p-adic Hodge theory and Bloch-Kato theory

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Outline

Basic principles of p-adic Hodge theory

- The aim of p-adic Hodge theory is to understand p-adic representations V of the Galois group G_K of a p-adic field $K/\mathbb{Q}_p$.
- Understanding ℓ-adic representations of G_K, for $\ell \neq p$, is considerably easier.
- The main strategy of p-adic Hodge theory is the construction of *rings of periods* B, equipped with an action of G_K, such that

$$D_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_K}$$

becomes an interesting invariant of the representation V.

- A ring of periods $B/\mathbb{Q}_p$ should satisfy the following requirements:
  - it should be a domain;
  - $\mathrm{Frac}(B)^{G_K} = B^{G_K}$ (in particular B^{G_K} is a field);
  - If $y \in B$ is such that $y \cdot \mathbb{Q}_p \subseteq B$ is stable under G_K, then $y \in B^{\times}$.
- If V is a p-adic representation of G_K, we define

$$D_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

- There is a natural map

$$B \otimes_{B^{G_K}} D_B(V) \to B \otimes_{\mathbb{Q}_p} V$$

  which is injective because B is a domain.
- Therefore $\dim_{B^{G_K}}(D_B(V)) \leq \dim_{\mathbb{Q}_p}(V)$. We say that V is B-admissible if we have equality (equivalently, if the above map is an isomorphism).

- The category of B-admissible representations of G_K is stable under subquotients, sums, tensor products and duals.
- However, it is generally *not* stable under extensions.
- If B has extra structure (a grading, filtration, an action of an operator), and this extra structure is compatible with Galois action, then $D_B(V)$ inherits this structure.

The ring B_{dR}

- Fontaine has constructed a field of periods B_{dR}. We list some of its properties.
- B_{dR} is the field of fractions of a complete discrete valuation ring B_{dR}^+ with residue field $\mathbb{C}_p$.
- The maximal ideal of B_{dR}^+ is generated by an element t which is a p-adic analogue of $2\pi i$ (depending on a choice $\epsilon = (\epsilon^{(n)})$ of compatible p-power roots of 1)
- $G_{\mathbb{Q}_p}$ acts on t via the cyclotomic character, i.e. $g(t) = \chi(g)t$ for $g \in G_{\mathbb{Q}_p}$.
- B_{dR} is therefore equipped with a descending filtration $\mathrm{Fil}^i = \mathfrak{m}^i B_{\mathrm{dR}}^+$, $\mathrm{Fil}^i \supseteq \mathrm{Fil}^{i+1}$, stable by the action of $G_{\mathbb{Q}_p}$

- By Hensel's lemma, $\overline{\mathbb{Q}_p} \subseteq B_{\mathrm{dR}}^+$, compatibly with the action of $G_{\mathbb{Q}_p}$.
- (However, B_{dR}^+ is not a $\mathbb{C}_p$-algebra in any natural way. In fact, a theorem of Colmez says that $\overline{\mathbb{Q}_p}$ is dense in B_{dR} in a suitable topology.)
- We have $(B_{\mathrm{dR}})^{G_K} = (B_{\mathrm{dR}}^+)^{G_K} = K$, so if V is a p-adic representation of G_K, $D_{\mathrm{dR}}(V) := D_{B_{\mathrm{dR}}}(V)$ is a filtered K-vector space.

de Rham representations

- We say that a representation V of G_K is de Rham if it is B_{dR}-admissible, that is if

$$B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V,$$

or which is equivalent, $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p}(V)$.

- We have the *de Rham comparison theorem*, conjectured by Fontaine and proved by Faltings:

Theorem

Let X/K be a smooth proper variety, and $V = H^i_{\text{ét}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$, so that V is a p-adic representation of G_K. Then V is de Rham, and there is a natural isomorphism of filtered K-vector spaces

$$D_{\mathrm{dR}}(V) \cong H^i_{\mathrm{dR}}(X/K).$$

- Here $H^i_{\mathrm{dR}}(X/K)$ is de Rham cohomology equipped with the Hodge filtration.

