Design of meshes adapted to the observation and control of discrete waves.

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Joint Work with
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Partial differential equations, optimal design and numerics
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Outline

Outline

The wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^N$:
Boundary Control

$$\begin{cases} \partial_{tt} y - \Delta y = 0, & (t,x) \in (0,T) \times \Omega, \\ y = v \chi_\Gamma & (t,x) \in (0,T) \times \partial\Omega, \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)) & x \in \Omega. \end{cases}$$

- y = the displacement of the waves.
- Initial datum (y^0, y^1) = initial displacement and velocity.
- The control function v acts on $\Gamma \subset \partial\Omega$.

The control problem

Given (y^0, y^1) and (y_T^0, y_T^1), find a control function v such that the solution y of the controlled wave eq with datum (y^0, y^1) satisfies

$$(y(T), \partial_t y(T)) = (y_T^0, y_T^1).$$

Remark

The control property may depend on the time horizon $T > 0$!

The wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^N$:

Boundary Control

$$\begin{cases} \partial_{tt} y - \Delta y = 0, & (t,x) \in (0,T) \times \Omega, \\ y = v\chi_\Gamma & (t,x) \in (0,T) \times \partial\Omega, \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)) & x \in \Omega. \end{cases}$$

- y = the displacement of the waves.
- Initial datum (y^0, y^1) = initial displacement and velocity.
- The control function v acts on $\Gamma \subset \partial\Omega$.

The control problem (simpler version using linearity+reversibility)

Given (y^0, y^1), find a control function v such that the solution y of the controlled wave eq with datum (y^0, y^1) satisfies

$$(y(T), \partial_t y(T)) = (0, 0).$$

Remark

The control property may depend on the time horizon $T > 0$!

Functional Setting (boundary case):

- $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$
- $v \in L^2((0,T) \times \partial\Omega)$.

Cauchy problem

If $v \in L^2((0,T) \times \partial\Omega)$, there exists a unique solution y in

$$C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$$

of the wave equation in the sense of transposition with initial data $(0,0)$ and boundary conditions v.

Duality result (Dolecki-Russell '77, Lions '88)

The wave equation is exactly controllable at time $T > 0$ if and only if there exists a constant $C_{obs} > 0$ such that all solution φ of

$$\begin{cases} \partial_{tt}\varphi - \Delta\varphi = 0, & (t,x) \in (0,T) \times \Omega, \\ \varphi = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ (\varphi(0), \partial_t\varphi(0)) = (\varphi^0, \varphi^1), & x \in \Omega, \end{cases}$$

with initial data $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies the following observability inequality:

$$\left\|(\varphi^0, \varphi^1)\right\|^2_{H_0^1 \times L^2} \leq C_{obs}^2 \iint_{(0,T)\times\partial\Omega} |\chi_\Gamma \partial_n \varphi|^2 \, dt d\sigma,$$

where ∂_n represents the normal derivative on $\partial\Omega$.

Remarks

Admissibility/Hidden regularity (Lions '88)

$\exists C > 0$, s.t. $\forall (\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\iint_{(0,T)\times\partial\Omega} |\chi_\Gamma \partial_n \varphi|^2 \, dt d\sigma \leq C \left\| (\varphi^0, \varphi^1) \right\|^2_{H_0^1 \times L^2}.$$

Hilbert Uniqueness Method (Lions '88)

Observability $\Rightarrow$ Controllability *with a constructive approach.*

$\rightsquigarrow$ In the following, focus on the observability property.

Let us come back to

$$\begin{cases} \partial_{tt}\varphi - \Delta\varphi = 0, & (t,x) \in (0,T) \times \Omega, \\ \varphi = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ (\varphi(0), \partial_t\varphi(0)) = (\varphi^0, \varphi^1), \end{cases}$$

with initial data $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and the following observability inequality:

$$\left\|(\varphi^0, \varphi^1)\right\|^2_{H_0^1 \times L^2} \leq C_{obs}^2 \iint_{(0,T)\times\partial\Omega} |\chi_\Gamma \partial_n \varphi|^2 \, dt d\sigma,$$

"Geometry", in a broad sense, matters.

