Heavy ions — exercises

(Dated: September 20, 2014)

THE BJORKEN MODEL

1) Write t and z expressed in terms of the proper time $\tau = \sqrt{t^2 - z^2}$ and the space-time rapidity $\eta = \frac{1}{2} \ln \frac{t+z}{t-z}$. Write the expression for the local flow velocity $u^\mu \equiv \mathrm{d}x^m u / \mathrm{d}\tau$.

2) Given the stress-energy tensor for a perfect fluid $T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - P g^{\mu\nu}$, where ϵ is the energy density and P the pressure, write an evolution equation in propert time for ϵ. Solve the equation for a typical value of the speed of the sound $c_s^2 = \mathrm{d}P/\mathrm{d}\epsilon$. **Hint:** Use the conservation equation $\partial_\mu T^{\mu\nu} = 0$.

3) Using the fundamental thermodynamical relation (at fixed volume), $\mathrm{d}\epsilon = T \mathrm{d}s$, write and solve the evolution equation for the entropy density s. Furthermore, using the relations $\mathrm{d}P = s \mathrm{d}T$ (at fixed volume) write and solve the evolution equation for the temperature.

Solution

1) The Bjorken model is a 1D solvable hydrodynamical model where all thermodynamical quantities only depend on the proper time. After straightforward manipulations we find that $t = \tau \cosh \eta$, $z = \tau \sinh \eta$ and the flow velocity is simply $u^\mu = \big(\cosh \eta, 0, 0, \sinh \eta \big) = x^\mu / \tau$.

2) Applying the differential operator on the stress-energy tensor and taking advantage of the chain rule $\partial / \partial x^\mu = \partial \tau / \partial x^\mu \, \partial / \partial \tau$, we obtain

$$\partial_\mu T^{\mu\nu} = \frac{\partial \big(\epsilon + P \big)}{\partial \tau} u^\mu \frac{\partial \tau}{\partial x^\mu} u^\nu - g^{\mu\nu} \frac{\partial P}{\partial \tau} \frac{\partial \tau}{\partial x^\mu} \tag{1}$$

$$+ \big(\epsilon + P \big) \left[\frac{\partial u^\mu}{\partial x^\mu} u^\nu + u^\mu \frac{\partial u^\nu}{\partial x^\mu} \right] . \tag{2}$$

Working out the details, we obtain

$$u^\mu \frac{\partial \tau}{\partial x^\mu} = \cosh \eta \frac{\partial \tau}{\partial t} + \sinh \eta \frac{\partial \tau}{\partial z} = 1 \tag{3}$$

$$\frac{\partial u^\mu}{\partial x^\mu} = \frac{1}{\tau} \tag{4}$$

$$u^\mu \frac{\partial u^\nu}{\partial x^\mu} = 0 \tag{5}$$

so that all terms in the equation are proportional to u^ν. This implies that

$$\frac{\partial \epsilon}{\partial \tau} + \frac{\epsilon + P}{\tau} = 0 \, . \tag{6}$$

Using that $\mathrm{d}P/\mathrm{d}\epsilon = P/\epsilon = c_s^2$, the solution is $\epsilon(\tau) = (\tau_0/\tau)^{1+c_s^2}\epsilon(\tau_0)$. Recall, that $c_s^2 = 1/3$ in a perfect fluid.

3) Using the definition of the entropy density $sT = \epsilon + P$ and the relation above, we get that

$$\frac{\partial s}{\partial \tau} + \frac{s}{\tau} = 0 \tag{7}$$

which is solved by $s(\tau) = \big(\tau_0/\tau\big)s(\tau_0)$. Note that the entropy density in the comoving frame, $s^\mu = s u^\mu$, is conserved, $\partial_\mu s^\mu = 0$! Finally, since

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = \frac{\mathrm{d}\epsilon}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}\tau} \tag{8}$$

we find the evolution equation for the temperature, $\partial T/\partial\tau + c_s^2 T/\tau = 0$, which is solved by $T(\tau) = \big(\tau_0/\tau\big)^{c_s^2}T(\tau_0)$.

THE EIKONAL APPROXIMATION AND THE PATH-ORDERED WILSON LINE

Definitions: Light-cone coordinates $x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^3)$, and $p\cdot x = p^+x^- + p^-x^+ - \boldsymbol{p}\cdot\boldsymbol{x}$.

