Anti-cyclotomic p-adic L-functions and the Exceptional Zero phenomenon

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Cyclotomic p-adic L-functions

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- A/F elliptic curve.
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$$L_p(A,s) = \int_{\mathcal{G}_p} \exp_p(s\ell(\gamma))d\mu_p(\gamma), \qquad s \in \mathbb{C}_p,$$

$\ell : \mathcal{G}_p \to \mathbb{Z}_p$ canonical isomorphism, μ_p cyclotomic p-adic measure attached to A.

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$\ell : \mathcal{G}_p \to \mathbb{Z}_p$ canonical isomorphism, μ_p cyclotomic p-adic measure attached to A.
$L_p(A,s)$ *interpolates* $L(A,\psi,1)$:

$$\int_{\mathcal{G}_p} \psi(\gamma)d\mu_p(\gamma) = e_p(A,\psi)L(A,\psi,1),$$

$e_p(A,\psi)$ Euler factor.

Cyclotomic Exceptional Zero Phenomenon

If A split multiplicative reduction at some $\mathcal{P} \mid p \Rightarrow e_p(A,1) = 0$

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Conjecture:

- $\mathrm{ord}_{s=0} L_p(A,s) \geq r := \#\{\mathcal{P} \mid p,\ A \text{ split mult. red. mod } \mathcal{P}\}$
- $\mathcal{L}_{\mathcal{P}_k}(A) = \ell_{\mathcal{P}_k}(q_{A/F_{\mathcal{P}_k}})/\mathrm{ord}_{\mathcal{P}_k}(q_{A/F_{\mathcal{P}_k}})$ $\mathcal{L}$-invariants

$$\frac{d^r L_p(A,s)}{ds^r}\Big|_{s=0} = \prod_{k=1}^{r} \mathcal{L}_{\mathcal{P}_k}(A) \cdot e_p'(A,1) \cdot L(A,1)$$

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→ $F = \mathbb{Q}$: **Greenberg-Stevens.**

→ $r = 1$: **Mook.**

→ Arbitrary F and r: **Spiess**, replacing $\mathcal{L}_{\mathcal{P}_k}(A)$ by *Automorphic $\mathcal{L}$-invariants*.

p-adic L-functions and Iwasawa algebras

Consider $\mu_p \in \mathbb{Z}_p[[\mathcal{G}_p]]$, Iwasawa Algebra

$$\deg : \mathbb{Z}_p[[\mathcal{G}_p]] \longrightarrow \mathbb{Z}_p,$$

$\deg(\mu_p) = L_p(A, 0) = \int_{\mathcal{G}_p} d\mu_p$.

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- $\mathcal{I} = \ker(\deg)$ *augmentation ideal*
- Order vanishing: $\mathrm{ord}_{s=0} L_p(A,s) = \max\{r : \mu_p \in \mathcal{I}^r\}$
- Dericative: If $\mu_p \in \mathcal{I}^r$,

$$\frac{d^r L_p(A,s)}{ds^r} \,|_{s=0} = \left(\text{image of } \mu_p \text{ in } \mathcal{I}^r/\mathcal{I}^{r+1}\right)$$

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1. Cyclotomic p-adic L-functions
2. Anti-cyclotomic p-adic measures
3. Exceptional Zero Phenomenon

Setting

- E/F totally imaginary quadratic extension.
- $\mathcal{P}$ prime of F above p.
- $\mathcal{G}_{E,\mathcal{P}}$ Galois group anti-cyclotomic extension of E.

$$\mathcal{G}_{E,\mathcal{P}} \simeq \mathbb{Z}_p^s, \qquad s = [F_{\mathcal{P}} : \mathbb{Q}_p].$$

- π automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$

$$\pi \subset L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/\mathbb{A}_F^\times), \quad \text{parallel weight 2}$$

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p-adic measure of $\mathcal{G}_{E,\mathcal{P}}$ attached to π with good interpolation properties?