$\rightsquigarrow$ In 1-d, $\Omega = (0, L)$, $\Gamma = \{L\}$, $T \geq 2L$.

Are Geometric Conditions needed to get observability estimates ?

An easy example in which it can be shown that, whatever the time $T > 0$ is, there is no observability inequality is the case of $\Omega = (0,1)^2$ and $\Gamma = \{0\} \times (0,1)$.
Consider the solutions:

$$\varphi_{k,\ell}(t,x) = \frac{1}{k} e^{it\pi\sqrt{k^2+\ell^2}} \sin(k\pi x_1) \sin(\ell \pi x_2) - \frac{1}{k+1} e^{it\pi\sqrt{(k+1)^2+\ell^2}} \sin((k+1)\pi x_1) \sin(\ell \pi x_2).$$

For $\ell \to \infty$, one can check that the energy of $\varphi_{k,\ell}$ blows up as $\ell \to \infty$ but the norm of the observation goes to 0 as $\ell \to \infty$, whatever the time $T > 0$ is.

Combination of solutions oscillating at very close frequencies with similar observations on $\Gamma \rightsquigarrow$ Failure of observability inequality.

The Geometric Control Condition

$\rightsquigarrow$ A necessary and sufficient condition for observability.

The Geometric Control Condition (Bardos Lebeau Rauch '92)

All the rays of Geometric Optics in Ω should meet the observation region $\{\chi_\Gamma > 0\}$ in a time less than T in a non-diffractive point.

Roughly speaking, the rays of Geometric Optics are straight lines, going at velocity one in the domain Ω, and bouncing on the boundary $\partial\Omega$ according to Descartes-Snell laws.

• the Hamiltonian is given by the principal symbol of the waves:

$$H(t, x, \tau, \xi) = |\tau|^2 - |\xi|^2, \quad (t, x) \in (0, T) \times \Omega, \quad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

Here, the parameters τ, ξ denote the Fourier parameters corresponding to t, x, respectively.

• The wave front set of solutions of the waves is supported on the bicharacteristic set, i.e. the set (t, x, τ, ξ) such that $H(t, x, \tau, \xi) = 0$.

• Singularities are transported by the Hamiltonian flow.

• Bicharasteristics (see Hörmander) are the trajectories $s \mapsto (t(s), x(s), \tau(s), \xi(s))$ given by

$$\frac{dx}{ds} = -\nabla_\xi H(t,x,\tau,\xi) = 2\xi, \qquad \frac{dt}{ds} = -\partial_\tau H(t,x,\tau,\xi) = -2\tau,$$
$$\frac{d\xi}{ds} = \nabla_x H(t,x,\tau,\xi) = 0, \qquad \frac{d\tau}{ds} = \partial_t H(t,x,\tau,\xi) = 0,$$

for initial data lying in the bicharasteristic set.

• The projection of the bicharasteristics on the physical space are the rays of geometric optics.
Simply given by $t \mapsto x(t) = x(0) + \frac{\xi_0}{|\xi_0|} t$ away from the boundary.

Outline

The 1d wave equation:

$$\begin{cases} \partial_{tt} y - \partial_{xx} y = 0, & (t,x) \in (0,T) \times (0,1) \\ y(t,0) = 0, \quad y(t,1) = v(t), & t \geq 0. \end{cases}$$

Controllable for $T \geq 2$.

Consider its corresponding finite difference semi-discretization, $N \in \mathbb{N}$, $h = \frac{1}{N+1}$.

$$\begin{cases} \partial_{tt} y_j - \dfrac{1}{h^2}(y_{j-1} + y_{j+1} - 2y_j) = 0, & j \in \{1, \cdots, N\}, t \geq 0, \\ y_0(t) = 0, \quad y_{N+1}(t) = v_h(t), & t \geq 0. \end{cases}$$

$\rightsquigarrow$ finite-dimensional problem.