The ring B_{HT} and Hodge-Tate representations

- The ring B_{HT} is the graded ring associated to the filtered ring B_{dR}
- We have $B_{\mathrm{HT}} = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}_p(j)$, since $\mathfrak{m}^j/\mathfrak{m}^{j+1} = \mathbb{C}_p(j)$
- A p-adic representation V of G_K is Hodge-Tate if it is B_{HT}-admissible.
- If V is Hodge-Tate, then $D_{\mathrm{HT}}(V) = D_{B_{\mathrm{HT}}}(V)$ is a graded K-vector space.
- If V is de Rham, then $D_{\mathrm{HT}}(V)$ is the graded vector space associated to the filtered vector space $D_{\mathrm{dR}}(V)$. In particular, a de Rham representation is Hodge-Tate.
- A Hodge-Tate weight of V is an integer j such that $D_{\mathrm{HT}}(V)_{-j} \neq 0$. Equivalently, if V is de Rham, $\mathrm{Fil}^{-j} D_{\mathrm{dR}}(V) \neq \mathrm{Fil}^{-j+1} D_{\mathrm{dR}}(V)$.
- With this convention, $\mathbb{Q}_p(j)$ has Hodge-Tate weight j.

The ring B_{cris} and crystalline representations

- Fontaine has defined a ring of periods B_{cris}, which is a subring of B_{dR} equipped with an induced filtration and Galois action.
- The ring B_{cris} contains the element t, and have $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t]$, where $B^+_{\mathrm{cris}} = B^+_{\mathrm{dR}} \cap B_{\mathrm{cris}}$.
- We have $B_{\mathrm{cris}}^{G_K} = \mathbb{Q}_p^{\mathrm{unr}} \cap K =: F$, and the action of φ on $B_{\mathrm{cris}}^{G_K}$ is the Frobenius on F.
- The ring B_{cris} is equipped with a Frobenius $\varphi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ commuting with the action of $G_{\mathbb{Q}_p}$, and inducing the usual Frobenius on $\mathbb{Q}_p^{\mathrm{unr}} \subseteq B_{\mathrm{cris}}$.
- Frobenius acts on t by $\varphi(t) = pt$.

Crystalline representations

- We say that a p-adic representation V of G_K is *crystalline* if it is B_{cris}-admissible.
- If V is a crystalline representation, then

$$D_{\mathrm{cris}}(V) := D_{B_{\mathrm{cris}}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

  is an F-vector space equipped with a filtration and a semi-linear Frobenius.
- If V is crystalline, then $K \otimes_F D_{\mathrm{cris}}(V) = D_{\mathrm{dR}}(V)$, as filtered K-vector spaces. Hence a crystalline representation is also de Rham.
- To summarize, we have

$$\text{crystalline} \Rightarrow \text{de Rham} \Rightarrow \text{Hodge-Tate}$$

- The property of being crystalline, for a p-adic representation V, is analogous to the property of being unramified for an ℓ-adic representation (for $\ell \neq p$).
- For example, if V is the p-adic Tate module of an abelian variety A/K, then V is crystalline if and only if V has good reduction (Iovita).
- One has the *crystalline comparison isomorphism*:

Theorem

Let k be a perfect field of characteristic p, let $\mathcal{O}_F = W(k)$ and $F = \mathrm{Frac}(\mathcal{O}_F)$. Let $X/\mathcal{O}_F$ be a smooth, proper scheme, geometrically irreducible. Then for every $i \geq 0$, there is a functorial isomorphism

$$H^i_{\mathrm{et}}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong H^i_{\mathrm{cris}}(X_k/\mathcal{O}_F) \otimes_{\mathcal{O}_F} B_{\mathrm{cris}},$$

compatibly with filtrations, G_F-action and action of Frobenius.

- A classical result of crystalline cohomology states that $H^i_{\mathrm{cris}}(X_k/\mathcal{O}_F) \cong H^i_{\mathrm{dR}}(X/\mathcal{O}_F)$. Thus $H^i_{\mathrm{cris}}(X_k/\mathcal{O}_F)$ has a natural filtration and $H^i_{\mathrm{dR}}(X/\mathcal{O}_F)$ has a natural action of a Frobenius.
- The crystalline comparison isomorphism implies that $V = H^i_{\mathrm{et}}(X_{\overline{F}}, \mathbb{Q}_p)$ is crystalline, and moreover that $D_{\mathrm{cris}}(V) \cong H^i_{\mathrm{cris}}(X_k/\mathcal{O}_F) \otimes_{\mathcal{O}_F} F \cong H^i_{\mathrm{dR}}(X/F)$.