$\rightarrow$ $F = \mathbb{Q}$ **Bertolini-Darmon**

$\rightarrow$ Other constructions by **Van Order**

Local repesentation (good ordinary)

$G_{\mathcal{P}} = \mathrm{GL}_2(F_{\mathcal{P}})$, $K_{\mathcal{P}} = \mathrm{GL}_2(\mathcal{O}_{F_{\mathcal{P}}})$,

$$Z_{\mathcal{P}} = \left\{ \left(\begin{array}{cc} t & 0 \\ 0 & t \end{array} \right) \right\} \simeq F_{\mathcal{P}}^{\times}, \quad B_{\mathcal{P}} = \left\{ \left(\begin{array}{cc} u_1 & x \\ 0 & u_2 \end{array} \right) \right\}.$$

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Two descriptions of $\pi_{\mathcal{P}}$ ordinary good reduction:

(i) $\pi_{\mathcal{P}} = \mathrm{Ind}_{B_{\mathcal{P}}}^{G_{\mathcal{P}}}(\mu_\alpha)$, $\alpha \in \bar{\mathbb{Q}} \cap \mathcal{O}_{\mathbb{C}_p}^{\times}$, $|\alpha| = q^{1/2}$, $q = \#\mathcal{O}_F/\mathcal{P}$

$$\mathrm{Ind}_{B_{\mathcal{P}}}^{G_{\mathcal{P}}}(\mu_\alpha) = \left\{ \phi : G_{\mathcal{P}} \to \mathbb{C} : \phi\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} g \right) = \alpha^{v(t_2/t_1)} \phi(g) \right\}.$$

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(ii) $\pi_{\mathcal{P}} = \mathrm{Ind}_{K_{\mathcal{P}} Z_{\mathcal{P}}}^{G_{\mathcal{P}}}(1_{\mathbb{C}})/(\mathcal{T}_{\mathcal{P}} - a_{\mathcal{P}})$, $a_{\mathcal{P}} = \alpha + q\alpha^{-1}$

$$\mathrm{Ind}_{K_{\mathcal{P}} Z_{\mathcal{P}}}^{G_{\mathcal{P}}}(1_{\mathbb{C}}) = \{\phi : K_{\mathcal{P}} Z_{\mathcal{P}} \backslash G_{\mathcal{P}} \to \mathbb{C}\}, \quad \mathcal{T}_{\mathcal{P}}\phi(\cdot) = \sum_{\substack{K_{\mathcal{P}} g \in K_{\mathcal{P}} g_{\mathcal{P}} K_{\mathcal{P}} \\ v(\det(g_{\mathcal{P}}))=1}} \phi(g\ \cdot).$$

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(ii) $\pi_{\mathcal{P}} = \mathrm{Ind}_{K_0(\mathcal{P})Z_{\mathcal{P}}}^{G_{\mathcal{P}}}(1_{\mathbb{C}})/(U_{\mathcal{P}} - \alpha)$,

$$\mathrm{Ind}_{K_0(\mathcal{P})Z_{\mathcal{P}}}^{G}(1_{\mathbb{C}}) = \{\phi : K_0(\mathcal{P})Z_{\mathcal{P}} \backslash G_{\mathcal{P}} \to \mathbb{C}\}.$$

Local distributions attached to torus

$T^2_{\mathcal{P}} \subset G_{\mathcal{P}}$ maximal torus, $T^2_{\mathcal{P}} \not\subset B_{\mathcal{P}}$, $T_{\mathcal{P}} = T^2_{\mathcal{P}}/Z_{\mathcal{P}}$,

$$B_{\mathcal{P}} \cap T^2_{\mathcal{P}} = Z_{\mathcal{P}}, \quad B_{\mathcal{P}} T^2_{\mathcal{P}} \subseteq G_{\mathcal{P}} \text{ open}$$

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Description (i):

$$\begin{array}{rcl} \delta_{T_{\mathcal{P}}} : C_c(T_{\mathcal{P}}, \mathbb{C}) & \longrightarrow & \mathrm{Ind}_{B_{\mathcal{P}} Z_{\mathcal{P}}}^{G_{\mathcal{P}}}(\mu_\alpha) \longrightarrow \pi_{\mathcal{P}} \\ f & \longmapsto & \phi_f\left(\left(\begin{array}{cc} t_1 & x \\ 0 & t_2 \end{array}\right) t\right) = \alpha^{v(t_2/t_1)} f(t), \end{array}$$

$t \in T_{\mathcal{P}}$.