Semi-discretization (finite differences):

$$\begin{cases} \partial_{tt} y_j - \dfrac{1}{h^2}(y_{j-1} + y_{j+1} - 2y_j) = 0, \quad j \in \{1, \cdots, N\}, t \geq 0, \\ y_0(t) = 0, \quad y_{N+1}(t) = v_h(t), \quad t \geq 0. \end{cases}$$

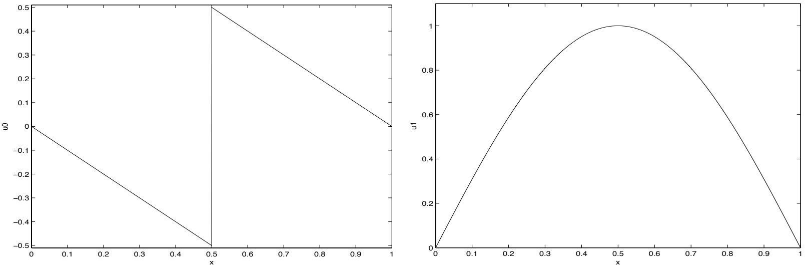


Figure : Left, initial displacement $y(0)$. Right, initial velocity $\partial_t y(0)$.

Numerical control (I)

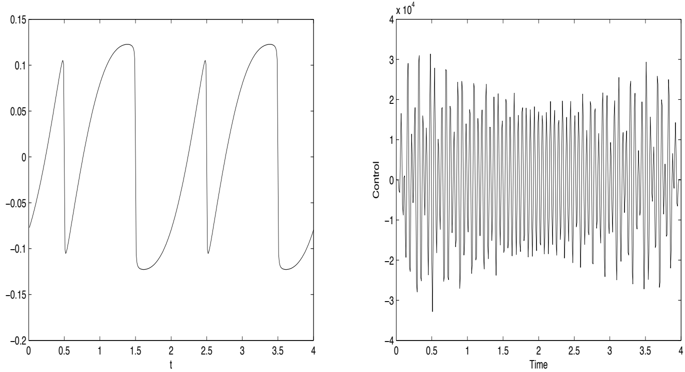


Figure : Computations of the control for $T = 4$: Left, the explicit formula. Right, the control computed on the discrete systems for $N = 50$.

Numerical control (II)

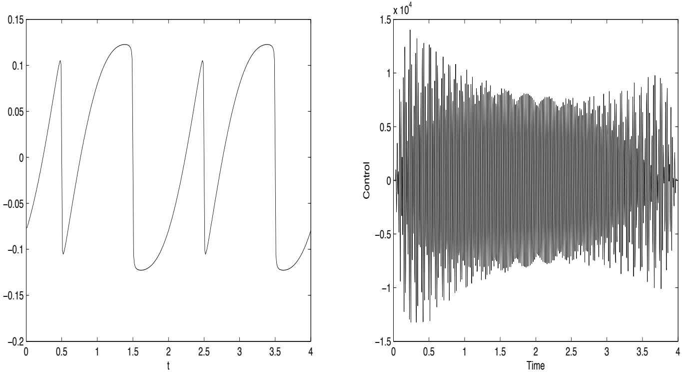


Figure : Computations of the control: Left, the explicit formula. Right, the control computed on the discrete systems for $N = 150$.

$\Rightarrow$ Instabilities at high-frequencies

Two explanations:

- Spectrally
- with discrete rays

The corresponding observability property cannot be true uniformly with respect to $h > 0$.

Difficulties

- From finite-dimensional systems to infinite dimensional ones $\rightsquigarrow$ Conditions on the time of observability for the wave eq.
- Observability with respect to a parameter.