1) Assuming the dominance of the +-momentum, calculate the S-matrix of an on-shell quark scattering off *one* color potential $A_\mu(x)$ (in the fundamental representation), $S_1(p',p)$. Show that $A_\mu(x) = A_\mu(x^+,\boldsymbol{x})$ results in the conservation of +-momentum, $2\pi\delta(p'^+ - p^+)$, and discuss why. **Hint:** Make use of the fact that $\frac{1}{2}\sum_\lambda \bar{u}^\lambda(p')\gamma^\mu u^\lambda(p) = 2p^\mu$ in the eikonal approximation.

2) Using the path ordering property,

$$\int \mathrm{d}x_1\ldots\mathrm{d}x_n\Theta(x_2-x_1)\ldots\Theta(x_n-x_{n-1})A(x_1)\ldots A(x_n) = \frac{1}{n!}\mathcal{P}\left[\int\mathrm{d}x\,A(x)\right]^n \tag{9}$$

calculate the S-matrix of 2 and n scatterings in the medium. Resum the S-matrices and identify the path-ordered Wilson line

$$\mathcal{U}(\boldsymbol{x}) = \mathcal{P}\exp\left[ig\int\mathrm{d}x^+A^-(x^+,\boldsymbol{x})\right]. \tag{10}$$

Solutions

1) We calcute the scattering of an intial quark with momentum p on a medium potential, ending up with a momentum p'. Using standard Feynman rules, the S-matrix of one scattering with the medium becomes

$$S_1(p',p) = \frac{1}{2}\sum_{\lambda,\lambda'}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}^{\lambda'}(p')\,ig\gamma^\mu\delta^{\lambda',\lambda}(2\pi)^4\delta^{(4)}\big(p'-p-k\big)\,A_\mu^a(k)T^a\,u^\lambda(p)\,, \tag{11}$$

where we have averaged over incoming spins and summed over outgoing ones. After simplifying and Fourier transforming the potential to configuration space and approximating $p^\mu A^a_\mu \approx p^+ A^{a,-}$, we get

$$S_1(p',p) = ig\, 2p^+ \int \mathrm{d}^4x\, e^{i(p'-p)\cdot x} A^{a,-}(x) T^a\,, \tag{12}$$

where $\mathrm{d}^4x = \mathrm{d}x^+\mathrm{d}x^-\mathrm{d}^2\boldsymbol{x}$. Note that is $A^{a,-}(x) \simeq A^{a,-}(x^+,0,\boldsymbol{x})$, which physically means that the medium is strongly boosted along the x^- direction, opposite to the direction of the quark, then

$$\int \mathrm{d}x^- e^{i(p'-p)^+x^-} = 2\pi\delta(p'^+ - p^+)\,, \tag{13}$$

and we conserve the +-momentum along the trajectory. Also, we will work in a approximation where $p^- = \boldsymbol{p}^2/(2p^+) \to 0$ (which follows from $p^2 = 2p^+p^- - \boldsymbol{p}^2 = 0$) when $p^+ \to \infty$. Then, finally, we obtain

$$S_1(p',p) = 2\pi\,\delta\big(p'^+ - p^+\big)\, 2p^+ \int \mathrm{d}^2\boldsymbol{x} e^{-i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}} \int \mathrm{d}x^+\, igA^-(x^+,\boldsymbol{x})\,, \tag{14}$$

where we have used the shorthand $A^aT^a = A$.

2) Using some of the steps developed above we immediately obtain the S-matrix for two scattering off potentials

$$\begin{aligned} S_2(p',p) = \frac{1}{2}\sum_\lambda \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int \mathrm{d}^4x_1\mathrm{d}^4x_2\, e^{i(p'-k)\cdot x_2 + i(k-p)\cdot x_1} \bar{u}^\lambda(p')\, ig\not{A}(x_2) \\ \times \frac{i\not{k}}{k^2+i\epsilon}\, ig\not{A}(x_2) u^\lambda(p)\,, \end{aligned} \tag{15}$$

where $\not{x} \equiv \gamma^\mu x_\mu$. Using the conservation and dominance of +-momentum, we can use the Dirac equation $\not{p}u(p) = 0$ and the relation $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$, to simplify

$$\bar{u}(p')\gamma^\mu\not{k}\gamma^\nu u(p) \simeq 2p^\nu\, \bar{u}(p')\gamma^\mu u(p)\,, \tag{16}$$

where we have suppressed the spin index. Then

$$S_2(p',p) = -ig^2(2p^+)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int \mathrm{d}^4x_1\mathrm{d}^4x_2\, \frac{e^{i(p'-k)\cdot x_2 + i(k-p)\cdot x_1}}{2k^+k^- + i\epsilon} A^-(x_1)A^-(x_2)\,. \tag{17}$$

