Neutrino physics

$$P_R = \frac{1}{2}(1+\gamma_5)$$

1. $\psi_L = P_L \psi$ with $P_L = \frac{1}{2}(1-\gamma_5)$, $P_L^2 = P_L$

Lorentz transformation: $\psi'(x') = S(\Lambda)\psi(x)$

with $S(\Lambda) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)$

(* flips sign twice)

i) Check that $\psi_L \to S(\Lambda)\psi_L$

since $[\gamma_5, \sigma_{\mu\nu}] \overset{*}{=} 0 \Rightarrow [P_L, \sigma_{\mu\nu}] = 0 \Rightarrow [P_L, (\sigma^{\mu\nu})^n] = 0$

hence $[P_L, S(\Lambda)] = 0$

$$\boxed{\psi_L' = P_L S(\Lambda)\psi = S(\Lambda) P_L \psi = S(\Lambda)\psi_L} \checkmark$$

defining $\psi_L^c \equiv C(\overline{\psi_L})^T$ check that $\psi_L^c \to S(\Lambda)\psi_L^c$

$$= C\gamma_0^T \psi_L^* = -\gamma_0 C \psi_L^*$$

$$\boxed{\overline{\psi}' = (S(\Lambda)\psi)^\dagger \gamma^0 = \psi^\dagger \underbrace{S(\Lambda)^\dagger \gamma^0}_{\gamma^0 S(\Lambda^{-1})} = \overline{\psi}\, S(\Lambda^{-1})}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$\sigma^{\mu\nu\dagger} = -\frac{i}{2}\left(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger}\right) = \gamma^0\sigma^{\mu\nu}\gamma^0 \quad \left[(\gamma^0\sigma^{\mu\nu}\gamma^0)^n = \gamma^0(\sigma^{\mu\nu})^n\gamma^0 \right]$$

$$S(\Lambda)^\dagger = \gamma^0 \exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\gamma^0 = \gamma^0 S(\Lambda^{-1})\gamma^0 = \gamma^0 S(\Lambda)^{-1}\gamma^0$$

$$\overline{\psi}_L' = \overline{\psi}_L S(\Lambda^{-1}) \quad \checkmark$$

$$\boxed{\psi_L^{c\prime} = C\, S(\Lambda^{-1})^T (\overline{\psi}_L)^T = S(\Lambda) \underbrace{C(\overline{\psi}_L)^T}_{\psi_L^c} = S(\Lambda)\psi_L^c} \checkmark$$

(see details on next page)

One has to use the following properties of C:

$$C\gamma^\mu C = \gamma^{\mu T} \Longrightarrow \gamma^\mu = C\gamma^{\mu T} C$$

$$C^2 = -1 \qquad C^T = C^+ = C^{-1} = -C$$

$$\sigma^{\mu\nu T} = \frac{i}{2}\left(\gamma^{\nu T}\gamma^{\mu T} - \gamma^{\mu T}\gamma^{\nu T}\right) = C\sigma^{\mu\nu}C = -C\sigma^{\mu\nu}C^+$$

$$S(\Lambda)^T = -C\, S(\Lambda^{-1})\, C = -C\, S(\Lambda)^{-1} C$$

[I have used $\left(C\sigma^{\mu\nu}C^+\right)^n = C(\sigma^{\mu\nu})^n C^+$]

(ii) Check that $\gamma_5 \psi_L^c = \psi_L^c$

$$\boxed{C\gamma^5 C} = i\, C\gamma^0\gamma^1\gamma^2\gamma^3 C = -i\,\gamma^{0T}\gamma^{1T}\gamma^{2T}\gamma^{3T}$$

$$= -i\,(\underbrace{\gamma^3\gamma^2\gamma^1}\gamma^0)^T = +i\,(\gamma^0\gamma^3\underbrace{\gamma^2\gamma^1})^T = i\,(\gamma^0\gamma^1\gamma^3\gamma^2)$$

$$= -i\,(\gamma^0\gamma^1\gamma^2\gamma^3)^T = \boxed{-\gamma_5^T} \Rightarrow P_{L,R}^T = -C P_{L,R} C$$

$$\overline{\psi_L} = \psi^+ \underbrace{P_L^+}_{P_L \text{ (since } \gamma_5^+ = \gamma_5)} \gamma^0 = \overline{\psi} P_R \qquad (\text{since } \gamma_5\gamma^0 = -\gamma^0\gamma_5)$$

$$\psi_L^c = C(\overline{\psi} P_R)^T = C P_R^T \overline{\psi}^T = P_R C \overline{\psi}^T \text{ (right handed)}$$

