Filiberto Ares
Departamento de Física Teórica
Universidad de Zaragoza

Rényi entanglement entropy in fermionic/spin chains

with José G. Esteve & Fernando Falceto

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What is entanglement?

Schrödinger (1935):

- Entanglement reflects *"the best possible knowledge of a whole doesn't necessarily include the best possible knowledge of its parts"*

- This is *"not one but rather the characteristic trait of Quantum Mechanics"*. There isn't a classical analogue

E.g. bipartite system $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, $|0\rangle_i, |1\rangle_i$, $i = X, Y$

- $\frac{1}{\sqrt{2}} \left(|0\rangle_X \otimes |1\rangle_Y - |1\rangle_X \otimes |0\rangle_Y \right) \Rightarrow$ entangled

- $|0\rangle_X \otimes |1\rangle_Y$, $|1\rangle_X \otimes |0\rangle_Y$ $\Rightarrow$ separable

How quantify entanglement? Entanglement entropy

- State of a quantum system: density matrix ρ.
- In a bipartite system, $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$, state of X: reduced density matrix $\rho_X = \mathrm{Tr}_Y\, \rho$.
- Rényi entropy of X:

$$S_\alpha(X) = \frac{1}{1-\alpha} \log \mathrm{Tr}\, \rho_X^\alpha; \quad \alpha > 1$$

- In the limit $\alpha \to 1$, von Neumann entropy of X:

$$S_1(X) = -\mathrm{Tr}(\rho_X \log \rho_X)$$

Entanglement entropy

- Rényi entropy of a reduced state: measures the correlations between the subsystems.
- Pure state: $\rho = |\psi\rangle \langle\psi| \Rightarrow S_\alpha(X) = S_\alpha(Y)$
  - If $|\psi\rangle = |X\rangle \otimes |Y\rangle \Rightarrow S_\alpha(X) = 0$, no correlations
  - If $|\psi\rangle \neq |X\rangle \otimes |Y\rangle \Rightarrow S_\alpha(X) > 0$, X and Y are entangled
- Gibbs state: system at temperature $1/\beta$

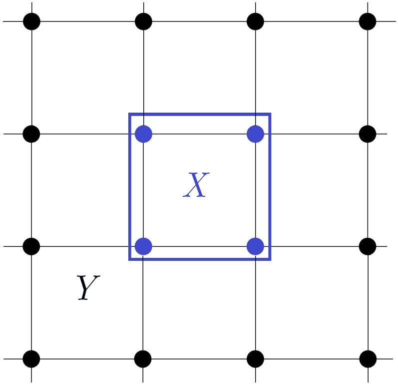
$$\rho = Z^{-1} e^{-\beta H}; \quad Z = \mathrm{Tr}(e^{-\beta H}); \quad H : \text{Hamiltonian}$$

  Mixed state $\Rightarrow S_\alpha(X) \neq S_\alpha(Y)$

  $S_\alpha(X)$: entanglement between X and Y + thermal correlations

Entanglement entropy

- Area law: ground state entanglement $\propto$ bonds broken isolating subsystem. E.g. in a 2-D lattice:



It's violated in excited states!

Our systems: fermionic and spin chains

- 1D lattice of N sites: $\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = (\mathbb{C}^2)^N$
- *Fermionic chain:* at each site n we define the annihilation/creation fermionic operators: $a_n, a_n^\dagger$

$$\{a_n^\dagger, a_m\} = \delta_{nm}; \quad \{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} = 0$$

- *Spin chains:* at each site n we define Pauli operators σ_n^μ, $\mu = x, y, z$

$$[\sigma_n^\mu, \sigma_m^\nu] = 2i\delta_{nm} \sum_{\tau=x,y,z} \varepsilon^{\mu\nu\tau} \sigma_n^\tau; \quad \{\sigma_n^\mu, \sigma_n^\nu\} = 2\delta^{\mu\nu}$$

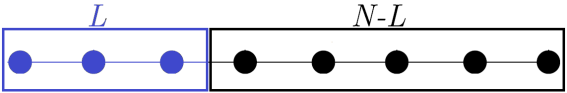
- Both systems are related through a non-local *Jordan-Wigner transformation*

Our goal

- Divide our chain in two subsystems X and Y:

$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$$

E. g. two intervals of contiguous sites:



- Compute $S_\alpha(X) = \dfrac{1}{1-\alpha} \log \mathrm{Tr}\, \rho_X^\alpha$ for a stationary state i.e. $\rho = |\Psi\rangle\langle\Psi|$ such that $H|\Psi\rangle = E_\Psi |\Psi\rangle$

$$\Downarrow$$

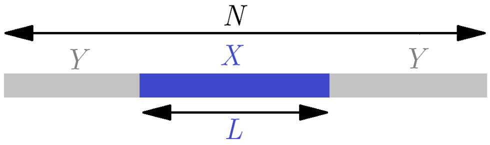
We must know $\rho_X = \mathrm{Tr}_Y\, \rho$, with dimension 2^L

$$\Downarrow$$

Its diagonalization is complicated!