The integral over the internal momentum k can be performed in the high-energy approximation to give

$$\int \mathrm{d}k^- \frac{e^{i(x_1-x_2)^+k^-}}{2k^+k^- + i\epsilon} = -2\pi i \frac{\Theta(x_2^+ - x_1^+)}{2p^+} \tag{18}$$

$$\int \mathrm{d}k^+ e^{i(x_1-x_2)^-k^+} = 2\pi\,\delta(x_1^- - x_2^-) \tag{19}$$

$$\int \mathrm{d}^2\boldsymbol{k} e^{i(\boldsymbol{x}_1-\boldsymbol{x}_2)\cdot\boldsymbol{k}} = (2\pi)^2\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)\,. \tag{20}$$

Then, after performing the remaining simplifications and integrals we get

$$S_2(p',p) = 2\pi\delta(p'^+ - p^+)\, 2p^+ \int \mathrm{d}^2\boldsymbol{x} e^{-i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}} \frac{1}{2}\mathcal{P}\left[\int \mathrm{d}x^+\, igA^-(x^+,\boldsymbol{x})\right]^2 , \tag{21}$$

where we have used the hint given above. By analogy, $S_n(p',p)$ is found by repacing the factor $1/2$ by $1/n!$ and $[\ldots]^2 \to [\ldots]^n$. When summing all the amplitudes, the factor in square brackets simply exponantiates, so that the final, re-summed S-matrix becomes

$$\begin{aligned} S(p,p') &= \sum_{n=0}^{\infty} S_n(p',p) \\ &= 2\pi\delta(p'^+ - p^+)\, 2p^+ \int \mathrm{d}^2\boldsymbol{x} e^{-i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}} \mathcal{U}(\boldsymbol{x})\,, \end{aligned} \tag{22}$$

where we have used the definition for the Wilson line given above.

ENERGY LOSS

1) Assuming that every particle looses a *constant* amount of energy in the plasma, calculate the high-$p_\perp$ behavior of the nuclear modification factor

$$R_{AA} = \frac{\mathrm{d}N_{AA}/\mathrm{d}\eta \mathrm{d}p_\perp}{N_{\mathrm{coll}}\, \mathrm{d}N_{pp}/\mathrm{d}\eta \mathrm{d}p_\perp} \tag{23}$$

when the underlying (pp) spectrum was **a)** a power-like or **b)** exponential.

2) How is this behavior modified if interaction in the plasma simply absorb particles. Discuss both cases.

Solutions

1) The modified spectrum in nucleus-nucleus (AA) collisions can be written as

$$\frac{1}{N_{\mathrm{coll}}}\frac{\mathrm{d}N_{AA}}{\mathrm{d}p_\perp} = \left|\frac{\mathrm{d}p'_\perp}{\mathrm{d}p_\perp}\right| \frac{1}{N_{\mathrm{coll}}}\frac{\mathrm{d}N}{\mathrm{d}p'_\perp}\,, \tag{24}$$

where $p'_\perp = p_\perp + \delta p_\perp$ means that all particles loose a certain amount $\delta p_\perp$ of transverse momentum as they pass through the plasma. In case of a constant shift, $\delta p_\perp = \mathrm{const}$, the Jacobian of the transformation is 1. For a power-like spectrum $\mathrm{d}N/\mathrm{d}p_\perp = A p_\perp^{-n}$ the nuclear modification factor becomes

$$R_{AA} = \left(\frac{1}{1+\delta p_\perp/p_\perp}\right)^n , \tag{25}$$

which goes to 1 as $p_\perp \to \infty$. In case of an exponential spectrum $\mathrm{d}N/\mathrm{d}p_\perp = B \exp\left(-p_\perp/T_{\mathrm{eff}}\right)$, the nuclear modification factor is a constant,

$$R_{AA} = e^{-\delta p_\perp / T_{\mathrm{eff}}} \,. \tag{26}$$

In both cases, if there is no medium interaction, $\delta p_\perp = 0$, $R_{AA} = 1$.

2) If the particles are absorbed in the medium we simply have to rescale the prefactor of the spectrum, $A \to A' < A$ and $B \to B' < B$. Both for the power-like and exponential spectra, the nuclear modification factor is a constant.