Shape derivatives for minima of integral functionals

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First and second order shape derivatives

Given a shape functional

$$J : \Omega \to \mathbb{R},$$

the typical question of shape optimization is to look for minimizers of J under some constraints, namely to study

$$\inf_{\Omega \in \mathscr{A}} J(\Omega),$$

over some class $\mathscr{A}$ of subsets of $\mathbb{R}^n$.

Determine minimizers is often a difficult task, therefore, especially at the beginning of the study, one can be satisfied with qualitative results.

A useful tool in this direction is represented by shape derivatives :

* $J' = 0$ detects critical shapes, where J', if it exists, is given by

$$J'(\Omega, V) := \lim_{\varepsilon \to 0} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon};$$

* $J'' > 0$ gives optimality among the critical shapes, where J'', if it exists, is given by

$$J''(\Omega, V) := \lim_{\varepsilon \to 0} 2 \, \frac{[J(\Omega_\varepsilon) - J(\Omega) - \varepsilon J'(\Omega, V)]}{\varepsilon^2};$$

with $\Omega_\varepsilon := \{x + \varepsilon V(x) \ : \ x \in \Omega\}$, being V a fixed Lipschitz vector field.

Available results

* structure theorems [Novruzi & Pierre 2005,...]
  $\rightsquigarrow$ theoretical
* explicit formulas [Henrot & Pierre 2005, ...]
  $\rightsquigarrow$ specific to the problem under study, strong regularity assumptions
* numerical results

In the following, I will present

** the case of minima of integral functionals : the convex setting

All the results can be found in [Bouchittè-Fragalà-L. 2014,2015]

Standing assumptions (I)

Given

* $\Omega \subset \mathbb{R}^n$ an open bounded connected set,
* $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ two continuous convex functions satisfying a (p, q)-growth condition (with $p > 1$),
* V a vector field in $W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$,

we consider

$$J(\Omega) := -\inf_{u \in W(\Omega)} \int_\Omega [f(\nabla u) + g(u)]\, dx \qquad (J)$$

where $W(\Omega)$ denotes one of the following function spaces : $W_0^{1,p}(\Omega)$ (Dirichlet case), $W^{1,p}(\Omega)$ (Neumann case), or the space $W_\gamma^{1,p}(\Omega)$ of Sobolev functions vanishing on a Borel subset γ of $\partial\Omega$.

Model examples

For simplicity, in the following examples we assume that Ω is a Lipschitz domain and we take $W(\Omega) = W_0^{1,p}(\Omega)$.

Ex 1. The p-torsion problem $(p \geq 2)$

We consider

$$f(z) := \frac{|z|^p}{p}, \qquad g(t) := -t.$$

* for $p = 2$, $J', J'' =$ known [Henrot & Pierre 2005];
* for $p > 2$, $J' =$ known [Chorwadwala & Mahadevan 2012], $J'' =?$

Ex 2. A non strictly convex case

We consider

$$f(z) := \begin{cases} |z| & \text{if } |z| \leq 1 \\ \frac{1}{2}(|z|^2 + 1) & \text{if } |z| > 1 \end{cases}, \qquad g(t) := -t.$$

* $J' =?$

Some remarks :

(Ex1)
Notice that $J(\Omega)$ can be rewritten as

$$J(\Omega) = -\int_\Omega \Big[\frac{|\nabla u_p|^p}{p} - u_p\Big]\, dx\,,$$

where u_p is the unique minimizer of (J) in $W_0^{1,p}(\Omega)$, which is characterized by the equation

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = 1 \quad \text{in } \Omega\,.$$

(Ex2)
Unlike the previous example, since f is convex but not strictly convex, problem (J) can have more than one solution, characterized by a variational inequality.
Due to the lack of regularity of f (non Gateaux differentiability at the origin !) the classical techniques can't be applied for the computation of J'. (Note that the theory of shape derivatives for problems governed by variational inequalities has been developed, see e.g. [Hintermüller & Laurain 2011], but without explicit formulas).

Strategy (I)

For the computation of J', the main idea is to find upper and lower bounds of the quotient $\frac{[J(\Omega_\varepsilon)-J(\Omega)]}{\varepsilon}$, and then pass to the limit as $\varepsilon \to 0$.

Our approach is based on the combined use of Convex Analysis and Γ-convergence, more precisely :

* upper/lower bounds $\rightsquigarrow$ duality
  The key tool is the following equivalent formulation of the shape functional :

$$J(\Omega) = J^*(\Omega) := \inf_{\sigma \in X(\Omega)} \int_\Omega [f^*(\sigma) + g^*(\operatorname{div}\sigma)]\,dx\,,$$

  where f^* and g^* denote the Fenchel conjugates of f and g respectively, and $X(\Omega)$ is the space of $L^{p'}$ vector fields with $L^{q'}$ divergence, being p' and q' the conjugate exponents of of p and q respectively. When $W(\Omega) = W^{1,p}(\Omega)$ or $W^{1,p}_\gamma(\Omega)$ we also assume that the elements of X have zero normal trace on $\partial\Omega$ and $\partial\Omega \setminus \gamma$ respectively.
* vanishing $\varepsilon \rightsquigarrow \Gamma$-convergence

Main result (I)

Theorem

Under the standing assumptions (I), the first order shape derivative at Ω in direction V exists and is given by the following formula :

$$J'(\Omega, V) = \sup_{\substack{u \ \text{opt} \\ \text{for } J(\Omega)}} \ \inf_{\substack{\sigma \ \text{opt} \\ \text{for } J^*(\Omega)}} \int_\Omega A(u,\sigma) : DV\, dx\,,$$

where A is an explicit tensor field. In the equality above, the order of inf *and* sup *can be switched, moreover there exists an optimal pair $(\overline{u},\overline{\sigma})$.*

Some remarks :

* in general J' is not linear with respect to V : an optimal pair may depend on V ($\exists$ counterexample) ;
* under additional assumptions, we recover the expected result : $J' = \int_{\partial\Omega} \alpha V \cdot n$, for some density α independent of V.

