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A nonconforming substructuring method for first-order systems in space-time

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Institute for Applied and Numerical Mathematics

Overview and Challenge

Maxims:

- How to discretize?

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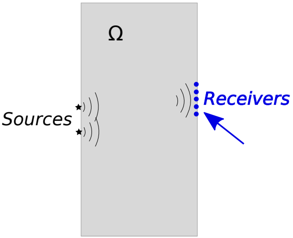
time-dependent wave



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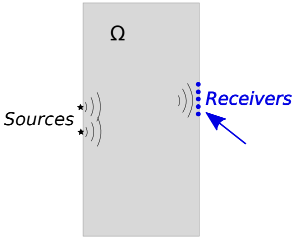
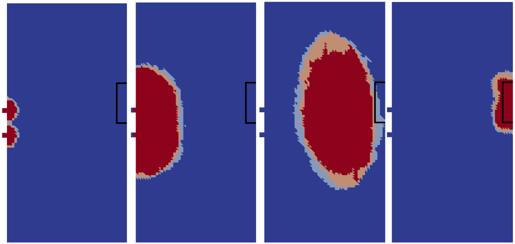


local polynomial degree over time



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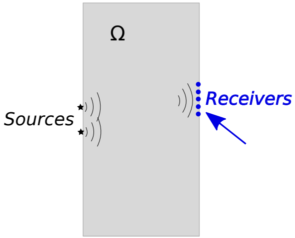

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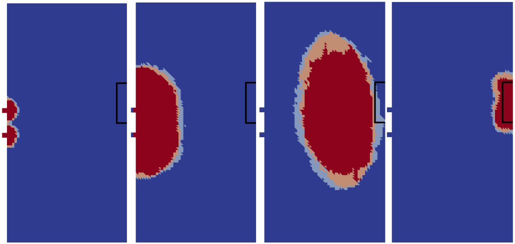

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Challenge:
Find discretization scheme

- that is versatile
- that is practical

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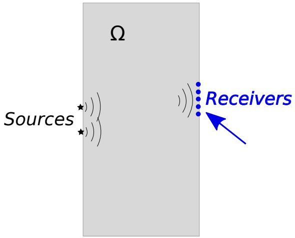

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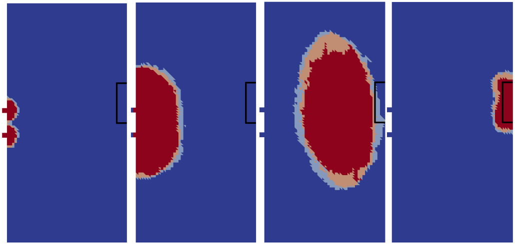

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Wave Equation – Low regularity solution

- Domain $\Omega \subset \mathbb{R}^D$, time interval $(0, T)$, space-time cylinder $Q = \Omega \times (0, T)$
- Wave equation

$$\partial_t^2 \phi = \Delta \phi \quad \text{in } Q, \qquad + \text{ initial/boundary conditions}$$

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- Solution with low regularity (1D case):

$$\Omega = (0, 1), \quad \phi_0(\cdot) := \phi(\cdot, 0) = 1, \quad \partial_t \phi(\cdot, 0) = 0, \quad \text{homogeneous bnd conditions}$$

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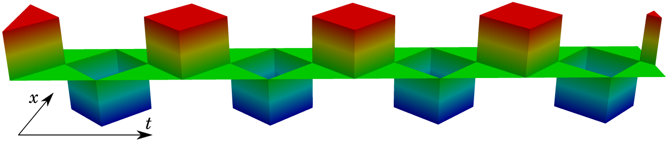

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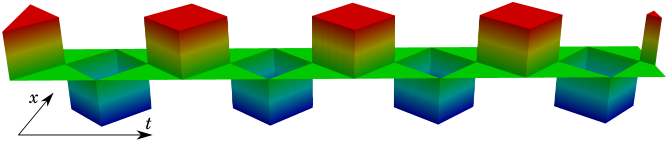

- How to approximate low regularity solutions?