Using $\gamma_5 P_R = P_R$ $\Rightarrow$ $\boxed{\gamma_5 \psi_L^c = +\psi_L^c}$ ✓

$\gamma_5 P_L = -P_L$

Aside $\quad S(\Lambda^{-1}) = S(\Lambda)^{-1}$

since $\quad S(\Lambda_1)\, S(\Lambda_2) = S(\Lambda_1 \Lambda_2)$

(iii) $\mathcal{L}_M = i\bar{\psi}_L \not{\partial} \psi_L - \frac{m}{2}\left(\overline{\psi_L^c}\psi_L + \bar{\psi}_L \psi_L^c\right)$

(iv) $\bar{\psi}_L \psi_L^c = \psi_L^\dagger \gamma^0 C \bar{\psi}_L^T = \psi_L^\dagger \gamma^0 C \gamma^{0T} \underbrace{(\psi_L^\dagger)^T}_{\psi_L^*}$

$= -\psi_L^\dagger C \psi_L^* = -(\psi_L^*)^T C \psi_L^* = -\sum_{\alpha\beta} (\psi_L^*)_\alpha C_{\alpha\beta} (\psi_L^*)_\beta$

↳ Dirac indices

given that $C_{\alpha\beta} = -C_{\beta\alpha}$ (since $C^T = -C$) antisymmetric

we need $(\psi_L^*)_\alpha (\psi_L^*)_\beta = -(\psi_L^*)_\beta (\psi_L^*)_\alpha$ ☆

in order to have a non-zero sum ✓

(v) Majorana field $\psi_M = \psi_L + \psi_L^c$; $\psi_M^c = \psi_M$

$\bar{\psi}_M \psi_M = \bar{\psi}_L \psi_L^c + \overline{\psi_L^c} \psi_L$ since other crossed terms vanish due to chirality

$\bar{\psi}_M \not{\partial} \psi_M = \bar{\psi}_L \not{\partial} \psi_L + \underbrace{\overline{\psi_L^c} \not{\partial} \psi_L^c}_{\circledast} = 2\bar{\psi}_L \not{\partial} \psi_L$

see next page

$\boxed{\overline{\psi_L^c} = (C\bar{\psi}_L^T)^\dagger \gamma^0 = -(\gamma^{0T}\psi_L^*)^\dagger C\gamma^0 = -(C\gamma^0 C\psi_L^*)^\dagger C\gamma^0}$

$\boxed{= \psi_L^T C}$

finish iv $\quad \overline{\psi_L^c}\psi_L = \psi_L^T C \psi_L = \sum_{\alpha\beta} (\psi_L)_\alpha C_{\alpha\beta} (\psi_L)_\beta$

☆ only non-zero for anticommuting ψ_L'

$$\circledast \quad \boxed{\overline{\psi_L^c}\, \not\partial\, \psi_L^c} = \psi_L^T \underbrace{C \not\partial C}_{\not\partial^T} \gamma_0^T \psi_L^* \underset{\substack{\uparrow \\ \text{integrate by parts}}}{=} -(\gamma_0 \not\partial \psi_L)^T (\psi_L^\dagger)^T \underset{\substack{\uparrow \\ \text{Grassmann variables}}}{=}$$

$$= \psi_L^\dagger \gamma_0 \not\partial \psi_L = \boxed{\overline{\psi}_L \not\partial \psi_L} \quad \checkmark$$

putting things together $\boxed{\mathcal{L}_M = \frac{i}{2} \overline{\psi}_M \not\partial \psi_M - m \overline{\psi}_M \psi_M}$

