Zitterbewegung in External Magnetic Field

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The Dirac Equation

The Dirac equation describes spin $\frac{1}{2}$ particles.

$$H\psi = (\boldsymbol{\alpha}\cdot\mathbf{P} + \beta m)\psi$$

$$\psi = u(\mathbf{p})e^{-ip\cdot x}$$

$$Hu = \begin{pmatrix} m & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\boldsymbol{\sigma}\cdot\mathbf{p}u_B = (E - m)u_A$$

$$\boldsymbol{\sigma}\cdot\mathbf{p}u_A = (E + m)u_B$$

$$u^{(s)} = N\begin{pmatrix} \chi^{(s)} \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi^{(s)} \end{pmatrix}, \qquad E > 0 \quad u^{(s+2)} = N\begin{pmatrix} \dfrac{-\boldsymbol{\sigma}\cdot\mathbf{p}}{|E|+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}, \qquad E < 0$$

Zitterbewegung

Erwin Schrödinger in 1930

$$\vec{v} \;=\; \hbar\frac{\partial x_k}{\partial t} = i[H, x_k] = c\alpha_k$$

$$(\vec{\alpha})_k^2 \;=\; \mathbf{1}$$

$$v_k \;=\; \pm\, c$$

$$[(\vec{\alpha})_k,\ (\vec{\alpha})_\ell] \;\neq\; 0 \qquad \text{for} \quad k \;\neq\; \ell$$

$$\hbar\frac{\partial \alpha_k}{\partial t} = i[H, \alpha_k] = 2[i\gamma_k m - \sigma_{kl}p^l] = 2i[p_k - \alpha_k H]$$

$$\alpha_k(t) = \alpha_k(0)\exp[-2iHt/\hbar] + cp_k H^{-1}$$

$$x_k(t) = x_k(0) + c^2 p_k H^{-1} + \\ + \frac{1}{2} i\hbar c H^{-1}\left(\alpha_k(0) - cp_k H^{-1}\right)\left(\exp\left(-2iHt/\hbar\right) - 1\right)$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{S} = \frac{1}{2}\,\vec{\Sigma}$$

$$\frac{d}{dt}\,\vec{L} = i\left[H, \vec{L}\right] = i(\vec{\alpha})_\ell \; \left[p_\ell, \vec{L}\right]$$

$$\frac{d}{dt}\,\vec{L} = \vec{\alpha} \times \vec{p}$$

$$\frac{d}{dt}\,\vec{S} = -\vec{\alpha} \times \vec{p}$$

$$\vec{J} := \vec{L} + \vec{S}$$

$$\tfrac{d}{dt}\,\vec{J} = 0$$

$$\begin{aligned}\langle\vec{v}\rangle &= \langle\vec{\alpha}\rangle \\ &= u^{\dagger}\ (\vec{p})\vec{\alpha}\ u(\vec{p}) \\ &= N^2\ \eta^{\dagger}\ \frac{\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\ +\ (\vec{\sigma}\cdot\vec{p})\ \vec{\sigma}}{p^0\ +\ m}\ \eta \\ &= N^2\ \frac{2\vec{p}}{p^0\ +\ m}\ \eta^{\dagger}\ \eta\ .\end{aligned}$$

$$\langle\vec{v}\rangle\ =\ \frac{\vec{p}}{p_0}\ =\ \vec{v}_{classical}$$

$$\langle\frac{d\vec{L}}{dt}\rangle\ =\ \vec{0}\ =\ \langle\frac{d\vec{S}}{dt}\rangle$$

It turns out that if one only looks at a the expectation value of only positive or only negative energy waves the Zitterbewegung term vanishes.

The most common interpretation of the Zitterbewegung-term is usually interpreted as an interference between positive- and negative-energy waves.

Phenomena usually mentioned in advanced books of Quantum Mechanics [1]

Often mentioned in passing to the interpretation of negative energy states and the "Dirac Sea" as antiparticles.