Corresponding observability property

There exists a constant C_{obs} independent of $h > 0$ such that any solution of

$$\begin{cases} \partial_{tt}\varphi_j - \frac{1}{h^2}(\varphi_{j-1} + \varphi_{j+1} - 2\varphi_j) = 0, \quad j \in \{1, \cdots, N\}, t \geq 0, \\ \varphi_0(t) = \varphi_{N+1}(t) = 0, \quad t \geq 0. \end{cases}$$

satisfies

$$E_h \leq C_{obs}^2 \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt,$$

with

$$E_h(t) = E_h(0) = h \sum_{j=1}^{N} |\partial_t \varphi_j(t)|^2 + h \sum_{j=0}^{N} \left(\frac{\varphi_{j+1} - \varphi_j}{h} \right)^2$$

Spectral explanation

$$\begin{cases} (\Delta_h \phi)_j = \frac{1}{h^2}(\phi_{j-1} + \phi_{j+1} - 2\phi_j), \\ \phi_0 = \phi_{N+1} = 0 \end{cases}$$

Spectrum of $-\Delta_h$

For $k \in \{1, \cdots, N\}$,

- Eigenvalues $\lambda_h^k = \frac{4}{h^2} \sin\left(\frac{k\pi h}{2}\right)^2$
- Eigenvectors $(w^k)_j = \sqrt{2}\sin(k\pi jh)$.

In particular, $\varphi(t) = e^{it\sqrt{\lambda_h^k}} w^k$ solves the discrete wave eq. and

$$E_h(\varphi) = 2(\lambda_h^k)^2, \quad \left|\frac{\varphi_N}{h}\right| = \sqrt{2}\cos\left(\frac{k\pi h}{2}\right)\sqrt{\lambda_h^k}.$$

$\Rightarrow$ Observability fails ($k = N = 1/h - 1$), blows up at least as h^{-1}

Besides, the observability constant blows up faster than any polynomial in h:
Take

$$\varphi(t) = e^{it\sqrt{\lambda_h^N}} \frac{w^N}{w_N^N} - e^{it\sqrt{\lambda_h^{N-1}}} \frac{w^{N-1}}{w_N^{N-1}}.$$

$\rightsquigarrow$ observability blows up at least as h^{-2} !

Choosing suitable combinations of eigenvectors corresponding to the largest eigenvalues, the observability blows up faster than any polynomial.

Close eigenvalues deteriorates the observability property.

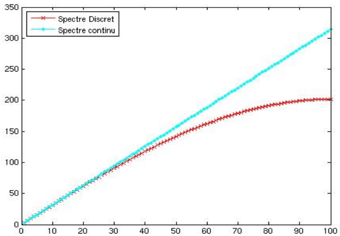


Figure : Dispersion diagram for the finite differences semi-discrete wave equation with $N = 100$: blue, the continuous eigenvalues $\sqrt{\lambda^k} = k\pi$; red, the discrete ones $\sqrt{\lambda_h^k} = \frac{2}{h}\sin\left(\frac{k\pi h}{2}\right)$

Horizontal tangent for $k \simeq N$

$\simeq$ Accumulation point in the spectrum.

Propagation of discrete rays

Discrete Hamiltonian (Trefethen '82, Macia '05):

$$\tau^2 - \frac{4}{h^2} \sin\left(\frac{\xi h}{2}\right)^2$$

$\rightsquigarrow$ yields rays of the form

$$t \mapsto x(t) = x_0 \pm \cos\left(\frac{\xi_0 h}{2}\right) t.$$

At high frequencies $\xi_0 \simeq 1/h$, high-frequency waves travel at a velocity imposed by the discretization.

Discrete and continuous rays

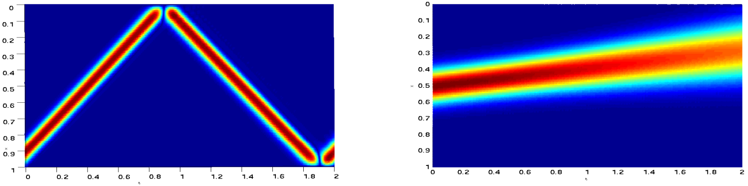


Figure : The wave propagation in continuous/discrete media in dimension one: right, the ray is a high-frequency Gaussian beam.