(vi) Weyl representation
$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix}$$

$$\sigma^\mu = (\mathbb{1}, \sigma^i)$$

$$\hat{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

$$\psi_M = \begin{pmatrix} \chi_M \\ \xi_M \end{pmatrix}$$

$$\overline{\psi}_M = (\xi_M^\dagger, \chi_M^\dagger)$$

$$\frac{i}{2} \overline{\psi}_M \not\partial \psi_M = \frac{i}{2} \left(\xi_M^\dagger \sigma_\mu \partial^\mu \xi_M + \chi_M^\dagger \hat{\sigma}_\mu \partial^\mu \chi_M \right) - m \left(\chi_M^\dagger \xi_M + \xi_M^\dagger \chi_M \right)$$

$$- m \overline{\psi}_M \psi_M$$

EOM's
$$\frac{i}{2} \sigma_\mu \partial^\mu \xi_M = m \chi_M$$

$$\frac{i}{2} \hat{\sigma}_\mu \partial^\mu \chi_M = m \xi_M$$

$\boxed{\text{Coda}}$ $\boxed{(\psi_L^c)^c} = -\gamma_0 C (\psi_L^c)^* = \gamma_0 C \underbrace{\gamma_0^*}_{\gamma_0^T} \underbrace{C^*}_{C} \psi_L =$

$$= \gamma_0 \underbrace{C \gamma_0^T C}_{\gamma_0} \psi_L = \boxed{\psi_L} \quad \checkmark$$

I have used $C^* = (C^\dagger)^T$
$= -C^T = C$

2. See-saw Lagrangian

$$\mathcal{L}_M = i\bar\psi_L \not\partial \psi_L + i\bar\psi_R \not\partial \psi_R - d\bar\psi_R\psi_L - \frac{1}{2} m \bar\psi_R \psi_R^c + h.c.$$

$$-d\bar\psi_L\psi_R - \frac{1}{2} m \underbrace{\overline{\psi_R^c}\psi_R}_{(1)}$$

$$\Psi_L = \begin{pmatrix}\psi_L \\ \psi_R^c\end{pmatrix} \qquad M = \begin{pmatrix}0 & d \\ d & m\end{pmatrix}$$

$$i\Psi_L \not\partial \Psi_L = i\bar\psi_L \not\partial \psi_L + i\bar\psi_R \not\partial \psi_R$$

$$M\Psi_L = \begin{pmatrix} d\psi_R^c \\ d\psi_L + m\psi_R^c \end{pmatrix}$$

$$\overline{\Psi_L^c} = (\overline{\psi_L^c}, \bar\psi_R)$$

$$\boxed{\overline{\Psi_L^c} M \Psi_L} = d\overbrace{\overline{\psi_L^c}\psi_R^c}^{\bar\psi_R\psi_L \ (2)} + d\bar\psi_R\psi_L + m\bar\psi_R\psi_R^c$$

$$\boxed{= 2d\bar\psi_R\psi_L + m\bar\psi_R\psi_R^c}$$

(1) $$(\bar\psi_R\psi_R^c)^\dagger = \psi_R^{c\dagger}(\bar\psi_R)^\dagger = \overbrace{\psi_R^T C}^{\overline{\psi_R^c}}\overbrace{\gamma_0\gamma_0}^{=1}\psi_R = \overline{\psi_R^c}\psi_R$$

$$\boxed{\begin{aligned}\mathcal{L}_M = i\Psi_L \not\partial \Psi_L - \frac{1}{2}\overline{\Psi_L^c} M \Psi_L + h.c. &= i\bar\psi_L \not\partial \psi_L + i\bar\psi_R \not\partial \psi_R - d\bar\psi_R\psi_L \\ &- d\bar\psi_L\psi_R - \frac{1}{2} m\bar\psi_R\psi_R^c - \frac{1}{2}\overline{\psi_R^c}\psi_R\end{aligned}}$$

(2) $$\overline{\psi_L^c}\psi_R^c = -\psi_L^T \underbrace{C\gamma_0 C}_{\gamma_0^T} \psi_R^* = -(\gamma_0\psi_L)^T\psi_R^*$$

$$= \psi_R^\dagger\gamma_0\psi_L = \bar\psi_R\psi_L$$

Diagonalize M

$$M - \lambda \mathbb{1} = \begin{pmatrix} -\lambda & d \\ d & m-\lambda \end{pmatrix}$$

$$|M - \lambda \mathbb{1}| = 0 \Rightarrow \lambda^2 - \lambda m - d^2 = 0$$

$$\lambda_\pm = \frac{m \pm \sqrt{m^2 + 4d^2}}{2} = \frac{m}{2}\left(1 \pm \sqrt{1 + 4\frac{d^2}{m^2}}\right)$$

(i) $m \gg d$

$$\lambda_+ = m$$

$$\lambda_- = \frac{m}{2}\left(-2\frac{d^2}{m^2}\right) = -\frac{d^2}{m} < 0 \text{ ?}$$

(ii) $m \ll d$

$$\lambda_\pm = \pm d$$