two string-net condensations → two topological orders Levin-Wen 05

Universal wavefunction overlap: "X-ray" for topo. order

- Consider the ground states $|\Psi_\alpha\rangle$ on torus T^2, and two maps, $\hat{S} = 90^\circ$ rotation and $\hat{T} =$ Dehn twist.

Heidar-Wen 13, He-Heidar-Wen 13

$$S_{\alpha\beta} e^{-f_S L^2 + o(L^{-1})} = \langle \Psi_\alpha | \hat{S} | \Psi_\beta \rangle$$
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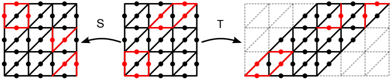

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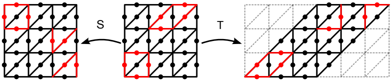

- For the first topo. order:

$\Psi_1($  $) = g^{\text{string-length}}$

$\Psi_2($  $) = (-)^{W_x} g^{\text{str-len}}$

$\Psi_3($  $) = (-)^{W_y} g^{\text{str-len}}$

$\Psi_4($  $) = (-)^{W_x+W_y} g^{\text{str-len}}$

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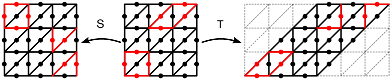

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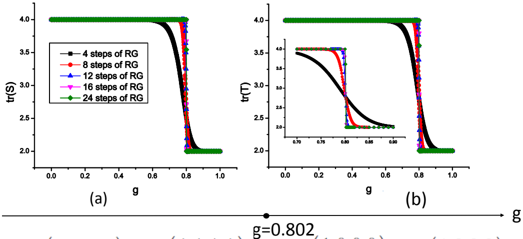
- $g < 0.8$ small-loop phase
  $|\Psi_\alpha\rangle$ are the same state
- $g > 0.8$ large-loop phase
  $|\Psi_\alpha\rangle$ are four diff. states



$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c)

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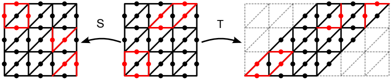

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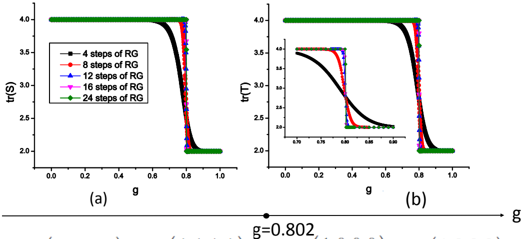

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(c)

- For the second topo. order:

$$S = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1 \end{pmatrix}, \quad T = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0 \end{pmatrix}.$$

Short-range entanglements w/ symmetry $\to$ SPT phases

For gapped systems with a symmetry $H = U_g H U_g^\dagger,\ g \in G$
Consider only states that do not break the symmetry G

- there are **LRE symmetric states** $\to$ many different phases
- there are **SRE symmetric states** $\to$ one phase (no symm. breaking)

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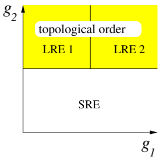
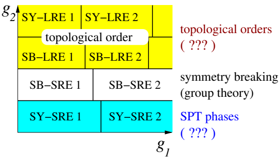
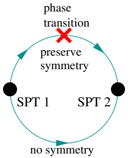
We may call them **symmetry protected** **trivial** **(SPT)** phase







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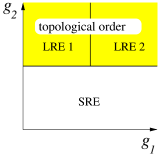
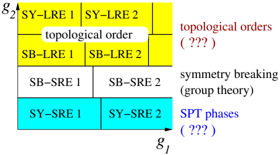
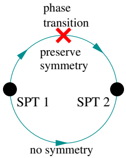
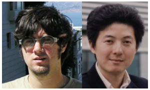
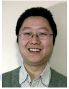
We may call them **symmetry protected trivial (SPT)** phase
or **symmetry protected topological (SPT)** phase







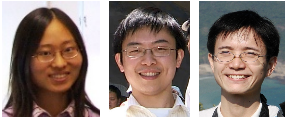
- SPT phases = equivalent class of *symmetric* LU transformations
- 1D Haldane phase, Haldane 83 2D/3D topological insulators, Kane-Mele 05; Bernevig-Zhang 06 Moore-Balents 07; Fu-Kane-Mele 07 are examples of SPT phases.

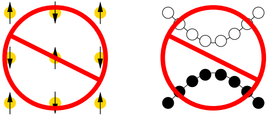
1D  2D   3D  

A group-cohomology theory of SPT phase (a new math)

Chen-Liu-Wen 11, Chen-Gu-Liu-Wen 11

- Group theory → symm. breaking phases
- What math → SPT phases?