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[1] For example: A. Messiah, Quantum Mechanics Volume II
J. J. Sakurai: Advanced Quantum Mechanics

Shroedinger's interpretation of his results from the Dirac equation:

Local circulatory motion $\Rightarrow$ Electron spin and magnetic moment

$$\omega_0 = 2mc^2/\hbar = 1.6 \cdot 10^{21} s^{-1}$$
So $\lambda_0 = c/\omega_0 = \hbar/2mc = 1.9 \cdot 10^{-13m}$

KERSON HUANG 1952

$$\psi(\mathbf{r}, t) = h^{-\frac{3}{2}} \int [C^{+}(\mathbf{p}) e^{-i\omega t} + C^{-}(\mathbf{p}) e^{i\omega t}] \exp(i\mathbf{p}\cdot\mathbf{r}/\hbar) d^3\mathbf{p},$$

$$C^{+}(\mathbf{p}) = a_1 u_1 + a_2 u_2$$
$$C^{-}(\mathbf{p}) = a_3 u_3 + a_4 u_4$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ kp_3 \\ kp_+ \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ kp_- \\ -kp_3 \end{pmatrix},$$

$$u_3 = \begin{pmatrix} -kp_3 \\ -kp_+ \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} -kp_- \\ kp_3 \\ 0 \\ 1 \end{pmatrix},$$

$$\psi(\mathbf{r}, 0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(r/r_0)$$

$$f(r/r_0) = (2/\pi r_0{}^2)^{\frac{3}{4}} \exp[-(r/r_0)^2]$$

$$\psi(\mathbf{r}, t) = \hbar^{-\frac{3}{2}} \int \left[\begin{bmatrix} 1 \\ 0 \\ kp_3 \\ kp_+ \end{bmatrix} e^{-i\omega t} + \begin{bmatrix} k^2p^2 \\ 0 \\ -kp_3 \\ -kp_+ \end{bmatrix} e^{i\omega t} \right] \frac{f(p/\eta)}{1+k^2p^2} \exp(i\mathbf{p}\cdot\mathbf{r}/\hbar) d^3\mathbf{p}$$

$$\langle \dot{\mathbf{r}} \rangle = \bar{\mathbf{v}} + 2c \int \mathbf{K}(\mathbf{p}) \sin(2\omega t + \phi(\mathbf{p})) d^3\mathbf{p}$$

$$\mathbf{K}(\mathbf{p}) = | C^{-*} \boldsymbol{\alpha} C^{+} |,$$

$$\phi(\mathbf{p}) = \tan^{-1}\{\mathrm{Re}(C^{-*} \boldsymbol{\alpha} C^{+}) / \mathrm{Im}(C^{-*} \boldsymbol{\alpha} C^{+})\}$$

$$\langle x_1 \rangle = -I \int_0^{2\pi} \lambda \sin(2\omega t + \varphi) d\varphi \qquad \langle x_2 \rangle = -I \int_0^{2\pi} \lambda \cos(2\omega t + \varphi) d\varphi$$

$$I = -(32\pi)^{-\frac{1}{2}}(\lambda / r_0)$$

$$\langle x_1 \rangle_\varphi = -\lambda \sin(2\omega t + \varphi) \qquad \langle x_2 \rangle_\varphi = -\lambda \cos(2\omega t + \varphi)$$

$$\langle \mathbf{r} \times \dot{\mathbf{r}} \rangle_1 = 0, \quad \langle \mathbf{r} \times \dot{\mathbf{r}} \rangle_2 = 0$$

$$\langle \mathbf{r} \times \dot{\mathbf{r}} \rangle_3 = (\hbar / m)(1 - \cos 2\omega t)$$

$$\langle \mu_1 \rangle = 0, \quad \langle \mu_2 \rangle = 0$$

$$\langle \mu_3 \rangle = (e\hbar / 2mc)(1 - \cos 2\omega t)$$

Zitterbewegung in the presence of an external uniform static magnetic field

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[c\alpha_i(p^i - \frac{e}{c}A^i) + \beta mc^2 + eA_0\right]\Psi$$

$$E_{\mathrm{ln}}^2 = m^2c^4 + p_z^2c^2 + ceB(n - l + 1 - 2s_z)$$

$$\frac{E_{\mathrm{B}\pm}}{\hbar} \simeq \pm\left(\frac{mc^2}{\hbar} \pm \frac{|e|\,B\sigma_\pm}{2mc}\right) \equiv \pm(\omega \pm \Omega\sigma_\pm)$$

$$\Psi(\boldsymbol{r},t) = h^{-3/2}\int\Bigg\{\mathcal{A}\begin{pmatrix}\phi^{\uparrow}\\ k_{+}^{\uparrow}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\phi^{\uparrow}\end{pmatrix}\exp\Big[-i(\omega+\Omega)t\Big] + \mathcal{B}\begin{pmatrix}\phi^{\downarrow}\\ k_{+}^{\downarrow}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\phi^{\downarrow}\end{pmatrix}\exp\Big[-i(\omega-\Omega)t\Big]$$