The subgroups H^1_e, H^1_f, H^1_g

- Recall that there is a natural identification of $H^1(K, V)$ with the set of isomorphism classes of extensions of $\mathbb{Q}_p$ by V.
- Given an extension

$$0 \to V \to W \xrightarrow{p} \mathbb{Q}_p \to 0,$$

  choose $w \in W$ with $p(w) = 1$. Then for all $\sigma \in G_K$, $p(\sigma w) = \sigma p(w) = \sigma \cdot 1 = 1$, so $p(w - \sigma w) = 0$. In other words $w - \sigma w \in V$.
- The map $\sigma \mapsto w - \sigma w$ is a cocycle $G_K \to V$, whose cohomology class depends only on the isomorphism class of the extension.

- Therefore, it is natural, given a property of p-adic representations, to look at the subset of $H^1(K, V)$ consisting of those extensions of $\mathbb{Q}_p$ by V having that property.
- We define subgroups

$$H^1_e(K, V) \subseteq H^1_f(K, V) \subseteq H^1_g(K, V) \subseteq H^1(K, V)$$

as follows:

- The subgroup $H^1_e(K, V) = \ker(H^1(K, V) \to H^1(K, B_{\mathrm{cris}}^{\varphi=1} \otimes V))$;
- The subgroup $H^1_f(K, V) = \ker(H^1(K, V) \to H^1(K, B_{\mathrm{cris}} \otimes V))$ consisting of the extensions of $\mathbb{Q}_p$ by V which are crystalline;
- The subgroup $H^1_g(K, V) = \ker(H^1(K, V) \to H^1(K, B_{\mathrm{dR}} \otimes V))$ consisting of the extensions of $\mathbb{Q}_p$ by V which are de Rham.

The exponential map

- Let V be a de Rham representation. Then Bloch and Kato have defined an "exponential map"

$$\exp_{K,V} : D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V) \to H^1(K,V).$$

- If $G/\mathcal{O}_K$ is a formal Lie group of finite height (eg. $G = \widehat{A}$ for $A/\mathcal{O}_K$), and $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p G$, then

$$D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V) \cong \tan(G(K)),$$

and we have a commutative diagram

$$\begin{array}{ccc} D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V) & \xrightarrow{\exp_{K,V}} & H^1(K,V) \\ \cong\uparrow & & \uparrow\delta_G \\ \tan(G(K)) & \xrightarrow{\exp} & G(K) \end{array}$$

- Here δ_G is the Kummer map and exp is the usual exponential map.

- In order to construct the Bloch-Kato exponential for an arbitrary de Rham representation V, one uses the *fundamental exact sequence*

$$0 \to \mathbb{Q}_p \to B_{\text{cris}}^{\varphi=1} \to B_{\text{dR}}/B_{\text{dR}}^+ \to 0$$

- The second map is simply the composite of

$$B_{\text{cris}}^{\varphi=1} \hookrightarrow B_{\text{cris}} \hookrightarrow B_{\text{dR}} \to B_{\text{dR}}/B_{\text{dR}}^+$$

- Tensoring the fundamental exact sequence with V over $\mathbb{Q}_p$ and taking invariants under G_K, we get an exact sequence

$$0 \to V^{G_K} \to D_{\text{cris}}^{\varphi=1}(V) \to ((B_{\text{dR}}/B_{\text{dR}}^+) \otimes V)^{G_K} \to H^1_e(K, V) \to 0$$

where $H^1_e(K, V) = \ker(H^1(K, V) \to H^1(K, B_{\text{cris}}^{\varphi=1} \otimes V))$.