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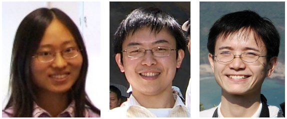

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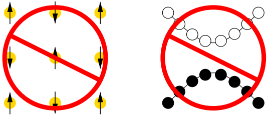
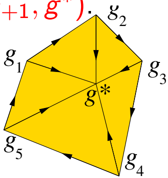


- *We want construct a state that is*
  - symmetric under symmetry G
  - connected to a direct product state
  via LU transformations.
- Consider a 1D system, states on each site given by $|g_i\rangle$, $g_i \in G$
- Start with a product state $\otimes_i|\phi_i\rangle$, $|\phi_i\rangle \equiv \sum_{g_i}|g_i\rangle$, or $\Phi_0(\{g_i\}) = 1$
- Perform LU trans: $U_{i,i+1}|g_i, g_{i+1}\rangle = \nu_2(g_i, g_{i+1}, g^*)|g_i, g_{i+1}\rangle$
  $\rightarrow$ We obtain a wave function $\Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$.



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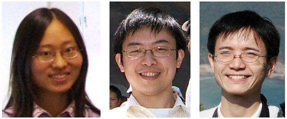
Chen-Liu-Wen 11, Chen-Gu-Liu-Wen 11



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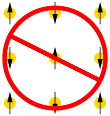

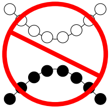

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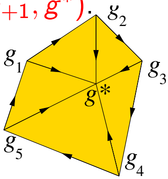
  $\to$ We obtain a wave function $\Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$.
- The wave function is symmetric $\Psi(\{hg_i\}) = \Psi(\{g_i\})$
  if the phase factor $\nu_2(g_0, g_1, g_2)$ satisfies:
  (1) $\nu_2(hg_0, hg_1, hg_2) = \nu_2(g_0, g_1, g_2), h \in G$
  (2) $\frac{\nu_2(g_1,g_2,g_3)\nu_2(g_0,g_1,g_3)}{\nu_2(g_0,g_2,g_3)\nu_2(g_0,g_1,g_2)} = 1$



- The LU trans. is not symmetric $\nu_2(hg_0, hg_1, g^*) \neq \nu_2(g_0, g_1, g^*)$

Group cohomology theory for SPT states

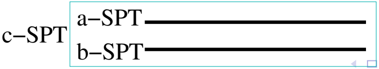
- $\nu_2(g_0, g_1, g_2)$ satisfying eqn. (1) and (2) are 2-cocycles $\mathcal{Z}^2[G, U(1)] = \{\nu_2(g_0, g_1, g_2)\}$.
- Every cocycle gives rise to a SPT state.
- But, do two SPT states from two different cocycles belong to the same phase?

  Two cocycles that can be connected within the space of cocycles are equivalent, and describe the same phase.
- The equivalent classes of cocycles = group cohomology classes $\mathcal{H}^2[G, U(1)]$.

  A subset of bosonic SPT states with symmetry G in d spatial dimensions are one-to-one described by group cohomology classes $\mathcal{H}^{d+1}[G, U(1)]$.

  $\mathcal{H}^{d+1}[G, U(1)] = \{a, b, c, \cdots\}$ is a set, whose elements label SPT phases. It is also an Abelian group: multiplication = stacking