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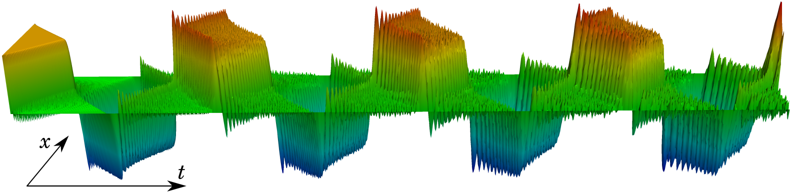

Figure: FDTD Scheme with 83 000 DoFs (space: 86, time: 969)

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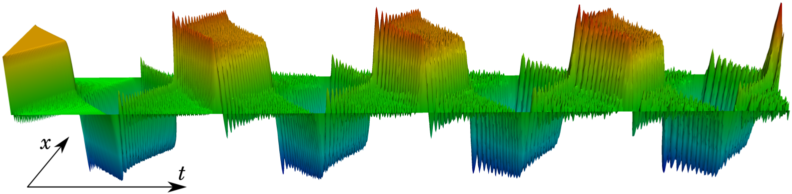

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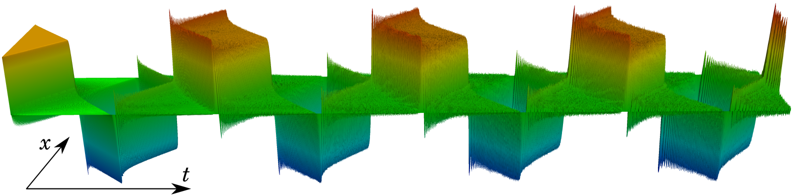

Figure: FDTD Scheme with 1 300 000 DoFs (space: 340, time: 3865)

$\longrightarrow$ CFL-condition fulfilled: $c = 1$, $\frac{c\Delta t}{h} = 0.67$

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- Space-Time Least Squares: test space $X = L(V)$, solve: $\min_{v\in V} \frac{1}{2}\|Lv - f\|^2_{\mathrm{L}_2(Q)}$

What does Least-Squares yield?

Space-Time Least Squares: $V_h \subset V$, solve: $\min_{v_h \in V_h} \frac{1}{2} \| L v_h - f \|^2_{\mathrm{L}_2(Q)}$

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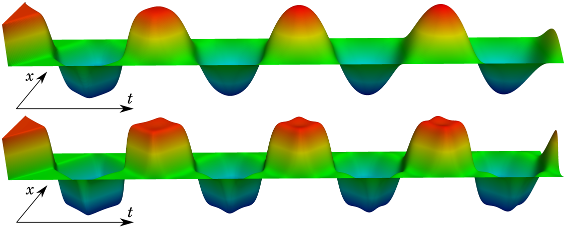

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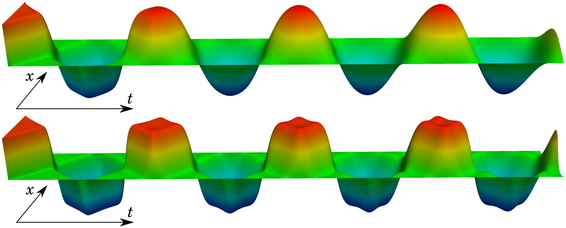

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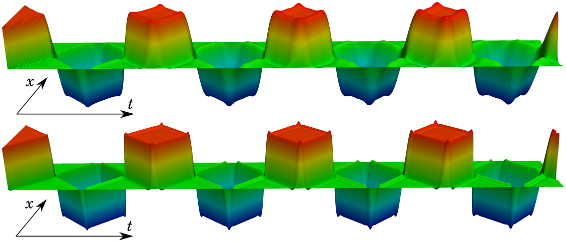

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Q1	260 000		1 100 000		4 200 000		17 000 000		67 000 000
e_h	0.340		0.255		0.192		0.141		0.103
$e_h/e_{h/2}$		1.334		1.328		1.357		1.374	
$\log_2 e_h/e_{h/2}$		0.415		0.410		0.440		0.458	
Q2	260 000		1 100 000		4 200 000		17 000 000		67 000 000
e_h	0.143		0.094		0.061		0.039		0.025
$e_h/e_{h/2}$		1.522		1.546		1.562		1.571	
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- DoFs per accuracy (estimate for uniform refinement)

e_h	0.1	0.01	0.001
Q1	70 000 000	>1e12	>1e17
Q2	850 000	>1e9	>1e12

How can this be improved?