- Since V is de Rham, we have $((B_{\mathrm{dR}}/B_{\mathrm{dR}}^{+}) \otimes V)^{G_K} = D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V)$ and therefore we deduce an isomorphism

$$\exp_{K,V} : \frac{D_{\mathrm{dR}}(V)}{\operatorname{Fil}^0 D_{\mathrm{dR}}(V) + D_{\mathrm{cris}}^{\varphi=1}(V)} \xrightarrow{\sim} H^1_e(K,V)$$

- We write

$$\log_{K,V} : H^1_e(K,V) \xrightarrow{\sim} \frac{D_{\mathrm{dR}}(V)}{\operatorname{Fil}^0 D_{\mathrm{dR}}(V) + D_{\mathrm{cris}}^{\varphi=1}(V)}$$

for the inverse of this isomorphism.

The dual exponential map

- Before going further, we describe in more detail the filtered φ-module $D_{\mathrm{cris}}(\mathbb{Q}_p(j))$.
- The choice of $\epsilon = (\epsilon^{(n)})$ determines the element $t \in B_{\mathrm{dR}}$, and a basis e_j of $\mathbb{Q}_p(j)$ for each j, such that $e_j \otimes e_{j'} = e_{j+j'}$.
- The element $t^{-j} e_j \in B_{\mathrm{cris}} \otimes \mathbb{Q}_p(j)$ is Galois invariant and determines a canonical basis of $D_{\mathrm{cris}}(\mathbb{Q}_p(j))$, which does not depend on the choice of ϵ.
- We have

$$\mathrm{Fil}^k D_{\mathrm{cris}}(\mathbb{Q}_p(j)) = \begin{cases} D_{\mathrm{cris}}(\mathbb{Q}_p(j)) = K & \text{if } k \leq -j \\ 0 & \text{if } k > -j \end{cases}$$

- The Frobenius φ on $D_{\mathrm{cris}}(\mathbb{Q}_p(j))$ acts as multiplication by p^{-j}, because $\varphi(t^{-j} e_j) = \varphi(t^{-j}) e_j = p^{-j} t^{-j} e_j$.

- In particular the element $t^{-1}e_1 \in D_{\mathrm{dR}}(\mathbb{Q}_p(1))$ gives an isomorphism $D_{\mathrm{dR}}(\mathbb{Q}_p(1)) \xrightarrow{\sim} K$; thus we obtain a perfect pairing

$$D_{\mathrm{dR}}(V) \otimes D_{\mathrm{dR}}(V^*(1)) \cong D_{\mathrm{dR}}(V \otimes V^*(1)) \to D_{\mathrm{dR}}(\mathbb{Q}_p(1)) \cong K \xrightarrow{\mathrm{Tr}} \mathbb{Q}_p$$

- Thus we identify $D_{\mathrm{dR}}(V)^*$ with $D_{\mathrm{dR}}(V^*(1))$.
- Moreover, the cup-product

$$H^1(K, V) \otimes H^1(K, V^*(1)) \to H^2(K, \mathbb{Q}_p(1)) = \mathbb{Q}_p$$

identifies $H^1(K, V)^*$ with $H^1(K, V^*(1))$.

- Therefore, we can view

$$(\exp_{K,V})^* : H^1(K, V)^* \to (D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V))^*$$

as a map

$$\exp^*_{K,V} : H^1(K, V^*(1)) \to \operatorname{Fil}^0 D_{\mathrm{dR}}(V^*(1))$$

This is the dual exponential map of V.

Some notation

- $K_n = K(\mu_{p^n})$
- $K_\infty = \cup K_n$
- $\Gamma_K = \mathrm{Gal}\,(K_\infty/K)$
- $\chi : \Gamma_K \to \mathbb{Z}_p^\times$ the cyclotomic character, identifies Γ_K with an open subgroup of $\mathbb{Z}_p^\times$
- $H_F = \ker\chi = \mathrm{Gal}\,(\overline{K}/K_\infty)$.