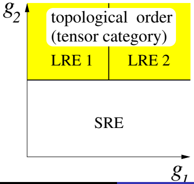
  $a \otimes b = c \rightarrow$

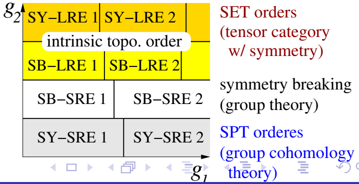


Group cohomology $\mathcal{H}^{d+1}[G, U_T(1)] \to$ bosonic SPT phases

Symmetry G	$d=0$	$d=1$	$d=2$	$d=3$	systems
$U(1) \rtimes Z_2^T$ $U(1) \rtimes Z_2^T \times$ trn	$\mathbb{Z}$ $\mathbb{Z}$	$\mathbb{Z}_2$ (0) $\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$ $\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^2$ $(\mathbb{Z}_2)$ $\mathbb{Z} \times \mathbb{Z}_2^8$	bosonic topo. ins.
$U(1) \times Z_2^T$ $U(1) \times Z_2^T \times$ trn	0 0	$\mathbb{Z}_2^2$ $\mathbb{Z}_2^2$	0 $\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$ $\mathbb{Z}_2^9$	T-symmetric spin systems
Z_2^T $Z_2^T \times$ trn	0 0	$\mathbb{Z}_2$ $(\mathbb{Z})$ $\mathbb{Z}_2$	0 (0) $\mathbb{Z}_2^2$	$\mathbb{Z}_2$ (0) $\mathbb{Z}_2^4$	bosonic topo. SC
$U(1)$ $U(1) \times$ trn	$\mathbb{Z}$ $\mathbb{Z}$	0 $\mathbb{Z}$	$\mathbb{Z}$ $\mathbb{Z}^2$	0 $\mathbb{Z}^4$	Hall effect in 2+1D
Z_n $Z_n \times$ trn	$\mathbb{Z}_n$ $\mathbb{Z}_n$	0 $\mathbb{Z}_n$	$\mathbb{Z}_n$ $\mathbb{Z}_n^2$	0 $\mathbb{Z}_n^4$	
$D_{2h} = Z_2 \times Z_2 \times Z_2^T$ $SO(3)$ $SO(3) \times Z_2^T$	$\mathbb{Z}_2^2$ 0 0	$\mathbb{Z}_2^4$ $\mathbb{Z}_2$ $\mathbb{Z}_2^2$	$\mathbb{Z}_2^6$ $\mathbb{Z}$ $\mathbb{Z}_2$	$\mathbb{Z}_2^9$ 0 $\mathbb{Z}_2^3$	 spin systems

Table of $\mathcal{H}^{d+1}[G, U_T(1)]$
"Z_2^T": time reversal,
"trn": translation,
others: on-site symm.
$0 \to$ only trivial phase.
$(\mathbb{Z}_2) \to$ free fermion result

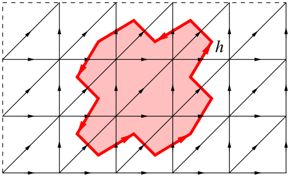


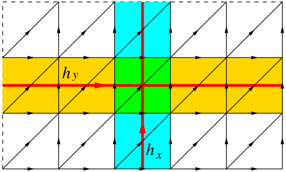


Universal wavefunction overlap: "X-ray" for SPT order

Universal wavefunction overlap for SPT state:
$S\ e^{-f_S L^2 + o(L^{-1})} = \langle \Psi_0 | \hat{S} | \Psi_0 \rangle$ and $T\ e^{-f_T L^2 + o(L^{-1})} = \langle \Psi_0 | \hat{T} | \Psi_0 \rangle$
$\to S = T = 1$, due to the trivial topological order in SPT state.

- To obtain non-trivial wavefunction overlap, we add symmetry twists: $H = \sum H_{ijk} \to H_h = \sum_{\text{in bulk}} H_{ijk} + \sum_{\text{on boundary}} H^h_{ijk}$



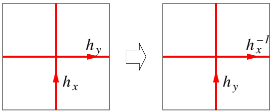


$\sum_{\text{on boundary}} H^h_{ijk}$ is the h-symmetry twist.

- Let $|\Psi_{(h_x,h_y)}\rangle$ be the ground state of H_{h_x,h_y} on T^2 with symmetry twists h_x, h_y in x- and y-directions.
$|\Psi_{(h_x,h_y)}\rangle$ simulate the degenerate ground states for topological ordered states.

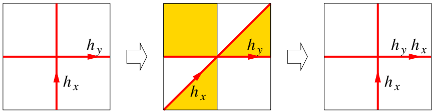
Universal wavefunction overlap: “X-ray” for SPT order

$\hat{S}$ move:



(a) (b)

$\hat{T}$ move:



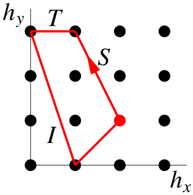
(a) (b) (c)

$\hat{S}$ move: $|\Psi_{(h_x,h_y)}\rangle \to \mathbf{S_{h_x,h_y}}\ e^{-f_S L^2+o(L^{-1})}|\Psi_{(h_y^{-1},h_x)}\rangle$

$\hat{T}$ move: $|\Psi_{(h_x,h_y)}\rangle \to \mathbf{T_{h_x,h_y}}\ e^{-f_T L^2+o(L^{-1})}|\Psi_{(h_x,h_y h_x)}\rangle$

$\hat{U}(h_t)$ move: $|\Psi_{(h_x,h_y)}\rangle \to \mathbf{U_{h_x,h_y}(h_t)}\ |\Psi_{(h_t h_x h_t^{-1},h_t h_y h_t^{-1})}\rangle$ Hung-Wen 13

- The product of $S, T, U \in U(1)$ around a closed orbit is universal.
- Conjecture: those phases (and their generalization) completely characterize the SPT states.



