Low congested regions and networks and optimal reinforcement for a membrane

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Congestion effects have been extensively studied since '50 by Wardrop (discrete case) and Beckmann (continuous model).

Data:

- a bounded Lipschitz region $\Omega \subset \mathbb{R}^d$;
- two probability measures f^+ and f^-.

The model, in a stationary regime, reduces to:

$$\min \left\{ \int_\Omega H(\sigma)\, dx \ : \ -\operatorname{div} \sigma = f \text{ in } \Omega, \ \sigma \cdot n = 0 \text{ on } \partial\Omega \right\},$$

where:

- σ is the traffic flux;
- $H : \mathbb{R}^d \to [0, +\infty]$ is the congestion function, i.e. a convex nonnegative function with $\lim_{|s| \to +\infty} H(s)/|s| = +\infty$.

A continuous model

In the model:

$$\min \left\{ \int_\Omega H(\sigma)\, dx \ : \ -\operatorname{div} \sigma = f \text{ in } \Omega,\ \sigma \cdot n = 0 \text{ on } \partial\Omega \right\},$$

- the boundary condition $\sigma \cdot n = 0$ on $\partial\Omega$ models the zero normal flux on the boundary.
- the PDE $-\operatorname{div} \sigma = f^+ - f^-$ captures the equilibrium between the traffic flux and the difference f.

Recently, *Carlier, Jimenez, Santambrogio* have presented a model equivalent to Beckmann's problem.

Membrane model

We consider an elastic membrane under the action of an exterior load f and fixed at its boundary; this amounts to solve the variational problem

$$\min\left\{\int_\Omega\left(\frac{1}{2}|\nabla u|^2 - fu\right)dx \; : \; u\in H_0^1(\Omega)\right\}$$

or equivalently the elliptic PDE

$$-\Delta u = f \text{ in } \Omega, \qquad u\in H_0^1(\Omega).$$

The two previous models are connected by the following:

Lemma

X, Y Banach, $A : X \to Y$ linear, $F : Y \to \overline{\mathbb{R}}$ convex. Then for every $f \in X'$

$$\inf_{\sigma \in Y'} \Big\{ F^*(\sigma) \ : \ A^*\sigma = f \Big\} = - \inf_{u \in X} \Big\{ F(Au) - \langle f, u \rangle \Big\}.$$

Taking $Au = \nabla u$ the transport problem with congestion H can be written in its dual form

A new model

In Ω we consider

- two congestion functions $H_1 \leq H_2$;
- a penalization cost m.

For every region C we consider the cost function

$$F(C) = \min \left\{ \int_{\Omega \setminus C} H_2(\sigma)\, dx + \int_C H_1(\sigma)\, dx \ : \ \sigma \in \Gamma_f \right\}$$

where

$$\Gamma_f = \left\{ \sigma \in L^1(\Omega; \mathbb{R}^d) \ : \ -\operatorname{div} \sigma = f \text{ in } \Omega, \ \sigma \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

Goal: find a low congested region $C \subset \Omega$ solving

$$\min \left\{ F(C) + m(C) \ : \ C \subset \Omega \right\}.$$

Perimeter penalization

We consider the case $m(C) = kPer(C)$, $k > 0$

Theorem

Assume that the cost $F(C)$ is finite for at least a subset C of $\overline{\Omega}$ with finite perimeter. Then there exists at least an optimal set C_{opt}.

Regularity:

- since $H_2 \geq H_1$, implies that ∂C has nonnegative mean curvature;
- when $d = 2$ and Ω is convex, the optimal regions C are convex.

We consider the case $m(C) = k\,|C|$, $k > 0$

Passing from sets C to density function $0 \leq \theta(x) \leq 1$ we obtain the relaxed formulation

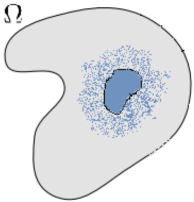
$$\min_{\sigma,\theta} \left\{ \int_\Omega \theta H_1(\sigma)\,dx + \int_\Omega (1-\theta) H_2(\sigma)\,dx + k \int_\Omega \theta\,dx \;:\; \sigma \in \Gamma_f \right\}.$$

After the elimination of the variable θ we end up with a non convex integrand.
A new relaxation is necessary, so we have

$$\min \left\{ \int_\Omega \Big(H_2(\sigma) \wedge \big(H_1(\sigma) + k\big) \Big)^{**} dx \;:\; \sigma \in \Gamma_f \right\}.$$

Recall: the functions H_1 and H_2 are superlinear.

