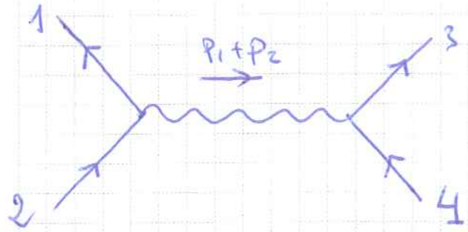
$\underline{e^+ e^- \longrightarrow q\bar{q}}$ in QED (all particles are massless)




$$p_1 + p_2 = p_3 + p_4$$

$$(p_1+p_2)^2 = 2p_1p_2 = S_{12} \equiv S = (p_3+p_4)^2$$

$$\boxed{\begin{aligned} p_1\cdot p_2 &= p_3\cdot p_4 \\ p_1\cdot p_3 &= p_2\cdot p_4 \\ p_1\cdot p_4 &= p_2\cdot p_3 \end{aligned}}$$

$$\mathcal{M} = \underbrace{\frac{i^2(-i)^2 g^2}{(p_1+p_2)^2}}_{\dfrac{g^2}{S_{12}} = \dfrac{g^2}{S}} \bar{u}_1 \gamma^\mu u_2 \, \bar{u}_3 \gamma_\mu v_4$$

$$\sum |\mathcal{M}|^2 = \frac{g^4}{S^2} \underbrace{Tr[\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1]}_{1} \underbrace{Tr[\gamma_\mu \not{p}_4 \gamma_\nu \not{p}_3]}_{2}$$

1 $4[p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - g^{\mu\nu} p_1\cdot p_2]$

2 $4[p_4^\mu p_3^\nu + p_3^\mu p_4^\nu - \underbrace{g^{\mu\nu} \, p_3\cdot p_4}_{\frac{1}{2}S}]$

1 · 2 $16\left[(p_1\cdot p_3)(p_2 p_4)\times 2 + (p_1\cdot p_4)(p_2\cdot p_3)\times 2 - 4\cancel{(p_1\cdot p_2)^2} + 4\cancel{(p_1\cdot p_2)^2}\right]$

$$= 32\left[(p_1\cdot p_3)^2 + (p_1\cdot p_4)^2\right] = 8(S_{13}^2 + S_{14}^2)$$

$$\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{2g^4}{S_{12}^2}(S_{13}^2 + S_{14}^2)$$

Helicity method    helicity computed individually

$$\left.\begin{array}{ll} ① & \langle 1|\gamma^\mu|2]\,\langle 3|\gamma_\mu|4] \\ ② & \langle 1|\gamma^\mu|2]\,[3|\gamma_\mu|4\rangle \\ ③ & [1|\gamma^\mu|2\rangle\,[3|\gamma_\mu|4\rangle \\ ④ & [1|\gamma^\mu|2\rangle\,\langle 3|\gamma_\mu|4] \end{array}\right\}$$ 4 possible combinations (rest are zero)

① $\langle 3|\gamma_\mu|4]\,[2|\gamma^\mu|1\rangle = 2\langle 31\rangle[24]$

② $2\langle 14\rangle[32]$

③ $\langle 2|\gamma^\mu|1]\,[3|\gamma_\mu|4\rangle = 2\langle 24\rangle[31]$

④ $\langle 3|\gamma_\mu|4]\,[1|\gamma^\mu|2\rangle = 2\langle 32\rangle[14]$

①①$^* = 4\langle 31\rangle[13]\,\langle 42\rangle[24] = 4 s_{13} s_{24} = 4 s_{13}^2$

②②$^* = 4\langle 14\rangle[41]\langle 23\rangle[32] = 4 s_{14} s_{23} = 4 s_{14}^2$

③③$^* = 4\langle 24\rangle[42]\langle 13\rangle[31] = 4 s_{24} s_{13} = 4 s_{13}^2$

④④$^* = 4\langle 32\rangle[23]\langle 41\rangle[14] = 4 s_{23} s_{14} = 4 s_{12}^2$

Sum over polarizations (helicities)

$$\boxed{\sum |M|^2 = 8\left(s_{13}^2 + s_{14}^2\right)}$$ Same result as before

Coda

① = ③$^*$    $\langle ij\rangle^* = [ji] = -[ij]$

② = ④$^*$

$$\bullet\ \boxed{\langle 1|\gamma^\mu|2][3|\gamma_\mu|4\rangle} = \langle 1|\gamma^\mu\underbrace{|2]\langle 2\,3\rangle[3|}_{\not p_2 \not p_3}\gamma_\mu|4\rangle \frac{1}{\langle 2\,3\rangle}$$

$$= \frac{1}{\langle 2\,3\rangle}\langle 1|\underbrace{\gamma^\mu \not p_2 \not p_3 \gamma_\mu}_{4 p_2\cdot p_3 = 2 s_{23} = 2\langle 2\,3\rangle[3\,2]}|4\rangle = \boxed{2\langle 1\,4\rangle[3\,2]}$$

$$\bullet\ \boxed{\langle 1\,2\rangle\langle 3\,4\rangle} = \langle 1\underbrace{2\rangle[2\,3]\langle 3}_{\not p_2 \not p_3}4\rangle \frac{1}{[2\,3]} = \frac{1}{[2\,3]}\langle 1|\not p_2 \not p_3|4\rangle$$

$$= \frac{\langle 1\,4\rangle\langle 3\,2\rangle\cancel{[2\,3]}}{\cancel{[2\,3]}} - \frac{1}{[2\,3]}\langle 1|\underbrace{\not p_3 \not p_2}_{|3\rangle[3\,2]\langle 2| = -|3\rangle[2\,3]\langle 2|}|4\rangle$$

$$= \boxed{\langle 1\,4\rangle\langle 3\,2\rangle + \langle 1\,3\rangle\langle 2\,4\rangle}$$