- For 2+1D Z_N SPT state labeled by $k \in H^3[Z_N, U(1)] = \mathbb{Z}_N$

$$\langle h, g|T^N|h, g\rangle = e^{2\pi i (h-1)^2 k/N}$$

A 2+1D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT state

$$H^3[\prod_{i=1}^{3} Z_{N_i}, U(1)] = \mathbb{Z}_{N_1} \oplus \mathbb{Z}_{N_2} \oplus \mathbb{Z}_{N_3} \oplus \mathbb{Z}_{N_{12}} \oplus \mathbb{Z}_{N_{23}} \oplus \mathbb{Z}_{N_{13}} \oplus \mathbb{Z}_{N_{123}}$$

where $N_{123} = \gcd(N_1, N_2, N_3)$.

- We consider a SPT state labeled by $k \in \mathbb{Z}_{N_{123}}$ and assume $N_1 = N_2 = N_3 = N$.
- **What is the SPT invariant $U_{h_x,h_y}(h_t)$?**

A 2+1D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT state

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- We consider a SPT state labeled by $k \in \mathbb{Z}_{N_{123}}$ and assume $N_1 = N_2 = N_3 = N$.
- **What is the SPT invariant $U_{h_x,h_y}(h_t)$?**

$U_{h_x,h_y}(h_t)$ is the fixed-point partition function on space-time $T^3 = (S^1)^3$ with symmetry twists in x, y, t directions:

$$Z_{\text{fixed-point}} = U_{h_x,h_y}(h_z) = e^{ik \frac{N_1 N_2 N_3}{(2\pi)^2 N_{123}} \int A_1 A_2 A_3}, \qquad \mathrm{d}A_i = 0,$$

where $\oint A_i = \frac{2\pi}{N_i} \times$ integer describes the Z_{N_i} twist. Wang-Gu-Wen 14

- **The physical meaning of the SPT inv.:** The intersection of the domain walls of Z_{N_1} and Z_{N_2} carries Z_{N_3}-charge k.
- **A mechanism for such a SPT state:** bind k Z_{N_3}-charge to the intersection of the domain walls of Z_{N_1} and Z_{N_2}.

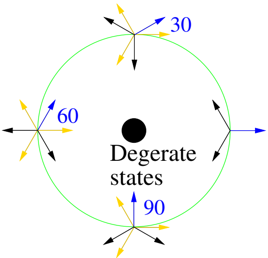
- Dimension reduction: $M^2_{\text{space}} \to S^1 \times I$ and $\oint_{S^1} A_3 = 2\pi/N_3$:

$$Z_{\text{fixed-point}} = e^{ik\frac{N}{2\pi}\int A_1 A_2}$$

$\to$ A 1+1D SPT state labeled by $k \in H^2[Z_{N_1} \times Z_{N_2}, U(1)] = \mathbb{Z}_{N_{12}}$.
$\to$ degenerated states at the end of I that form a projective representation of $Z_{N_1} \times Z_{N_2}$.

- A Z_{N_3} "vortex" (end of Z_{N_3} symmetry twist) carries degenerated states that form a projective representation of $Z_{N_1} \times Z_{N_2}$.

- **How to make Z_3-vortex**:
  1) Consider $U(1)$ symm. break down to Z_3 symm.
  2) A vortex of the order parameter $=$ Z_3-vortex.



- A mechanism for the above 1+1D $Z_{N_1} \times Z_{N_2}$ SPT state: bind k Z_{N_2}-charge to the domain wall of Z_{N_1}.

Gauge, gravity, and mixed gauge-gravity anomalies

- The SPT inv. is not gauge invariant when the space-time has a boundary: $A_3 \to A_3 + \mathrm{d}f_3$
$$k\frac{N_1N_2N_3}{(2\pi)^2N_{123}}\int_{M^3} A_1A_2A_3 \to k\frac{N_1N_2N_3}{(2\pi)^2N_{123}}\left[\int_{M^3} A_1A_2A_3 + \int_{\partial M^3} A_1A_2f_3\right]$$
We need a non-trivial boundary theory with *gauge anomaly* to cancel the non-gauge invariance of the SPT inv.
$$Z_{\text{fixed-point}}(A_i) = e^{ik\frac{N_1N_2N_3}{(2\pi)^2N_{123}}\int_{M^3} A_1A_2A_3 + \int_{\partial M^3} L^{\text{boundary}}_{\text{gauge anom.}}(A_i,\phi)}$$