Continuous dynamics $\neq$ Discrete dynamics

These rays concentrate more than any polynomial.

To sum up

Pathologies arise at high-frequency (order $1/h$).

For each $h > 0$, the finite-dimensional system obtained by discretization is observable in any time $T > 0$,
BUT the observability constant is not uniform with respect to $h > 0$, whatever $T > 0$ is.
(and blows up faster than any polynomial in $1/h$ *(Micu '02))*

Corollary

There exist initial data for which the sequence of discrete controls diverge.

How to re-establish observability ?

- Penalize the spurious high-frequencies of the discrete solutions:
  - Filtering techniques
    Glowinski & al '91, Infante Zuazua '99, Zuazua '99, SE '09, Miller '12, Marica-Zuazua '15, ...
  - Bi-grid techniques
    Asch Lebeau '98, Negreanu Zuazua '04, Ignat Zuazua '09, ...
  - Tychonoff regularization
    Glowinski Li Lions '90, Zuazua '05, SE '09, ...
- Use specific schemes (mainly mixed finite elements) which behaves well at high-frequency:
  Castro Micu '06, Münch '05, Castro Micu Münch '08, SE '08,
- Use the observability of the continuous equation:
  Continuous Approach
  Cindea Micu Tucsnak '11, SE Zuazua '13, Cindea Fernandez-Cara Münch '13, ...

Main idea

In all the aforementioned works, one considers discretization methods adapted to the resolution of the wave equation, not to the controllability problem at hand.

A different approach : **Adapting the mesh**

Instead of considering a discretization adapted to the Cauchy theory and study its observability/controllability properties, design a discretization method adapted to the considered control problem.

- Related to works on optimal grids for inverse problems by Borcea Drushkin Knizhnermann '02, '05 ...
- Seems new (?) in our context.

Outline

Meshes under consideration:

- $g : [0, 1] \to [0, 1]$ denotes a smooth diffeomorphism of the interval $[0, 1]$, $g(0) = 0$, $g(1) = 1$.
- $N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$.
- $x_j = g(jh)$,
- $h_{j+1/2} := x_{j+1} - x_j$, $h_j := \frac{h_{j-1/2}+h_{j+1/2}}{2}$.

Space semi-discrete wave equation on that mesh:

$$\begin{cases} h_j y_j''(t) - \left(\dfrac{y_{j+1}(t) - y_j(t)}{h_{j+1/2}} - \dfrac{y_j(t) - y_{j-1}(t)}{h_{j-1/2}} \right) = 0, & 1 \leq j \leq N, \\ y_0(t) = y_{N+1}(t) = 0, & t \in (0, T), \\ y_j(0) = y_j^0, \ y_j'(0) = y_j^1, & 1 \leq j \leq N. \end{cases}$$

Behaviors of discrete rays

Away from the boundary, the discrete Hamiltonian reads as (Marica Zuazua '14):

$$H(t,x,\tau,\xi) = \tau^2 - \frac{4}{(g'(x))^2 h^2} \sin\left(\frac{\xi h}{2}\right)^2 .$$

In particular, the rays $t \mapsto x(t)$ satisfies

$$\frac{d^2x}{dt^2}(t) = -\frac{g''(x(t))}{g'(x(t))^3}.$$

In particular, if $g'' < 0$, the rays are bent to the right.

To fix the ideas, $g_\theta(x) = \sqrt{(2\theta+1)x + \theta^2} - \theta$, for which