Iwasawa cohomology

- Let V be a p-adic representation of G_K of dimension d, and $T \subseteq V$ a $\mathbb{Z}_p$-lattice stable by G_K.
- The Iwasawa cohomology of V is defined as

$$H^i_{\mathrm{Iw}}(K, V) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim H^i(K_n, T)$$

  where the inverse limit is taken with respect to the corestriction maps $\mathrm{cor}_{K_{n+1},K_n} : H^i(K_{n+1}, T) \to H^i(K_n, T)$
- Each $H^i(K_n, T)$ is naturally a $\mathbb{Z}_p[\mathrm{Gal}(K_n/K)]$-module, so $H^i_{\mathrm{Iw}}(K, V)$ has a natural structure of a $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_K$-module, where $\Lambda_K = \varprojlim \mathbb{Z}_p[\mathrm{Gal}(K_n/K)]$.
- The module $H^i_{\mathrm{Iw}}(K, V)$ is idependent of the choice of lattice T.

Remarks on the Iwasawa algebra

- The ring $\mathbb{Q}_p \otimes \Lambda_K$ identifies with the space of p-adic measures on Γ_K, i.e.

$$\mathbb{Q}_p \otimes \Lambda_K = \mathrm{Hom}(C(\Gamma_K, \mathbb{Q}_p), \mathbb{Q}_p)$$

  where $C(\Gamma_K, \mathbb{Q}_p)$ is the Banach space of continuous $\mathbb{Q}_p$-valued functions on Γ_K.
- As such it is equipped with a structure of $C(\Gamma_K, \mathbb{Q}_p)$-module; if $f \in C(\Gamma_K, \mathbb{Q}_p)$ and $\mu \in \mathbb{Q}_p \otimes \Lambda_K$, then $f\mu$ is the measure $h \mapsto \int hf\,\mu$.
- In particular, if $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ is a continuous character and μ is a measure, the product $\eta\mu$ is a measure.
- Λ_K is is equipped with an action of G_K, given by $g(\mu) = [\overline{g}]\mu$, where $\overline{g} = g|_{K_\infty}$ is the image of g in Γ_K and $[\overline{g}]$ is the corresponding group-like element (the Dirac measure at $\overline{g}$).

- The structure of H^i_{Iw} has been determined by Perrin-Riou:

Theorem (Perrin-Riou)

We have $H^i_{\mathrm{Iw}}(K,V) = 0$ for $i \neq 1,2$; moreover:

1. *The torsion submodule of $H^1_{\mathrm{Iw}}(K,V)$ is a $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_K$-module isomorphic to V^{H_K}, and $H^1_{\mathrm{Iw}}(K,V)/V^{H_K}$ is a free $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_K$-module of rank $[K:\mathbb{Q}_p]d$*
2. $H^2_{\mathrm{Iw}}(K,V) = (V^*(1)^{H_K})^*$

- In particular the $H^i_{\mathrm{Iw}}(K,V)$ are $\mathbb{Q}_p \otimes \Lambda_K$-modules of finite type.
- There are natural projection maps $H^i_{\mathrm{Iw}}(K,V) \to H^i(K_n,V)$ compatible with corestriction maps.

Twisting Euler systems

- We can also describe $H^1_{\mathrm{Iw}}(K, V)$ as a single cohomology group, which can help us understand the "twisting" operation on Euler systems.
- Let $\mathrm{Meas}(\Gamma_K, V) = \Lambda_K \otimes_{\mathbb{Q}_p} V$ be the space of V-valued measures on Γ_K. It is equipped with the diagonal action of G_K
- Then we have a natural isomorphism $H^1(K, \Lambda_K \otimes_{\mathbb{Q}_p} V) \xrightarrow{\sim} H^1_{\mathrm{Iw}}(K, V)$, which we describe.
- If $\sigma \mapsto \mu_\sigma$ is a cocycle $G_K \to \mathrm{Meas}(\Gamma_K, V)$ representing a cohomology class $\zeta \in H^1(K, \mathrm{Meas}(\Gamma_K, V))$, then, for any n, the map $\sigma \mapsto \int_{\Gamma_{K_n}} \mu_\sigma$ is a cocycle $G_{K_n} \to V$ representing a cohomology class $\zeta_n \in H^1(K_n, V)$.
- The collection $(\zeta_n) \in H^1(K_n, V)$ is compatible under corestriction maps and determines an element of $H^1_{\mathrm{Iw}}(K, V)$.