$$+\,\mathcal{C}\begin{pmatrix}k_{-}^{\uparrow}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi^{\uparrow}\\ \chi^{\uparrow}\end{pmatrix}\exp\Big[i(\omega-\Omega)t\Big] + \mathcal{D}\begin{pmatrix}k_{-}^{\downarrow}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi^{\downarrow}\\ \chi^{\downarrow}\end{pmatrix}\exp\Big[i(\omega+\Omega)t\Big]\Bigg\}\exp[i(p_y y + p_z z)/\hbar]\, d^3\pi,$$

for the spin–up states

$$\begin{cases}\exp(i\omega t) \to \exp[i(\omega-\Omega)t] & \text{for negative energy}\\ \exp(-i\omega t) \to \exp[-i(\omega+\Omega)t] & \text{for positive energy}\end{cases}$$

for the spin–down states

$$\begin{cases}\exp(i\omega t) \to \exp[i(\omega+\Omega)t] & \text{for negative energy}\\ \exp(-i\omega t) \to \exp[-i(\omega-\Omega)t] & \text{for positive energy}\end{cases}$$

$$\Psi(\boldsymbol{r},0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} f\left(\frac{r}{r_o}\right)$$

$$f(\frac{\pi}{\pi_o}) = (\frac{2}{\bar{\pi}\pi_o^2})^{3/4} \exp\Big[-(\frac{\pi}{\pi_o})^2\Big]$$

$$\Psi(\boldsymbol{r},t) \simeq h^{-3/2} \int \Bigg\{ \begin{pmatrix} 1 \\ 0 \\ K\pi_z \\ K\pi_+ \end{pmatrix} \exp\Big[-i(\omega+\Omega)t\Big] + \begin{pmatrix} 0 \\ 0 \\ -K\pi_z \\ 0 \end{pmatrix} \exp\Big[i(\omega-\Omega)t\Big]$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -K\pi_+ \end{pmatrix} \exp\Big[i(\omega+\Omega)t\Big] \Bigg\} f(\pi/\pi_o) \exp[i(p_y y + p_z z)/\hbar]\, d^3\pi\,.$$

$$< \dot{\boldsymbol{r}} >= \int \Psi^*(\boldsymbol{r},t)(c\boldsymbol{\alpha})\Psi(\boldsymbol{r},t)d^3r$$

$$< x >\simeq I^{\uparrow}\frac{\lambdabar_c}{2}\int_0^{2\bar{\pi}} \sin\Big[(\omega_{\mathrm{zbw}}+\omega_c)\,t+\varphi\Big]\,d\varphi$$

$$< y >\simeq I^{\uparrow}\frac{\lambdabar_c}{2}\int_0^{2\bar{\pi}} \cos\Big[(\omega_{\mathrm{zbw}}+\omega_c)\,t+\varphi\Big]\,d\varphi$$

$$< z >\simeq J\lambdabar_c \sin\left(\omega_{\mathrm{zbw}}t\right)$$

$$I^{\uparrow} \equiv -(8\bar{\pi})^{-\frac{1}{2}}\frac{\lambdabar_c}{r_o}\left(1-\frac{\omega_c}{\omega_{\mathrm{zbw}}}\right)$$

$$J \equiv 0$$

$$< x >_{\varphi_o}\simeq \frac{\lambdabar_c}{2}\sin\Big[(\omega_{\mathrm{zbw}}+\omega_c)\,t+\varphi_o\Big]$$

$$< y >_{\varphi_o}\simeq -\frac{\lambdabar_c}{2}\cos\Big[(\omega_{\mathrm{zbw}}+\omega_c)\,t+\varphi_o\Big]$$