$$-\frac{g_\theta''(x(t))}{g_\theta'(x(t))^3} = \frac{2}{1+2\theta}.$$

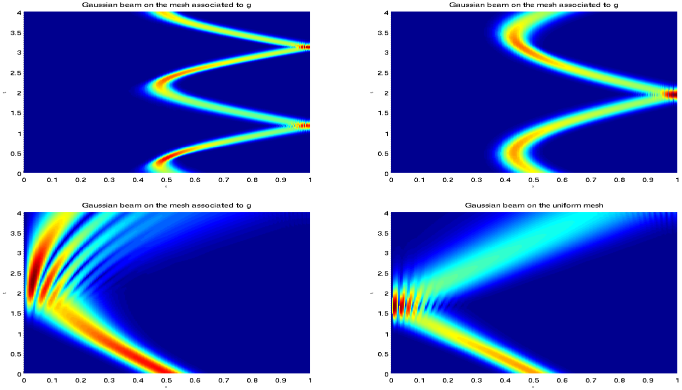


Figure : Gaussian beams for various choices of the function g starting from $x_0 = 0.5$, $\xi_0 = 0.8\pi/h$. From left to right and top to bottom, $g = g_{0,1}$, $g = g_1$, $g = g_{10}$ and $g(x) = x$.

Insights

- If g is strictly concave, high-frequency rays meet the boundary $x = 1$.
- Observability from $x = 1$ should hold for strictly concave diffeomorphism g corresponding to meshes which are refined close to $x = 1$.

Intuitively: Informations close to the observation set is more accurate and can be exploited more.

$\rightsquigarrow$ Remarks also valid for the finite element method.

More on the Hamiltonian flow

Setting

$$\omega(\xi) = 2\sin\left(\frac{\xi}{2}\right),$$

the bicharacteristics are given by

$$\begin{cases} \dfrac{dx}{dt}(t) = -\dfrac{1}{\tau_0 g'(x(t))^2}\omega(\xi(t))\partial_\xi\omega(\xi(t)), \\ \dfrac{d\xi}{dt}(t) = -\dfrac{1}{\tau_0}\dfrac{g''(x(t))}{g'(x(t))^3}\omega(\xi(t))^2, \end{cases}$$

with

$$\tau_0^2 = \frac{\omega(\xi(t))^2}{g'(x(t))^2}.$$

$\Rightarrow$ We can draw the phase portraits.

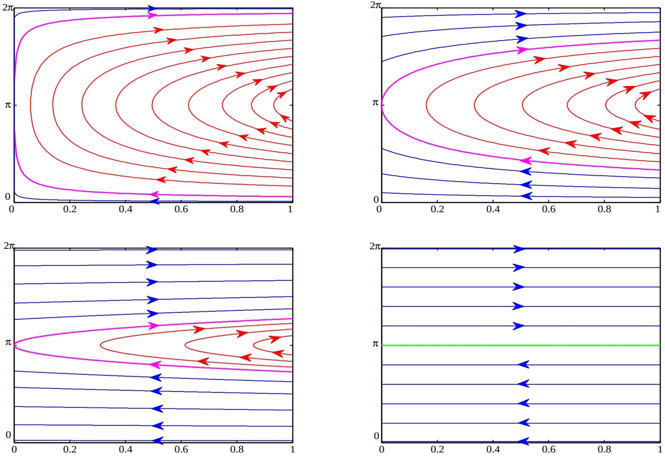


Figure : Phase portraits $(x, \xi h)$ for the Hamiltonian flow for various functions g: from left to right and top to bottom, $g = g_{0.1}$, $g = g_1$, $g = g_{10}$ and $g(x) = x$ (uniform case).

Theorem (SE Marica Zuazua '15)

Given $T > 2$, there exists a smooth diffeomorphism g such that the solutions $\mathbf{y}^{h,g}$ of the semi-discrete wave equation associated to g are uniformly observable through $x = 1$ in time T. A suitable g: $g_\theta(x) = \sqrt{(2\theta+1)x+\theta^2} - \theta$, for $\theta \in \left(0, \frac{T-2}{2}\right)$.

Theorem (SE Marica Zuazua '15)

Assume that g is strictly concave. Define

$$\theta_g := \max_{x\in[0,1]} \left\{ \frac{g'(x)^2 + g(x)g''(x)}{-g''(x)} \right\}.$$

Then the solutions $\mathbf{y}^{h,g}$ of the semi-discrete wave equation associated to g are uniformly observable through $x = 1$ in any time $T > T_g$, where T_g is given by $T_g := 2(1+\theta_g)$.