- Let $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ be a continuous character. Then there is a natural map
$$H^1_{\mathrm{Iw}}(K, V) \to H^1_{\mathrm{Iw}}(K, V(\eta^{-1})),$$
which we now describe.
- Remark that if $\mu \in \mathrm{Meas}(\Gamma_K, V)$ and $\sigma \in G_K$, we have $\sigma(\eta\mu) = \eta(\sigma^{-1}) \cdot \eta\mu$, so multiplication by η is a Galois-equivariant map when viewed as a map $\mathrm{Meas}(\Gamma_K, V) \to \mathrm{Meas}(\Gamma_K, V(\eta^{-1}))$.
- It induces a map $H^1_{\mathrm{Iw}}(K, V) \to H^1_{\mathrm{Iw}}(K, V(\eta^{-1}))$ on cohomology.
- If $x \in H^1_{\mathrm{Iw}}(K, V)$, we write x_η for the image of x in $H^1_{\mathrm{Iw}}(K, V(\eta^{-1}))$, and $x_{\eta,n}$ for its image in $H^1(K_n, V(\eta^{-1}))$
- If $\eta = \chi^j$ we simply write $x_{n,j} \in H^1_{\mathrm{Iw}}(K_n, V(-j))$, where we identify $V(\chi^{-j})$ and $V(-j)$ using the basis ϵ.

The Perrin-Riou big logarithm (or regulator) map

- Let p be an odd prime, and $F/\mathbb{Q}_p$ a finite unramified extension.
- Let V be a crystalline representation of G_F, with non-negative Hodge-Tate weights and no quotient isomorphic to the trivial representation.
- Let $\mathcal{H}_{\mathbb{Q}_p}(\Gamma_F)$ be the algebra of $\mathbb{Q}_p$-valued distributions on Γ_F (dual to the space of locally analytic $\mathbb{Q}_p$-valued functions on Γ_F)
- Then Perrin-Riou has constructed a "big logarithm" or "regulator" map

$$\mathcal{L}_{V,F} : H^1_{\mathrm{Iw}}(F, V) \to \mathcal{H}(\Gamma_F) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V)$$

- The map $\mathcal{L}_{V,F}$ interpolates the Bloch-Kato dual exponential and logarithm maps for twists of V in the cyclotomic tower.
- The big logarithm provides a bridge between arithmetic and analysis. It allows us to construct p-adic L-functions from Euler systems.

Explicit formulae for the regulator

- We will now explain how the regulator map interpolates the Bloch-Kato logarithm and dual exponential maps.
- We will assume for simplicity that $F = \mathbb{Q}$.
- We follow Appendix B of Loeffler-Zerbes (*Iwasawa Theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions*).
- If $\nu \in \mathcal{H}(\Gamma_K) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V)$, and $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ is a character, we shall write $\int_\Gamma \eta\nu$ for $\nu(\eta)$. It is an element of $D_{\mathrm{cris}}(V)$.

Theorem

Let V be a cristalline representation of $G_{\mathbb{Q}_p}$ with non-negative Hodge-Tate weights and no quotient isomorphic to $\mathbb{Q}_p$. Let η be a continuous character of Γ of the form $\chi^j\omega$, where ω is a finite-order character of conductor n. Let $x \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$. Then:

- *If $j \geq 0$, we have*

$$\int_\Gamma \eta \mathcal{L}_V(x) = j! \times$$

$$\begin{cases} (1-p^j\varphi)(1-p^{-1-j}\varphi^{-1})^{-1}\left(\exp^*_{\mathbb{Q}_p, V(\eta^{-1})^*(1)}(x_{\eta,0}) \otimes t^{-j}e_j\right) & \text{for } n=0 \\ \tau(\omega)^{-1}p^{n(j+1)}\varphi^n\left(\exp^*_{\mathbb{Q}_p, V(\eta^{-1})^*(1)}(x_{\eta,0}) \otimes t^{-j}e_j\right) & \text{for } n \geq 1 \end{cases}$$