Entanglement entropy and CFT

- Partition function 1D system $\equiv$ 1+1 field theory propagator
- Local, gapless Hamiltonian: (1+1) conformal field theory,



In the ground state

$$S_\alpha(X) \approx \frac{\alpha+1}{\alpha}\frac{c}{6}\log L + C_\alpha; \quad N \to \infty$$

(Holzhey, Larsen, Wilczek, hep-th/9403108)
(Calabrese & Cardy, hep-th/0405152)

c: central charge of the underlying CFT

Using holographic techniques (AdS/CFT correspondence)
(Ryu & Takayanagi, hep-th/0603001)

CFT in $1+1$-dim $\quad\equiv\quad$ Quantum gravity (string theory) in AdS $2+1$-dim

Alternatively...

Quadratic, translational invariant, periodic Hamiltonians of the form

$$H = \sum_{n=1}^{N} \sum_{l=1}^{N/2} J_l a_n^\dagger a_{n+l} + h.c. \quad a_{n+N} \equiv a_n, \quad J_l \in \mathbb{C}$$

Its eigenstates $H |\Psi_{\mathcal{K}}\rangle = E_{\mathcal{K}} |\Psi_{\mathcal{K}}\rangle$ are

$$|\Psi_{\mathcal{K}}\rangle = \prod_{k \in \mathcal{K}} b_k^\dagger |0\rangle$$

$$b_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i\theta_k n} a_n, \quad \theta_k = \frac{2\pi k}{N}, \quad k = -N/2, \ldots, N/2 - 1$$

$$\mathcal{K} \subset \{-N/2, \ldots, N/2 - 1\}$$

Wick decomposition and correlation matrix

- Eigenstates $|\Psi_{\mathcal{K}}\rangle = \prod_{k\in\mathcal{K}} b_k^\dagger |0\rangle$ satisfy Wick decomposition property. E. g.

$$\mathrm{Tr}(\rho a_{n_1}^\dagger a_{n_2}^\dagger a_{n_3} a_{n_4}) = \mathrm{Tr}(\rho a_{n_1}^\dagger a_{n_4})\, \mathrm{Tr}(\rho a_{n_2}^\dagger a_{n_3}) - \mathrm{Tr}(\rho a_{n_1}^\dagger a_{n_3})\, \mathrm{Tr}(\rho a_{n_2}^\dagger a_{n_4})$$

- Then we can write (Peschel, cond-mat/0212631, Vidal, Latorre, Rico, quant-ph/0211074)

$$S_\alpha(X) = \frac{1}{1-\alpha}\,\mathrm{Tr}\log\left[\left(\frac{I+V_X}{2}\right)^\alpha + \left(\frac{I-V_X}{2}\right)^\alpha\right]$$

$$(V_X)_{nm} = \mathrm{Tr}(\rho[a_n^\dagger, a_m]), \quad n, m \in X$$

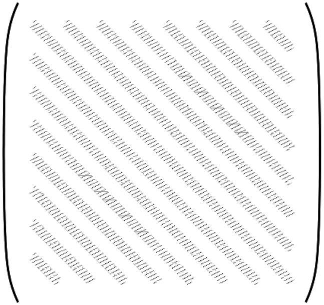
V_X is a $L \times L$ matrix!

Correlations and Toeplitz matrices

$$(V_X)_{nm} = \frac{1}{N}\left(\sum_{k\in\mathcal{K}} e^{i\theta_k(n-m)} - \sum_{k\notin\mathcal{K}} e^{i\theta_k(n-m)}\right)$$

Since X is an interval of contiguous sites, V_X is a Toeplitz matrix: $(V_X)_{nm} = \xi(n-m)$

$$V_X =$$



Fisher-Hartwig theorem

In the thermodynamic limit $N \to \infty$,

$$\theta_k \to \theta, \text{ configuration } \mathcal{K} \to \text{density } g(\theta)$$

$$(V_X)_{nm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{i\theta(n-m)} \mathrm{d}\theta$$