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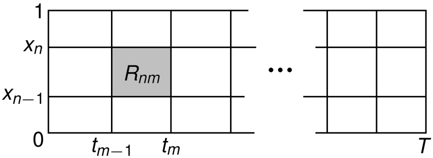

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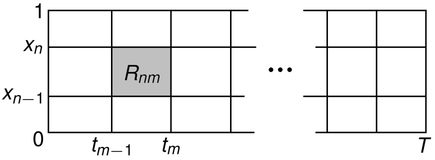

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1. approximate traces of $v \in V$ on the skeleton
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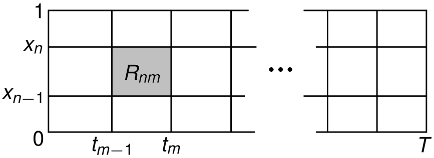

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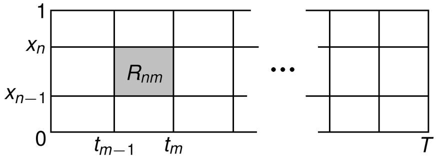

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- Choose trace operators $\gamma_R \colon V|_R \longrightarrow F(\partial R)$, $\gamma_R^{\mathrm{ad}} \colon W|_R \longrightarrow F(\partial R)$ with

$$\langle \gamma_R v, \gamma_R^{\mathrm{ad}} w \rangle_{\partial R} = (Lv, w)_R - (v, L^{\mathrm{ad}} w)_R, \qquad v \in V|_R,\ w \in W|_R,$$

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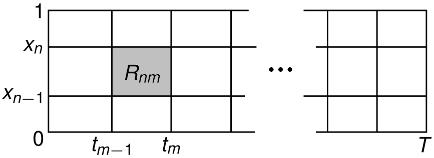

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- Observe $\gamma_R v \in \gamma_R^{\mathrm{ad}}(W|_R)'$ (dual space)

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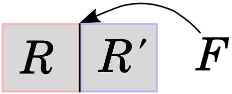

- On face $F = \partial R \cap \partial R'$ we have for $v_h \in V_h$

$$\langle \gamma_R v_R, \gamma_R^{\mathrm{ad}} w_R\rangle_F = -\langle \gamma_{R'} v_{R'}, \gamma_{R'}^{\mathrm{ad}} w_{R'}\rangle_F \qquad \forall w_h \in W_h \quad \text{(face bubbles)}$$

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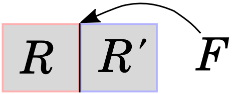
- Observe for $v \in \prod_{R\in\mathcal{R}} V|_R$

$$v \in V \quad \Longleftrightarrow \quad \langle \gamma v, \gamma^{\mathrm{ad}} w\rangle := \sum_{R\in\mathcal{R}} \langle \gamma_R v|_R, \gamma_R^{\mathrm{ad}} w|_R\rangle = 0 \quad \forall w \in W$$

- Choose $W_h \subset W$, $V_R \subset V|_R$ and set $V_{\mathcal{R}} = \prod_{R\in\mathcal{R}} V_R$

$$V_h := \Big\{ v_h \in V_{\mathcal{R}} : \sum_{R\in\mathcal{R}} \langle \gamma_R v_R, \gamma_R^{\mathrm{ad}} w_R\rangle = 0 \text{ for all } w_h \in W_h \Big\}$$



- On face $F = \partial R \cap \partial R'$ we have for $v_h \in V_h$

$$\langle \gamma_R v_R, \gamma_R^{\mathrm{ad}} w_R\rangle_F = -\langle \gamma_{R'} v_{R'}, \gamma_{R'}^{\mathrm{ad}} w_{R'}\rangle_F \qquad \forall w_h \in W_h \quad \text{(face bubbles)}$$