The same procedure for the spin–down

$$< x > \simeq I^{\downarrow} \frac{\lambda\!\!\!^{-}_c}{2} \int_0^{2\bar{\pi}} \sin\Big[(\omega_{\rm zbw} - \omega_c)\, t - \varphi\Big]\, d\varphi$$

$$< y > \simeq I^{\downarrow} \frac{\lambda\!\!\!^{-}_c}{2} \int_0^{2\bar{\pi}} \cos\Big[(\omega_{\rm zbw} - \omega_c)\, t - \varphi\Big]\, d\varphi$$

$$< z > \simeq J \lambda\!\!\!^{-}_c \sin\left(\omega_{\rm zbw} t\right)$$

$$I^{\downarrow} \equiv -(8\bar{\pi})^{-\frac{1}{2}} \frac{\lambda\!\!\!^{-}_c}{r_o} \left(1 + \frac{\omega_c}{\omega_{\rm zbw}}\right)$$

$$J \equiv 0$$

$$<\mu_x^{\uparrow}>= 0, \qquad <\mu_y^{\uparrow}>= 0 \qquad \text{and} \qquad <\mu_z^{\uparrow}>= -\frac{|e|\,\lambda\!\!\!^{-}_c}{2}\Big[1 - \cos(\omega_{\text{zbw}} + \omega_c)t\Big]$$

$$<\mu_x^{\downarrow}>= 0, \qquad <\mu_y^{\downarrow}>= 0 \qquad \text{and} \qquad <\mu_z^{\downarrow}>= \frac{|e|\,\lambda\!\!\!^{-}_c}{2}\Big[1 - \cos(\omega_{\text{zbw}} - \omega_c)t\Big]$$

$$v_{\text{zbw}} = \left[\frac{\lambda\!\!\!^{-}_c}{2}(1 \pm \varepsilon)\right]\left[\omega_{\text{zbw}}(1 \mp \varepsilon)\right] = c(1 - \varepsilon^2) \simeq c$$

References

[1] P.A.M. Dirac, *Proc. Roy. Soc.* **A117** (1928), 610-624; *ibid* **A118** (1928), 351-361.

[2] P.A.M. Dirac, *The Principles of Quantum Mechanics*, (Oxford University Press, Oxford, 4th ed. 1958).

[3] M.E. Rose, *Relativistic Electron Theory*, (John Wiley & Sons, New York, 1961).

[4] J.D. Bjorken & S.D. Drell, *Relativistic Quantum Mechanics*, (McGraw–Hill, New York, 1964).

[5] J.J. Sakurai, *Advanced Quantum Mechanics*, (Pearson Education, Inc., Delhi India, 1967).

[6] C. Itzykson & J.-B. Zuber, *Quantum Field Theory*, (McGraw–Hill, Singapore, 1980).

[7] W. Greiner, *Relativistic Quantum Mechanics*, (Springer–Verlag, Berlin, 1990).

[8] E. Merzbacher, *Quantum Mechanics*, (John Wiley & Sons, New York, 3rd ed. 1998).

[9] K. Huang, *Am. J. Phys.* **20** (1952), 479-484.

[10] E. Schrödinger, *Sitz. Preuss. Akad. Wiss. Phys. Math. Kl.* **24** (1930), 418-428; *ibid* **3** (1931), 1.

[11] A.O. Barut & A.J. Bracken, *Phys. Rev.* **D23** (1981), 2454-2463.

[12] T.D. Newton & P. Wigner, *Rev. Mod. Phys.* **21** (1949), 400-406.

[13] T.M. Rusin & W. Zawadzki, *J. Phys.: Cond. Matter* **19** (2007), 136219 [18 pages]; *Phys. Rev.* **B76** (2007), 195439 [7 pages].

[14] D. Bender & *et al.*, *Phys. Rev.* **D30** (1984), 515-527.

[15] D. Hestenes, *Found. Phys.* **23** (1993), 365-387.

[16] W. Gordon, *Z. Physik* **50** (1927), 630-632.

[17] J. Frenkel, *Wave Mechanics, Advanced General Theory*, (Oxford University Press, London, 1934).

[18] A.O. Barut & A.J. Bracken, *Phys. Rev.* **D24** (1981), 3333-3334.

[19] D. Hestenes, *Found. Phys.* **20** (1990), 1213-1232.

[20] D. Hestenes, *Found. Phys.* **40** (2010), 1-54.

[21] M. Mathissen, *Acta Phys. Polon* **6** (1937), 163-200; *ibid* (1937), 218.

[22] H. Hönl, *Ann. Physik* **33** (1938), 565-585.

[23] J.W. Weyssenhof, *Nature* **141** (1938), 328-329; *Acta Phys. Polon* **9** (1947), 47-53.

[24] J. Brandmüller, *Naturwiss.* **38** (1951), 139-139.

[25] G. Salesi & E. Recami, *Phys. Lett.* **A267** (2000), 219-224.