- $\delta_{T_{\mathcal{P}}}$ is $T_{\mathcal{P}}$-invariant.

Global distributions attached to torus

- G/F multiplicative group quaternion algebra, $G(F_{\mathcal{P}}) = \mathrm{GL}_2(F_{\mathcal{P}})$. $\pi^{JL} \in L^2(G(F)\backslash G(\mathbb{A}_F)/Z(\mathbb{A}_F))$ Jacquet-Langlands lift.

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- E/F quadratic extension. $T \subset G/Z$ torus attached to $E^\times/F^\times$. Artin-map:

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- $\bigotimes_v' \pi_v^{JL} \simeq \pi^{JL} \subset L^2(G(F)\backslash G(\mathbb{A}_F)/Z(\mathbb{A}_F))$

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$f \in C(\mathcal{G}_{E,\mathcal{P}}, \mathbb{C}_p)_0$, $H_{\mathcal{P}} \subseteq T(F_{\mathcal{P}})$ maximal compact subgroup, $H \subseteq H_{\mathcal{P}}$ small enough subgroup:

$$\int_{\mathcal{G}_{E,\mathcal{P}}} f(\gamma) d\mu_{E,\mathcal{P}}^I(\gamma) := [H_{\mathcal{P}} : H] \int_{T(\mathbb{A}_F)/T(F)} f(\rho(t))\delta^I(1_H)(t) d^\times t,$$

Global distributions: Heegner condition

- $\mu^{I}_{E,\mathcal{P}} = 0$ unless,

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- E/F totally imaginary: $\Sigma^\pi = \Sigma^\pi_\infty \cup \Sigma^\pi_f$,

$$\text{(parallel weight 2} \Rightarrow) \quad \Sigma^\pi_\infty = \{v \mid \infty\} \Rightarrow B \text{ totally definite.}$$
$$\text{(Heegner condition)} \quad \Rightarrow \quad \#\Sigma^\pi_f + [F : \mathbb{Q}] \ \text{ is even.}$$

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$\rightarrow$ If $\#\Sigma^\pi_f + [F:\mathbb{Q}]$ even, we define $\mu^{\pi,I}_{E,\mathcal{P}}$: **DEFINITE CASE.**

Theorem

If $\pi_{\mathcal{P}}$ ordinary then $\mu^{\pi,I}_{E,\mathcal{P}}$ is a measure, defines

$$L^I_{\mathcal{P}}(\pi, E) \in \mathbb{Z}_p[[\mathcal{G}_{E,\mathcal{P}}]].$$

Global distributions: Interpolation Property

Theorem

Non-zero constant C_E *and* $\chi : \mathcal{G}_{E,\mathcal{P}} \to \mathbb{C}^\times$ *continuous character,*

$$\left| \int_{\mathcal{G}_{E,\mathcal{P}}} \chi(\gamma) d\mu^{\pi,I}_{E,\mathcal{P}}(\gamma) \right|^2 = C_E C(\pi_{\mathcal{P}}, \chi_{\mathcal{P}}) \frac{L(1/2, \pi_E, \chi)}{L(1, \pi, ad)},$$

where

$$C(\pi_{\mathcal{P}}, \chi_{\mathcal{P}}) = \begin{cases} \frac{L(1,\pi_{\mathcal{P}},ad)}{L(1/2,\pi_{\mathcal{P}},\chi_{\mathcal{P}})}, & |\alpha|^2 = q, \\ \frac{L(-1,\pi_{\mathcal{P}},\chi_{\mathcal{P}})L(1,\pi_{\mathcal{P}},ad)}{L(0,\pi_{\mathcal{P}},\chi_{\mathcal{P}})L(1/2,\pi_{\mathcal{P}},\chi_{\mathcal{P}})}, & \alpha = \pm 1, \ \chi_{\mathcal{P}} \mid_{\mathcal{O}_E^\times} = 1, \\ q^{n_\chi} \frac{L(1,\pi_{\mathcal{P}},ad)}{L(1/2,\pi_{\mathcal{P}},\chi_{\mathcal{P}})}, & \alpha = \pm 1, \ \chi_{\mathcal{P}} \mid_{\mathcal{O}_E^\times} \neq 1, \end{cases}$$