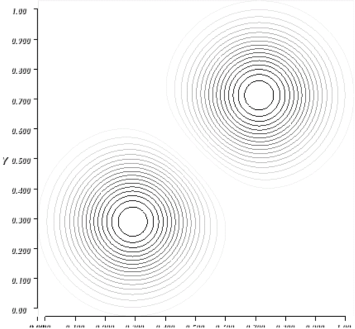
We denote by $\bar{\sigma}$ the optimal solution of the problem we have that:



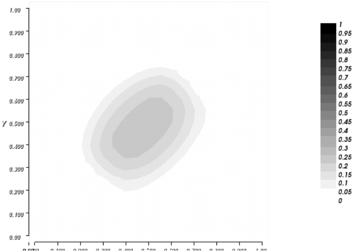
$$\Big(H_2 \wedge (H_1 + k)\Big)^{**}(\overline{\sigma}) = H_2(\overline{\sigma}) \Rightarrow \theta = 0;$$

$$\Big(H_2 \wedge (H_1 + k)\Big)^{**}(\overline{\sigma}) = H_1(\overline{\sigma}) + k \Rightarrow \theta = 1;$$

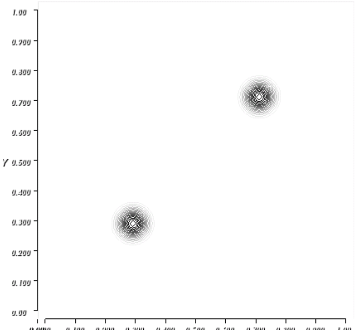
$$\Big(H_2 \wedge (H_1 + k)\Big)^{**}(\overline{\sigma}) < \Big(H_2 \wedge (H_1 + k)\Big)(\overline{\sigma}) \Rightarrow 0 < \theta < 1.$$



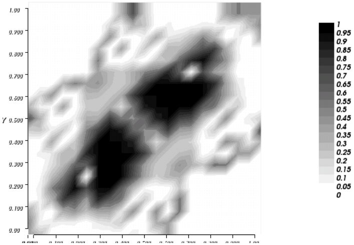
(a) $\lambda = 0.02$



(b) $k = 0.4$



(c) $\lambda = 0.001$



(d) $k = 0.01$

We consider the case $C \in \mathcal{A}_L$,

$$\mathcal{A}_L = \{S \subset \Omega,\ S \text{ closed connected},\ \mathcal{H}^1(S) \leq L\}$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. For every $S \in \mathcal{A}_L$ we define the energy functional

$$\mathcal{E}_f(S) = \inf\left\{ \int_\Omega \frac{1}{2}|\nabla u|^2\,dx + \int_S \frac{1}{2}|\nabla u|^2\,d\mathcal{H}^1 - \int_\Omega u\,df \ :\ u \in C_c^\infty(\Omega)\right\}$$

so that the optimization problem we deal with is

$$\max\left\{\mathcal{E}_f(S)\ :\ S \in \mathcal{A}_L\right\}.$$

One dimensional sets

Since limits of one-dimensional sets are in general measures, it is convenient to define the energy functional $\mathcal{E}_f$ even for a measure μ, by setting

$$\mathcal{E}_f(\mu) = \inf \left\{ \int_\Omega \frac{1}{2}|\nabla u|^2\, dx + \int_\Omega \frac{1}{2}|\nabla u|^2\, d\mu - \int_\Omega u\, df \ : \ u \in C_c^\infty(\Omega) \right\}.$$

- **Theorem:** Assume that $\mathcal{E}_f(\mu)$ is finite for at least a $\mu \in \mathcal{M}_L^+(\Omega)$. Then the maximization problem $\max \left\{ \mathcal{E}_f(\mu) \ : \ \mu \in \mathcal{M}_L^+(\Omega) \right\}$ admits at least a solution.
- **Theorem:** Let μ be a solution of the maximization problem. Then there exists a one-dimensional closed connected set S such that the absolutely continuous part of μ with respect to $\mathcal{H}^1 \lfloor S$ is also a solution. In other words the solution μ is of the form $\mu = a(x)\mathcal{H}^1 \lfloor S$, with $S \in \mathcal{A}_L$ and $a(x) \geq 1$.

Open question

It is possible to find an example in which we have the case $a(x) > 1$?

If we consider $f^+ = \delta_A$, $f^- = \delta_B$ and $L >> |A - B|$, what is the optimal μ? On which set is it concentrated?