- Trace space: $\hat V_h = \gamma^{\mathrm{ad}}(W_h)'$ (dual space)

- Representation of $\hat v_h \in \hat V_h$: $\Big(\langle \hat v_h, \gamma_R^{\mathrm{ad}} w_F\rangle\Big)_{F\subset\partial R,\ R\in\mathcal{R}}$ (by duality)

Substructuring – Saddle Point Formulation

- Define $J_R\colon\ V_R \longrightarrow \mathbb{R},\ v_R \longmapsto \frac{1}{2}\big\|Lv_R - f\big\|^2_{\mathrm{L}_2(R)}$
- Problem: find $v_h \in V_{\mathcal{R}}$ with

$$\sum_{R\in\mathcal{R}} J_R(v_R) \longrightarrow \text{Min!} \qquad \text{s.t.} \qquad \sum_{R\in\mathcal{R}} \langle \gamma_R v_R, \gamma_R^{\text{ad}} w_R\rangle = 0 \quad \forall w_h \in W_h$$

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Substructuring – Algorithm

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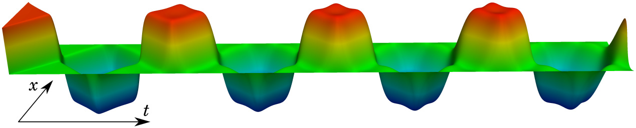
$\longrightarrow$ well suited for parallel implementation!

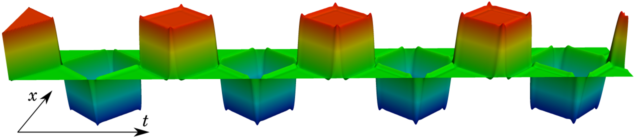
What can we do with that?

Comparison – Plots

$V_h = \mathbb{Q}_1(Q)$



$V_h = \mathbb{Q}_2(Q)$



Comparison – Plots

$V_h = \mathbb{Q}_1(Q)$

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$V_h = \text{ncLS}$

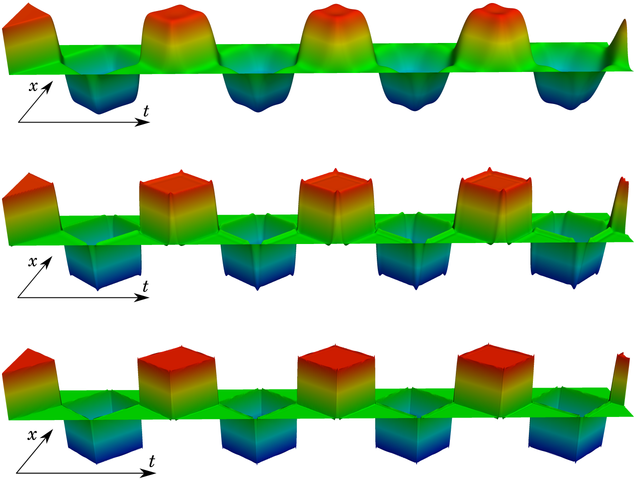

Figure: Example plots for p (LS: 1 316 354 DoFs, ncLS: 1 317 376 DoFs)

$$V_R = \big(\mathbb{P}_3(\mathbb{R}) \otimes \mathbb{P}_3(\mathbb{R})\big)^2, \qquad W_h|_R \subset \big(\mathbb{P}_4(\mathbb{R}) \otimes \mathbb{P}_3(\mathbb{R})\big)^2$$

Comparison in L_1-Norms ($T = 8 \cdot \frac{3}{\pi} \approx 7.64$)