Comments

- Proof done by multipliers techniques.
- Results valid for the finite-difference and finite-elements methods in 1d.
- Coincides with the insights provided by the analysis of the discrete Hamiltonian. But a careful analysis on the boundary is missing.
- We also have spectral insights based on numerical evidences of
  - A spectral gap $\rightsquigarrow$ Ingham's Lemma.
  - Localization of high-frequency eigenvectors on the refined parts of the mesh.
- Accurate results as well for computing discrete controls.

A spectral gap

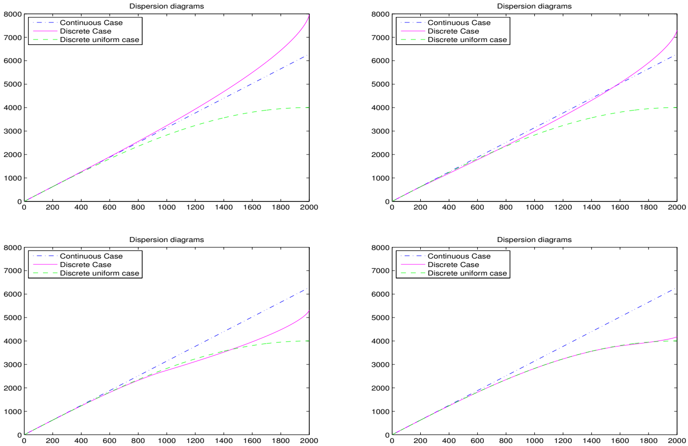


Figure : Dispersion diagram $k \to \sqrt{\lambda^{k,h,g}}$ for various g: from left to right and top to bottom, $g = g_0$, $g = g_{0.1}$, $g = g_1$, $g = g_{10}$.

Eigenvectors for eigenvalues in the bottom of the spectrum

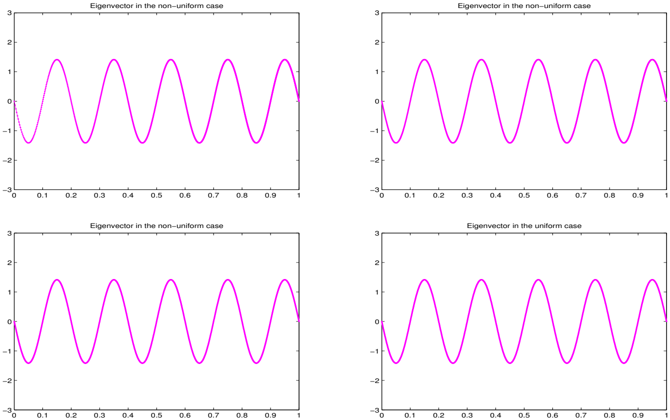


Figure : Plot of the eigenvectors $\mathbf{w}^{k,h,g}$ for $k = 10$, $N = 2000$, and various functions g: from left to right and top to bottom, $g = g_{0.1}$, $g = g_1$, $g = g_{10}$ and $g(x) = x$.

Eigenvectors for eigenvalues in the middle of the spectrum

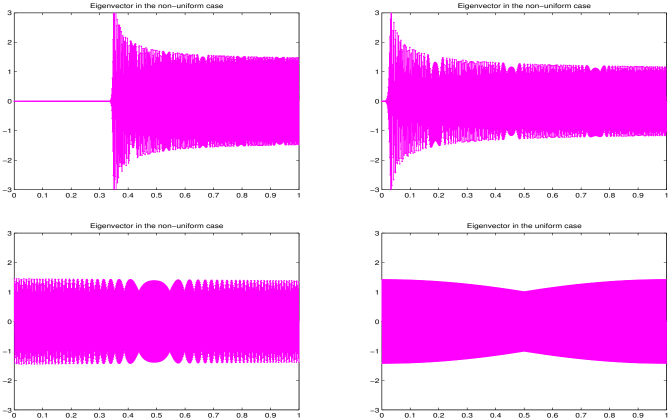


Figure : Plot of the eigenvectors $\mathbf{w}^{k,h,g}$ for $k = 1000$, $N = 2000$, and various functions g: from left to right and top to bottom, $g = g_{0.1}$, $g = g_1$, $g = g_{10}$ and $g(x) = x$.