and n_χ *is the conductor of* $\chi_{\mathcal{P}}$*.*

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- Mod. param.: $\phi \in \pi^{JL}$ automorphic new-form $\to \omega \in \Omega^1_{X_U}$,

$$X_U/F \longrightarrow \mathrm{Pic}(X_U)/F \longrightarrow A/F \stackrel{\log_\omega}{\longrightarrow} \mathbb{C}_p.$$

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- Heegner points,

$$T(F)\backslash G(\hat{F})/U\hat{F}^\times \simeq \mathrm{CM}(E) \subset X_U(E^{ab}).$$

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Description (ii) of $\pi_{\mathcal{P}}$ + Frobenius Reciprocity:

$$\delta^{II} : C_c(T_{\mathcal{P}}, \mathbb{C}) \xrightarrow{\delta_{T_{\mathcal{P}}}} \pi_{\mathcal{P}} \hookrightarrow \bigotimes_{v\nmid\infty}{}' \pi_v^{JL} \longrightarrow \mathcal{A}(T(F)\backslash G(\hat{F}))$$

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Global distributions: Interpolation Property

Theorem

If $\pi_{\mathcal{P}}$ ordinary then $\mu_{E,\mathcal{P}}^{\pi,II}$ is a measure, defines

$$L_{\mathcal{P}}^{II}(\pi, E) \in \mathbb{Z}_p[[\mathcal{G}_{E,\mathcal{P}}]].$$

Theorem

Let $\chi : \mathcal{G}_{E,\mathcal{P}} \to \mathbb{C}^\times$ continuous character,

$$\int_{\mathcal{G}_{E,\mathcal{P}}} \chi(\gamma) d\mu_{E,\mathcal{P}}^{\pi,II}(\gamma) = \log_\omega(P_\chi),$$

where $P_\chi \in A(E^{ab}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$ Heegner point,

$$\langle P_\chi, P_\chi \rangle_{NT} = C_E C(\pi_{\mathcal{P}}, \chi_{\mathcal{P}}) \frac{L'(1/2, \pi_E, \chi)}{L(1, \pi, ad)}.$$

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- $\kappa^i \in H^0(T(F), \mathrm{Meas}(T(\hat{F})/\Gamma, \mathbb{C}_p))$,

$$f_{\mathcal{P}} \otimes f^{\mathcal{P}} \mapsto \sum_{x \in T(F^{\mathcal{P}})/\Gamma} f^{\mathcal{P}}(x)\delta^i(f_{\mathcal{P}})(x),$$

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- $f \in C(\mathcal{G}_{E,\mathcal{P}}, \mathbb{C}_p)$

$$\int_{\mathcal{G}_{E,\mathcal{P}}} f d\mu_{E,\mathcal{P}}^{\pi,i} = \kappa^i \cap \partial f$$

Índex

Exceptional Zero

If $\pi_{\mathcal{P}}$ split multiplicative, $C(\pi_{\mathcal{P}},1)=0$, hence

$$\deg(L^{i}_{\mathcal{P}}(\pi,E))=\int_{\mathcal{G}_{E,\mathcal{P}}} d\mu^{\pi,i}_{E,\mathcal{P}}=0 \quad (i=I,II)$$