- Error: $e_h = \|u - u_h\|_{L_1(Q)} / \|u\|_{L_1(Q)}$

Q1	260 000		1 100 000		4 200 000		17 000 000		67 000 000
e_h	0.340		0.255		0.192		0.141		0.103
$e_h/e_{h/2}$		1.334		1.328		1.357		1.374	
$\log_2 e_h/e_{h/2}$		0.415		0.410		0.440		0.458	
Q2	260 000		1 100 000		4 200 000		17 000 000		67 000 000
e_h	0.143		0.094		0.061		0.039		0.025
$e_h/e_{h/2}$		1.522		1.546		1.562		1.571	
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ncLS	330 000	1 300 000	5 300 000	21 000 000	84 000 000
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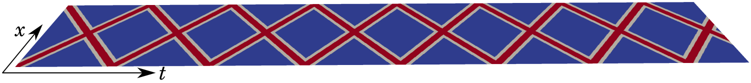
- DoFs per accuracy (estimate for uniform refinement)

e_h	0.1	0.01	0.001
Q1	70 000 000	>1e12	>1e17
Q2	850 000	>1e9	>1e12
ncLS	54 000	2e7	>1e10

p-Adaptivity – Proof of concept

- ad-hoc marking of faces



- uniform grid

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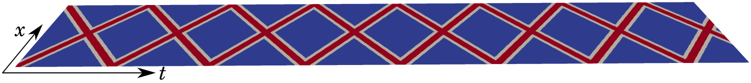

- uniform grid
- choose $W_h \longrightarrow$ choose DoFs on each face

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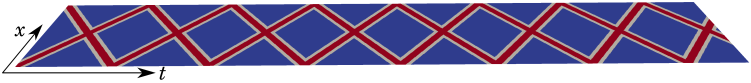

- uniform grid
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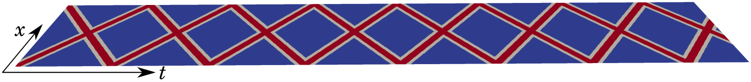

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ncLS (uniform)	330 000	1 300 000	5 300 000	21 000 000	84 000 000
e_h	0.050664	0.029965	0.017685	0.010438	0.006167
$e_h/e_{h/2}$		1.690786	1.694362	1.694260	1.692637
$\log_2 e_h/e_{h/2}$		0.757694	0.760742	0.760655	0.759273
ncLS (heterogenous)	180 000	720 000	2 900 000	12 000 000	46 000 000
e_h	0.061163	0.031287	0.018011	0.010614	0.006620
$e_h/e_{h/2}$		1.954922	1.737079	1.696971	1.603246
$\log_2 e_h/e_{h/2}$		0.967111	0.796663	0.762962	0.680996

- Similar accuracy with half number of DoFs

Summary and Future Work

Result: discretization that

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- numerical analysis
- multi-level preconditioner
- built-in error estimator
- other error functions J_R are possible

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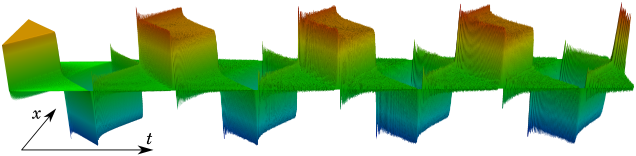
- multi-level preconditioner
- built-in error estimator
- other error functions J_R are possible
- use as a forward-solver for full waveform inversion
- higher dimensions (2D, 3D)

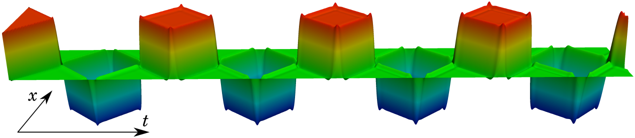
Thank you!

Questions?

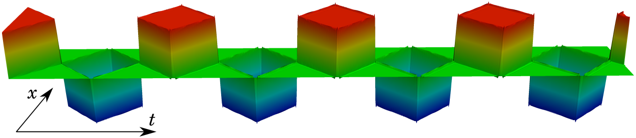
FDTD



$V_h = \mathbb{Q}_2(Q)$



$V_h = \text{ncLS}$



Conformity based error estimator

- Assume the conformity error is dominant, i.e.