[26] M.H.L. Pryce, *Proc. Roy. Soc. London* **A195** (1948), 62-81.

[27] L.L. Foldy & S.A. Wouthuysen, *Phys. Rev.* **78** (1950), 29-36.

[28] S. Tani, *Progr. Theor. Phys.* **6** (1951), 267-285.

[29] H. Feshbach & F. Villars, *Rev. Mod. Phys.* **30** (1958), 24-25.

[30] A.O. Barut & S. Malin, *Rev. Mod. Phys.* **40** (1968), 632-651.

[31] J.A. Lock, *Am. J. Phys.* **47** (1979), 797-802.

[32] P. Krekora, Q. Su & R. Grobe, *Phys. Rev. Lett.* **93** (2004), 043004 [4 pages].

[33] A.O. Barut & N. Zanghi, *Phys. Rev. Lett.* **52** (1984), 2009-2012.

[34] E. Recami & G. Salesi, *Phys. Rev.* **A57** (1998), 98-105.

[35] G. Salesi, *Int. J. Mod. Phys.* **A17** (2002), 347-374.

[36] A.O. Barut & M. Pavšič, *Class. Quant. Grav.* **4** (1987), L131-L136.

[37] E. Recami & G. Salesi, "Field theory of the electron: spin and Zitterbewegung", in *Particles, Gravity and Space–Time*, Ed. P.I. Pronin & G.A. Sardanashvily (World Scientific, Singapore, 1996), pp. 345-368, [hep-th/9508168].

[38] M. Pavšič, E. Recami & W.A. Rodrigues Jr., *Hadronic J.* **18** (1995), 97-118.

[39] W. Zawadzki, in *Optical Properties of Solides*, Ed. E.D. Haidemenakis (Gordon & Breach, New York, 1970), p. 179.

[40] F. Cannata, L. Ferrari & G. Russo, *Solid State Commun.* **74** (1990), 309-312.

[41] L. Ferrari & G. Russo, *Phys. Rev.* **B42** (1990), 7454-7461.

[42] W. Zawadzki, in *High Magnetic Fields in the Physics of Semiconductors II*, Ed. G. Landw W. Ossau (World Scientific, Singapore, 1997), p. 755.

[43] G. Salesi, *Int. J. Mod. Phys.* **A20** (2005), 2027-2036.

[44] Z.F. Jiang, R.D. Li, S.C. Zhang & W.M. Liu, *Phys. Rev.* **B72** (2005), 045201 [5 pages].

[45] W. Zawadzki, *Phys. Rev.* **B72** (2005), 085217 [4 pages].

[46] W. Zawadzki, "One–dimensional semirelativity for electrons in carbon nanotubes", *mat/0510184* [4 pages].

[47] J. Schliemann, D. Loss & R.M. Westervelt, *Phys. Rev. Lett.* **94** (2005), 206801 [4 pages]; *Rev.* **B73** (2006), 085323 [9 pages].

[48] M.I. Katsnelson, *Eur. Phys. J.* **B51** (2006), 157-160.

[49] J. Cserti & G. Dávid, *Phys. Rev.* **B74** (2006), 172305 [4 pages].

[50] J.Y. Vaishnav & C.W. Clark, *Phys. Rev. Lett.* **100** (2008), 153002 [4 pages].

[51] P. Brusheim & H.Q. Xu, "Catching the Zitterbewegung", *con-mat/0810.2186* [4 pages].

[52] L. Lamata, J. León, T. Schätz & E. Solano, *Phys. Rev. Lett.* **98** (2007), 253005 [4 pages].

[53] M. Villavicencio & J.A.E. Roa–Neri, *Eur. J. Phys.* **21** (2000), 119-123.

[54] J.W. van Holten, *Physica A.* **182** (1992), 279-292.

[55] J.W. van Holten, "Relativistic dynamics of spin in strong external fields", *hep-th/9303124* [11 pages].

[56] H. Thirring, *Principles of Quantum Electrodynamics*, (Academic Press, New York, 1958).

[57] R.S. Van Dyck Jr., P.B. Schwinberg & H.G. Dehmelt, *Phys. Rev.* **D34** (1986), 722-736.

[58] T.P. Heavner, S.R. Jefferts, E.A. Donley, J.H. Shirley & T.E. Parker, *Metrologia* **42** (2005), 411-422.

[59] G. Salesi & E. Recami, *Found. Phys. Lett.* **10** (1997), 533-546.