SPT order (group cohomology) $\leftrightarrow$ gauge anomaly in one lower dimension (on the boundary)

- Similarly, a topologically ordered state also has a topo. inv. that is not diffeomorphism invariant when the space-time has a boundary. We need a non-trivial boundary theory with *gravitational anomaly* to cancel the non-diffeomorphism invariance of the bulk topo. inv.
$$Z_{\text{fixed-point}}(g_{\mu\nu}) = e^{ik\int_{M^3} CS_3(g_{\mu\nu}) + \int_{\partial M^3} L^{\text{boundary}}_{\text{grav. anom.}}(g_{\mu\nu},\phi)}$$

Topological order $\leftrightarrow$ grav. anomaly in one lower dimension

SPT state from mixed gauge-grav. anomalies – beyond group cohomology

- Gauge topological term $\to$ SPT states of group cohomology
- Gravitational topological term $\to$ topologically ordered states
- Mixed gauge-grav. topological term $\to$ new SPT states

- A new class of $U(1)$ SPT state in 4+1D labeled by $k \in \mathbb{Z}$:
  $Z_{\text{fixed-point}}(A_i, g_{\mu\nu}) = e^{i \frac{k}{3(2\pi)} \int_{M^5} \mathrm{d}A \wedge CS_3(g_{\mu\nu})}$, CS_3 grav. CS term

  - Dimension reduction $M^5 = S^2 \times M^3$ and put a 2π-flux of $U(1)$ through $S^2 \to Z_{\text{fixed-point}}(g_{\mu\nu}) = e^{i \frac{k}{3} \int_{M^3} CS_3(g_{\mu\nu})} \to k$-copy of E_8 bosonic IQ state. Wang-Gu-Wen 14

- 3+1D Z_2^T SPT states, need to use non-orientable state to probe.
  - $Z_{\text{fixed-point}}(g_{\mu\nu}) = e^{i \frac{k_1}{2} \int_{M^4} [w_1(g_{\mu\nu})]^4}$
    $\to Z_2^T$ SPT state described by $k_1 \in \mathcal{H}^4(Z_2^T, U_T(1)] = \mathbb{Z}_2$
  - $Z_{\text{fixed-point}}(g_{\mu\nu}) = e^{i \frac{k_2}{2} \int_{M^4} [w_2(g_{\mu\nu})]^2}$
    $\to Z_2^T$ SPT state described by $k_2 = \mathbb{Z}_2$ beyond group cohomology.

  Senthil-Vishwanath 12, Kapustin 14 Here w_i is the i^{th} Stiefel-Whitney class.

A classification of gapped quantum liquids

- **Symmetry breaking phases**: **group theory**
  No fractional statistics, no fractional quantum numbers
  Example: Ferromagnets, superfluids, *etc*
  Key: Symmetry breaking
- **Topo. ordered phases**: **n-category theory (extended TQFT)**
  Have fractional statistics, and fractional quantum numbers
  Example: FQH states, Z_2 spin liquid states, chiral spin liquid states, *etc*
  Key: Long-range entanglement (topological order)
- **SPT ordered phases**: **group cohomology theory and beyond**
  No fractional statistics, no fractional quantum numbers
  Example: Haldane phase in 1+1D, topological insulators, *etc*
  Key: Symmetry protection
- The above three features can coexist.

Classify LRE/topological-order and gravitational anomaly

- 1+1D: there is no topological order Verstraete-Cirac-Latorre 05
  1+1D: anomalous topological order are classified by unitary fusion categories (UFC). Lan-Wen 13 (anomalous topological order = gapped 2D edge)
- 2+1D: Abelian topological order are classified by K-matrices
  2+1D: topo. order with gappable edge are classified by *UFC* Levin-Wen 05
  2+1D: topological order are classified by $(UMTC, c)$ (?)
- Topological order with no non-trivial topo. excitations: Kong-Wen 14

	$1+1D$	$2+1D$	$3+1D$	$4+1D$	$5+1D$	$6+1D$
Boson:	0	$\mathbb{Z}_{E_8}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}$
Fermion:	$\mathbb{Z}_2$	$\mathbb{Z}_{p+ip}$	?	?	?	?

- The boundary of topologically ordered states has *gravitational anomaly*. Topological orders (patterns of long-range entanglement) classify gravitational anomalies in one lower dimension.
  long-range entanglement $\leftrightarrow$ geometry