Applications (I)

(Ex1)

$$J'(\Omega, V) = \int_{\partial\Omega} \frac{|\nabla u_p|^p}{p'} V \cdot n \, d\mathscr{H}^{n-1}.$$

Notice that, if a Lipschitz domain Ω is a minimizer of J under a volume constraint, then $J'(\Omega, V) = 0$ for every V volume preserving, i.e., whenever $V \cdot n$ has zero average on $\partial\Omega$. Thus the associated function u_p satisfies an overdetermined problem, with

$$\left| \frac{\partial u_p}{\partial n} \right| = const. \quad \text{on } \partial\Omega.$$

(Ex2)

$$J'(\Omega, V) = \int_{\partial\Omega} \frac{1}{2}(|\nabla u_\Omega|^2 - 1)_+ V \cdot n \mathscr{H}^{n-1},$$

where u_Ω is an arbitrary solution of (J).
As in the previous example, if Ω is Lipschitz and minimizes J under a volume constraint, we infer that every u_Ω satisfies an overdetermined variational inequality, with

$$\left| \frac{\partial u_\Omega}{\partial n} \right| \leq 1 \quad \text{on } \partial\Omega.$$

Standing assumptions (II)

By applying the same strategy adopted for the first order shape derivative, we are able to treat also second order shape derivatives, assuming the following further regularity assumptions :

* Ω is an open bounded connected set with piecewise C^1 boundary ;
* $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two C^2 strongly convex functions, satisfying (p, q)-growth conditions with $p = 2$;
* $J(\Omega)$ admits a (unique) Lipschitz solution $\overline{u}$;
* V is a C^1 vector field.

Main result (II)

Theorem

Under the standing assumptions (II), the second order shape derivative of J at Ω in direction V exists and is a quadratic form in V. Moreover, J'' is of the form

$$J''(\Omega, V) = \int_{\partial\Omega} B(\overline{u}, V) \cdot n \, d\mathscr{H}^{n-1} + q(\overline{u}, V)\,,$$

being B an explicit vector field, and q an explicit non local term depending on the second differentials of f and g computed as $\nabla\overline{u}$ and $\overline{u}$ respectively.

Applications (II)

(Ex 1)

* $p = 2$: the previous theorem applies, and we recover the known formula.
* $p > 2$: due to the degeneracy of $\nabla^2 f(\nabla u_p)$ on the singular set $\{\nabla u_p = 0\}$, we cannot apply the same argument. Nevertheless it can be adapted (see next theorem), by following an approximation procedure.

Let f, g, u_p be as in Ex 1. For brevity we set $\theta := |\nabla u_p|^{p-2}$, and

$$P := \nabla^2 f(\nabla u_p) = \theta \left(I - (p-2) \frac{\nabla u_p}{|\nabla u_p|} \otimes \frac{\nabla u_p}{|\nabla u_p|} \right).$$

Theorem

Let $\partial\Omega$ be C^2. The p-torsion functional is twice differentiable at any direction V, provided that

$$W^{1,2}_\theta(\Omega) := \{ v \in W^{1,1}_{loc}(\Omega) \ : \ v \in L^2(\Omega),\ \nabla v \in L^2_\theta(\Omega; \mathbb{R}^n) \} = \overline{C^1(\overline{\Omega})}.$$

In this case, for V normal to the boundary, we have

$$J''(\Omega, V) = -\frac{1}{p} \int_{\partial\Omega} (V \cdot n)^2 (p \partial_n u_p + |\partial_n u_p|^p H_{\partial\Omega})\, d\mathscr{H}^{n-1} - \inf_{\substack{v \in W^{1,2}_p(\Omega) \\ v = -V \cdot \nabla u_p}} \int_\Omega (P \nabla v) \cdot \nabla v \, dx.$$

Note that condition (C) holds true when $\{\nabla u_p = 0\}$ has vanishing capacity in $\overline{C^1(\overline{\Omega})}$, which happens e.g. when the singular set is a singleton.

Open problems

* extension to convex non-smooth integrands ? Possibly replacing the second order differentials of f and g with... ?
* does the knowledge of J' and J'' for the p-torsion problem entail some useful information on the second order shape derivative of J_∞, limit of the p-torsion functionals as $p \to +\infty$? Note that, up to a multiplicative constant,

$$J_\infty(\Omega) = \int_\Omega d(x, \partial\Omega)\, dx\,, \quad \lim_{p\to+\infty} u_p(x) = d_{\partial\Omega}(x)\,.$$

Thanks for your attention !