$$\sup_{R\in\mathcal{R}} \inf_{v_R\in V_R} \|u - v_R\|_{V_R} \ll \inf_{v\in V} \|u_h - v\|_V,$$

with exact solution $u \in V$ and approximation $(u_R)_R \in V_h$

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$$\sum_{R\in\mathcal{R}} \langle \gamma_R v_R, \gamma_R^{\text{ad}} w_R \rangle = 0 \quad \forall w_h \in W_h$$

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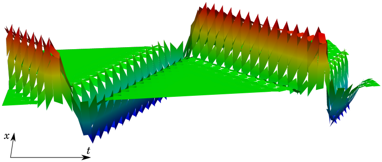

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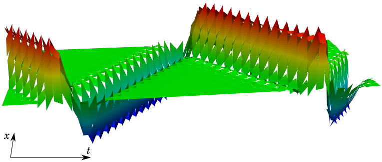

- Idea: choose $\tilde{W}_h = \text{span}\{\tilde{w}_F^1, \dots, \tilde{w}_F^N \colon F \subset \Gamma_h \text{ mesh-face}\} \subset W \setminus W_h$

$$\eta_F = \sum_n \left| \langle \gamma_R u_h, \gamma_R^{\text{ad}} \tilde{w}_F^n\rangle - \langle \gamma_{R'} u_h, \gamma_{R'}^{\text{ad}} \tilde{w}_F^n\rangle \right| \qquad \text{(error estimator)}$$

for $F = \partial R \cap \partial R'$

Convergence for a smooth solution

- Use smooth initial value

$$p_0(x,0) = \begin{cases} \cos\left(\frac{s-m}{w}\right)^2, & |s-m| < w \\ 0, & \text{else}, \end{cases}$$

with $w = 0.3 \cdot \frac{3}{\pi}$, $m = 0.85 \cdot \frac{3}{\pi}$.

- Error: $e_h = \|u - u_h\|_{L_2(Q)}$

ncLS	11 000		42 000		170 000		660 000		2 600 000
e_h	1.4e-3		3.8e-4		1.0e-4		2.9e-5		8.0e-6
$e_h/e_{h/2}$		3.75		3.67		3.60		3.58	
$\log_2 e_h/e_{h/2}$		1.91		1.87		1.85		1.84	

What can we expect?

- $N_p^{\mathrm{DG}} = 2 \cdot \dim \mathbb{Q}_p(R)$, $\quad \|e\|_{L_2(Q)} \le Ch^{p+1/2}$
- $N_1^{\mathrm{DG}} = 8$, rate: ≤ 1.5 $\quad N_2^{\mathrm{DG}} = 18$, rate: ≤ 2.5
- Here: $N_{3,3}^{\mathrm{ncLS}} = 10$, rate: $1.5 \le 1.84 \le 2.5$

Local Operators for the Wave Equation

Karlsruhe Institute of Technology

- For $u_R = (p_1, q_1)$, $v_R = (p_2, q_2)$ it holds

$$
\begin{aligned}
\langle A_R u_R, v_R \rangle &= (L u_R, L v_R)_{L_2(R)} \\
&= \left(\begin{pmatrix} \partial_t p_1 + \nabla \cdot q_1 \\ \partial_t q_1 + \nabla p_1 \end{pmatrix}, \begin{pmatrix} \partial_t p_2 + \nabla \cdot q_2 \\ \partial_t q_2 + \nabla p_2 \end{pmatrix} \right)_{L_2(R)} \\
\langle \ell_R v_R \rangle &= (f, L v_R)_{L_2(R)}
\end{aligned}
$$

- For $v_R = (p, q)$, $w_R = (\phi, \psi)$ we have

$$
\begin{aligned}
\langle B_R w_R, v_R \rangle &= \langle \gamma_R v_R, \gamma_R^{\mathrm{ad}} w_R \rangle \\
&= \big((p, q), (\phi, \psi)\big)_{C \times \{t_n\}} - \big((p, q), (\phi, \psi)\big)_{C \times \{t_{n-1}\}} \\
&\quad + \big\langle (p, q \cdot n), (\psi \cdot n, \phi) \big\rangle_{\partial C \times (t_{n-1}, t_n)}
\end{aligned}
$$