Eigenvectors for eigenvalues in the top of the spectrum

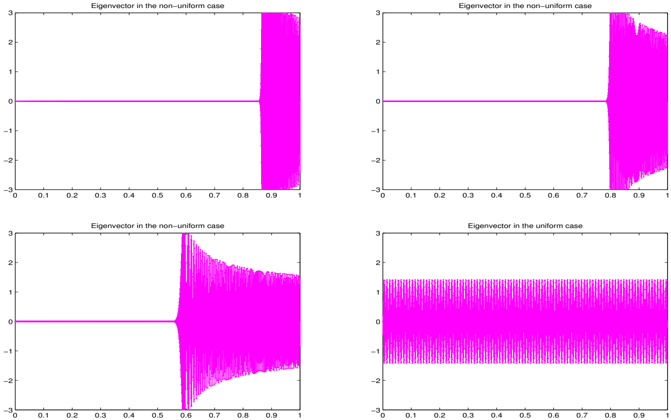


Figure : Plot of the eigenvectors $\mathbf{w}^{k,h,g}$ for $k = 1900$, $N = 2000$, and various functions g: from left to right and top to bottom, $g = g_{0.1}$, $g = g_1$, $g = g_{10}$ and $g(x) = x$.

Numerical experiment for computing discrete controls

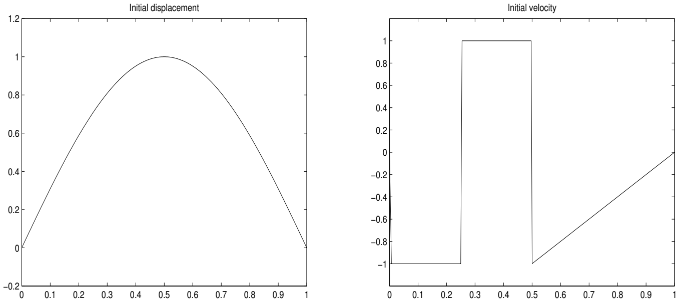
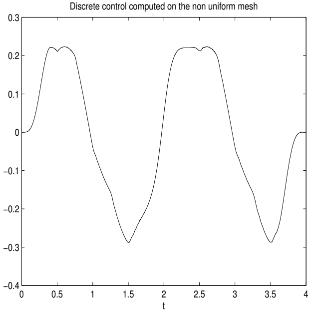


Figure : Initial datum to be controlled: left, y^0, right, y^1.



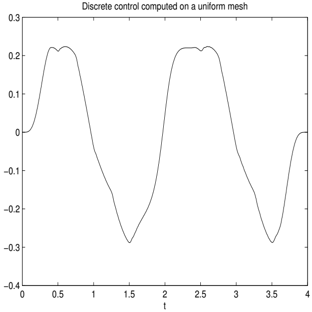


Figure : Discrete controls. Left,computed on the mesh associated to g_θ with $\theta = (T-2)/2$, $T = 4$ and $h = 1/301$. Right, computed on a uniform mesh with a filtering process.

Outline

Adapting the mesh to the control problem at hand is a natural idea

Open questions:

- What is the sharp time of uniform observability ?
  Probably given by the discrete Hamiltonian but the boundary needs to be handled carefully.
- Analysis limited to very structured meshes.
  Meshes that are not diffeomorphic images of the uniform mesh ?
- Analysis limited to the 1d case.
  Generation of suitable meshes in higher dimensions ?

Thank you for your attention!

Reference:

Numerical meshes ensuring uniform observability of 1d waves: construction and analysis,
Sylvain Ervedoza, Aurora Marica and Enrique Zuazua,
to appear in IMA Journal of Numerical Analysis.