- *If $j < 0$, we have*

$$\int_\Gamma \eta \mathcal{L}_V(x) = \frac{(-1)^{-j-1}}{(-j-1)!} \times$$

$$\begin{cases} (1-p^j\varphi)(1-p^{-1-j}\varphi^{-1})^{-1}\left(\log_{\mathbb{Q}_p, V(\eta^{-1})^*(1)}(x_{\eta,0}) \otimes t^{-j}e_j\right) & \text{for } n=0 \\ \tau(\omega)^{-1}p^{n(j+1)}\varphi^n\left(\log_{\mathbb{Q}_p, V(\eta^{-1})^*(1)}(x_{\eta,0}) \otimes t^{-j}e_j\right) & \text{for } n \geq 1 \end{cases}$$

- The second part of the theorem is due to Perrin-Riou; the first part is a consequence of Perrin-Riou's explicit reciprocity conjecture $\mathrm{Rec}(V)$, proved independently by Benois and Colmez.
- The explicit formulae for the regulator can be viewed as a generalization of Coleman's formulae for the special values at integers of the Kubota-Leopoldt p-adic L-function in terms of polylogarithms of cyclotomic units.
- The formulae allow us to deduce the nonvanishing of an Euler system from the nonvanishing of a p-adic L-function (when we know that this p-adic L-function arises from an Euler system).

A fundamental example

- In this section, we describe a classical result of Coleman which was Perrin-Riou's inspiration for the construction of the regulator map.
- Let $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\epsilon^{(n)})$.
- Then we have

Theorem (Coleman)

Let $u = (u_n)_{n\geq 0}$ be an element of $\varprojlim \mathcal{O}^{\times}_{\mathbb{Q}_{p,n}}$, where the inverse limit is taken with respect to the norm maps. Then there exists a unique series $\mathrm{Col}_u(T) \in \mathbb{Z}_p[[T]]^{\times}$ such that $\mathrm{Col}_u(\epsilon^{(n)} - 1) = u_n$ for every n. Moreover, if we set $G_u(T) = \log(\mathrm{Col}_u(T))$, then there exists a unique measure $\lambda_u \in \Lambda$ on Γ such that

$$\int_{\Gamma} (1+T)^{\chi(x)} \lambda_u(x) = G_u(T) - \frac{1}{p} \sum_{\zeta^p = 1} G_u(\zeta(1+T) - 1).$$

- Remark that $\varprojlim \mathcal{O}_{\mathbb{Q}_{p,n}}^{\times}$ is equipped with a structure of Λ-module.
- By Kummer theory, we have a natural map $\varprojlim \mathcal{O}_{\mathbb{Q}_{p,n}}^{\times} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$.
- The map $u \mapsto \lambda_u$ is almost an isomorphism of Λ-modules (its kernel and cokernel are $\mathbb{Z}_p$-modules of rank 1).
- Coleman's idea was that, for a suitable choice of u, one could construct the Kubota-Leopoldt p-adic L-function.

- Choose $\gamma \in \Gamma$, and define $u = (u_n)$ by

$$u_n = \frac{\gamma\epsilon^{(n)} - 1}{\epsilon^{(n)} - 1}.$$

- One can check that the u_n's are norm-compabible, and moreover that if $k \in \mathbb{N} - \{0\}$, we have

$$\int_\Gamma \chi(x)^k \lambda_u(x) = (\chi(\gamma)^k - 1)(1 - p^{k-1})\zeta(1 - k)$$

- If γ is chosen to have infinite order, then the Kubota-Leopoldt p-adic L-function is given by the (pseudo-) measure $(1 - \gamma)^{-1}\lambda_u$.

Constructing the regulator map: a sketch

- We briefly describe the construction of Perrin-Riou's big logarithm.
- We will need a number of ingredients:
  1. The theory of (φ, Γ)-modules
  2. Wach modules;
  3. Fontaine's isomorphism.

(φ, Γ)-modules

- Let $F/\mathbb{Q}_p$ be an unramified extension.
- We shall need the period rings A_F, A_F^+, B_F, B_F^+ and $B_{\mathrm{rig},F}^+$.
- Our choice of ϵ determines an element $\pi \in A_F^+$, and we have $A_F^+ = \mathcal{O}_F[[\pi]]$
- We set $A_F = \widehat{A_F[\pi^{-1}]}$
- We set $B_F^+ = A_F^+[1/p]$ and $B_F = A_F[1/p]$.
- We define $B_{\mathrm{rig},F}^+$ as the ring of power series $f \in F[[\pi]]$ which converge on the open unit disc.