If $g(\theta)$ is a piecewise function, Fisher-Hartwig theorem for Toeplitz matrices (Fisher & Hartwig '69, Basor '74) leads to

$$S_\alpha(X) = A_\alpha L + B_\alpha \log L + C_\alpha + \cdots$$

(Jin & Korepin, quant-ph/0304108v4; FA, Esteve, Falceto, Sánchez-Burillo, 1401.5922v2[quant-ph])

- If $g(\theta) = \pm 1 \Rightarrow A_\alpha = 0$
- If $g(\theta)$ has ν discontinuities $\Rightarrow B_\alpha, C_\alpha \neq 0$

Physical insights

Three different behaviours for $S_\alpha(X)$,

- $S_\alpha(X) = B_\alpha \log L + C_\alpha$
  $|\Psi_{\mathcal{K}}\rangle$: ground state local, gapless Hamiltonian (CFT prediction)

  $$B_\alpha = \frac{\alpha+1}{\alpha}\frac{\nu}{12}$$

  $c = \nu/2 \Rightarrow$ # massless particles CFT$=$ # discontinuities $g(\theta)$

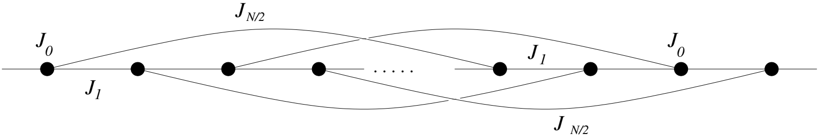
- If Hamiltonian is gapped: $S_\alpha(X) = 0$

- $S_\alpha(X) = A_\alpha L + B_\alpha \log L + C_\alpha$
  $|\Psi_{\mathcal{K}}\rangle$: ground state non-local Hamiltonian

Physical insights

- Consider

$$H = \sum_n J_0 a_n^\dagger a_n + J_1 a_n^\dagger a_{n+1} + J_{N/2} a_n^\dagger a_{n+N/2} + h.c., \quad J_{N/2} = J_0, \quad J_1 = -2J_0$$



- Its ground state follows $S_\alpha(X) = A_\alpha L + B_\alpha \log L + C_\alpha$

Why a linear term?

Physical insights

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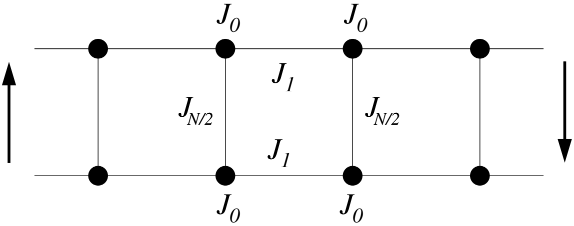

Physical insights

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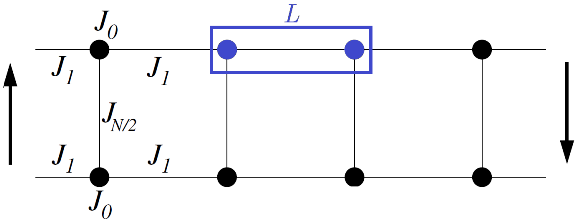


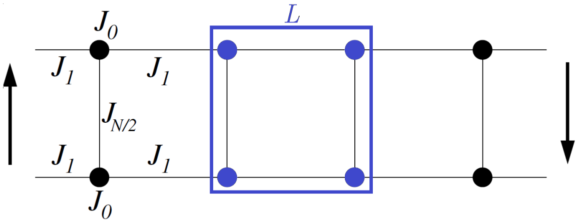
- Its ground state follows $S_\alpha(X) = A_\alpha L + B_\alpha \log L + C_\alpha$

Area law: entanglement $\propto$ bonds broken isolating subsystem
Agreement with $S_\alpha(X) \propto L =$ bonds broken

Far from one interval...

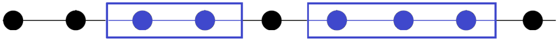
...V_X is not a Toeplitz matrix.

- A fragment of a ladder



$S_\alpha(X)$ follows CFT results

- Several intervals



CFT and holographic techniques for the ground state

(Calabrese, Cardy, Tonni, 0905.2069[hep-th], Hubeny & Rangamani, 0711.4118v2[hep-th])

We have a conjecture for $|\Psi_{\mathcal{K}}\rangle$ (FA, Esteve, Falceto, 1406.1668[quant-ph])

Thanks for your attention