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Description $\pi_{\mathcal{P}}$:

$$\begin{array}{ccccccc}
 & & & C_c(T_{\mathcal{P}},\mathbb{C}_p)_0 & & & \\
 & & & \downarrow \bar{\delta}_{T_{\mathcal{P}}} & \searrow^{\delta_{T_{\mathcal{P}}}} & & \\
0 \longrightarrow & \mathbb{C}1_{\mathbb{P}^1(F_{\mathcal{P}})} & \xrightarrow{\iota} & \mathrm{Ind}_{B_{\mathcal{P}}}^{G_{\mathcal{P}}}(1_{\mathbb{C}})=C(\mathbb{P}^1(F_{\mathcal{P}}),\mathbb{C}) & \longrightarrow & \pi_{\mathcal{P}} & \longrightarrow 0,
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$$0 \longrightarrow \mathbb{C}1_{\mathbb{P}^1(F_{\mathcal{P}})} \xrightarrow{\iota} \mathrm{Ind}_{B_{\mathcal{P}}}^{G_{\mathcal{P}}}(1_{\mathbb{C}}) = C(\mathbb{P}^1(F_{\mathcal{P}}),\mathbb{C}) \longrightarrow \pi_{\mathcal{P}} \longrightarrow 0,$$

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- This implies:

$$\int_{\mathcal{G}_{E,\mathcal{P}}} d\mu^{\pi,i}_{E,\mathcal{P}} = \kappa^i \cap \partial 1 = \iota^\vee(\kappa^i) \cap 1_{\mathbb{P}^1(F_{\mathcal{P}})} = 0.$$

Image in $\mathcal{I}/\mathcal{I}^2$

Let $\nabla L^i_{\mathcal{P}}(\pi, E)$ be the image of $L^i_{\mathcal{P}}(\pi, E) \in \mathcal{I}$ in $\mathcal{I}/\mathcal{I}^2$:

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Let $\nabla L_{\mathcal{P}}^{i}(\pi, E)$ be the image of $L_{\mathcal{P}}^{i}(\pi, E) \in \mathcal{I}$ in $\mathcal{I}/\mathcal{I}^2$:

Theorem

If $T_{\mathcal{P}}$ **splits** *then*

$$\begin{aligned} \nabla L_{\mathcal{P}}^{I}(\pi, E) &= \underline{\mathcal{L}_{\mathcal{P}}}(\pi) \left(C_E C(\pi_{\mathcal{P}}) \frac{L(1/2, \pi_E, 1)}{L(1, \pi, ad)} \right)^{1/2}, \\ \nabla L_{\mathcal{P}}^{II}(\pi, E) &= \underline{\mathcal{L}_{\mathcal{P}}}(\pi) \log_p(P_T), \end{aligned}$$

where $C(\pi_{\mathcal{P}}) = \frac{-L(1,\pi_{\mathcal{P}},ad)\zeta_{\mathcal{P}}(-1)^2}{L(1/2,\pi_{\mathcal{P}},1)} \neq 0$ *and* $P_T \in A(E^{ab}) \otimes \bar{\mathbb{Q}}$ *Heegner pt*

$$\langle P_T, P_T \rangle_{NT} = C_E C(\pi_{\mathcal{P}}) \frac{L'(1/2, \pi_E, 1)}{L(1, \pi, ad)}.$$

$\underline{\mathcal{L}_{\mathcal{P}}}(\pi) \in \mathcal{I}/\mathcal{I}^2$ **automorphic $\mathcal{L}$-invariant.**

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$$\frac{\partial}{\partial \ell} L_{\mathcal{P}}^I(\pi, E) := \int_{\mathcal{G}_{E,\mathcal{P}}} \ell(\gamma) d\mu_{E,\mathcal{P}}^{\pi,i}(\gamma) = \kappa^i \cap \partial \ell,$$

for all $\ell \in \mathcal{G}_{E,\mathcal{P}}^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{G}_{E,\mathcal{P}}, \mathbb{Z}_p)$.

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Open Problems

- **Bertolini-Darmon**: $F = \mathbb{Q}$ in definite case, but with $\underline{\mathcal{L}_p}(\pi)$ replaced by $\mathcal{L}_p(A) = \ell_p(q_{A/\mathbb{Q}_p})/\mathrm{ord}_p(q_{A/\mathbb{Q}_p})$
- Relation $\underline{\mathcal{L}_{\mathcal{P}}}(\pi)$ with geometry of A?
- Functorialtiy of $\mathcal{L}$-Invariants?
- $T_{\mathcal{P}}$ inert? (**Bertolini-Darmon** $F = \mathbb{Q}$, p-adic Gross-Zagier)
- $T_{\mathcal{P}}$ ramified?