- These rings are equipped with an $\mathcal{O}_F$-linear action of Γ, defined by $\gamma(\pi) = (\pi + 1)^{\chi(\gamma)} - 1$, extended by linearity and continuity.
- These rings are also equipped with a Frobenius φ, acting as the usual Frobenius on $\mathcal{O}_F$ and on π by $\varphi(\pi) = (\pi + 1)^p - 1$.
- We define a left inverse ψ for φ, characterized by the property that

$$(\varphi \circ \psi)(f(\pi)) = \frac{1}{p} \sum_{\zeta^p = 1} f(\zeta(1 + \pi) - 1).$$

- One can show that there is a natural identification of $(B^+_{\mathrm{rig},F})^{\psi=0}$ with $\mathcal{H}(\Gamma_F)$.

The (φ, Γ)-module

- If V is a p-adic representation of G_F, we define

$$D_F(V) = (B \otimes_{\mathbb{Q}_p} V)^{H_F},$$

where B a certain period ring with $B^{H_F} = B_F$.

- $D_F(V)$ is a B_F-module equipped with commuting actions of Γ and φ, i.e. it is a (φ, Γ)-module.
- One can recover V from $D_F(V)$ by

$$V = (B \otimes_{B_F} D_F(V))^{\varphi=1}.$$

The Wach module

- Let V be a crystalline representation of G_F, and $T \subseteq V$ be a stable lattice.
- Then Wach and Berger have shown that there exists a unique A_F^+-submodule $N_F(T) \subseteq D_F(T)$ such that:
  1. $N_F(T)$ is A_F^+-free of rank $\dim V$;
  2. Γ_F preserves $N_F(T)$ and acts trivially on $N_F(T)/\pi N_F(T)$;
  3. There exists $b \in \mathbb{Z}$ such that $\varphi(\pi^b N_F(T)) \subseteq \pi^b N_F(T)$, and $\pi^b N_F(T)/\varphi^*(\pi^b N_F(T))$ is killed by a power of $\varphi(\pi)/\pi$.
- We set $N_F(V) = B_F^+ \otimes_{A_F^+} N(T)$; it is independent of T.
- By a theorem of Berger, one can also recover $D_{\mathrm{cris}}(V)$ from $N_F(V)$ as

$$D_{\mathrm{cris}}(V) = (B_{\mathrm{rig},F}^+ \otimes_{B_F^+} N_F(V))^{G_F}.$$

The Fontaine isomorphism

- Fontaine has showed that we can recover the Iwasawa cohomology of V from its (φ, Γ)-module.
- More precisely, he has has proved that there exists a canonical isomorphism
$$D_F(T)^{\psi=1} \xrightarrow{\sim} H^1_{\mathrm{Iw}}(F, T).$$
- Moreover, if V has non-negative Hodge-Tate weights and no trivial quotient, then a result of Berger implies that
$$D_F(T)^{\psi=1} = N_F(T)^{\psi=1},$$
and therefore we have
$$N_F(T)^{\psi=1} \xrightarrow{\sim} H^1_{\mathrm{Iw}}(F, T).$$

- Let us now specialize to $F = \mathbb{Q}_p$, and let $x \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$.
- By Fontaine's isomorphism, we can view

$$1 \otimes x \in \left(B^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{B^+_{\mathbb{Q}_p}} N(V)\right)^{\psi=1} \subseteq \left(B^+_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{B^+_{\mathbb{Q}_p}} D_{\mathrm{cris}}(V)\right)^{\psi=1}$$

- Then $\mathcal{L}_V(x)$ is the unique element of $\mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V)$ such that

$$\mathcal{L}_V(x)(1+\pi) = (1-\varphi)x.$$

- I refer you to the paper of David and Sarah for a beautiful account of the construction, and for the proof of the explicit formulae.
- Thank you!

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