Control of 3D micro-swimmers

Benasque 2015

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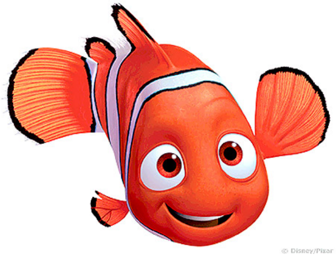
Motivations

Swimming is seen as a control problem.
Given two points in space, can the swimmer go from one point to the other?

The motion of the swimmer is due to fluid-structure interactions.



The fluid

Reynolds number: $Re = \dfrac{\rho U L}{\mu}$

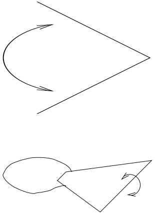
	$L\,(cm)$	$U\,(cm.s^{-1})$	$T\,(s)$	Re
Bacteria	10^{-5}	10^{-3}	10^{-4}	10^{-5}
Spermatozoon	10^{-3}	10^{-2}	10^{-2}	10^{-3}
Fish	50	100	0.5	5.10^{4}
Pigeon	25	10^{3}	5.10^{-1}	10^{5}

The deformations I

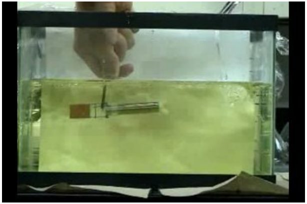
All the deformations are not interesting in order to swim.

Theorem (Scallop theorem, Purcell, 1977)

Given a time periodic deformation described by one physical geometric parameter, the net motion of the swimmer over one period is null.

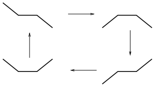


No net motion
$\Rightarrow$
in Stokes fluid

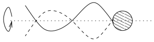


Taylor's experiences

The deformations II



Purcell's swimmer



Helical deformation

Net motion

$\Rightarrow$

in Stokes fluid

Taylor's experiences

State of art

- Swimmer description and modelling:
  - G. Taylor's experiences, 1951
  - Low Reynolds swimmers modelling, E. M. Purcell, 1977, and S. Childress, 1981
  - Foundations of Low Reynolds swimming, A. Shapere and F. Wilczek, 1989
- Controllability results:
  - In perfect fluid, T. Chambrion and A. Munnier, 2010
  - In Stokes fluid, for a n-sphere swimmer, F. Alouges, A. DeSimone and A. Lefebvre, 2009
  - In Stokes fluid, for a ciliated organism, J. San Martin, T. Takahashi and M. Tucsnak, 2007

1. Modelling
2. Low Reynolds number specificities
3. Controllability
4. Conclusion

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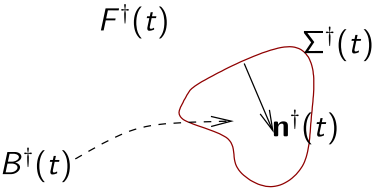
Domains

Let $B^\dagger(t)$ be the domain occupied by the swimmer, $\Sigma^\dagger(t)$ its boundary and $F^\dagger(t) = \mathbb{R}^3 \setminus \overline{B^\dagger(t)}$ the fluid domain.



The fluid

Stokes equations:

$$\begin{array}{rcl} -\Delta \mathbf{u}^\dagger + \nabla p^\dagger & = & 0 \quad \text{in } F^\dagger(t) \\ \operatorname{div} \mathbf{u}^\dagger & = & 0 \quad \text{in } F^\dagger(t) \end{array}$$

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Velocity continuity:

$$\mathbf{u}^\dagger = \mathbf{v}_s \qquad \text{on } \Sigma^\dagger(t)\,,$$

with $\mathbf{v}_s$ is the swimmer velocity.

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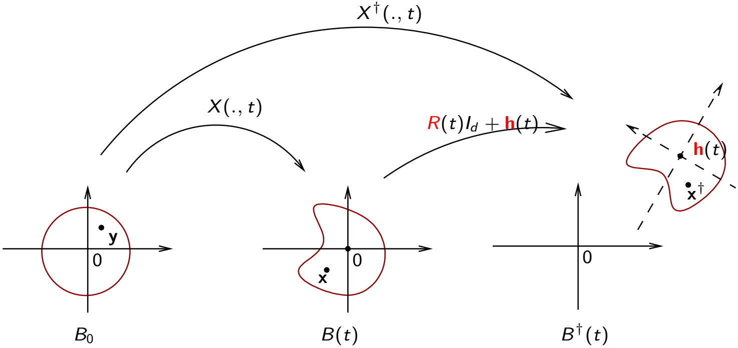
Set $\sigma(\mathbf{u}^\dagger, p^\dagger) = \left(\nabla \mathbf{u}^\dagger + (\nabla \mathbf{u}^\dagger)^T\right) - p^\dagger I_3 \in \mathbb{R}^{3\times 3}$, the Cauchy-stress tensor, the force exerted by the fluid on a part $\mathrm{d}\Gamma$ of $\Sigma^\dagger(t)$ is $\sigma \mathbf{n}^\dagger \, \mathrm{d}\Gamma$.

The swimmer

Deformations

The swimmer is located by:

- its mass center $\mathbf{h} \in \mathbb{R}^3$ and
- its orientation $R \in O^+(3)$.



The swimmer

Velocity of deformation

The velocity of a point $x^\dagger = X^\dagger(y,t) = RX(y,t) + \mathbf{h}$ of $B^\dagger(t)$ is:

$$\mathbf{v}_S = \dot{\mathbf{h}} + R\boldsymbol{\omega} \times (x^\dagger - \mathbf{h}) + R\,\mathbf{w}(x^\dagger, t)\,,$$

with:

- $\mathbf{w}$ the non-rigid deformation velocity of the swimmer,

$$\mathbf{w}(x^\dagger, t) = \dot{X}\left(X(.,t)^{-1}\left(R^T(x^\dagger - \mathbf{h}(t))\right),\ t\right)\ .$$

- $\boldsymbol{\omega}$ the angular velocity of the swimmer in a referential attached to him,

$$\dot{R} = R\hat{\boldsymbol{\omega}}\,,$$

where, $\hat{\boldsymbol{\omega}}$, a 3×3-skew symmetric matrix, is such that $\hat{\boldsymbol{\omega}}x = \boldsymbol{\omega} \times x$.

The swimmer

Deformations constraints

The deformation $X(t)$ shall be:

- a C^1-diffeomorphism of $\mathbb{R}^3$

and shall keep constant:

- the mass

$$\longrightarrow \qquad \rho(\cdot, t) = \frac{1}{\left|\det\left(\mathrm{Jac}X(\cdot, t)\right)\right|}$$

- the mass center position

$$0 = \int_{B(t)} \rho(x, t) x \, \mathrm{d}x$$

- the angular momentum

$$0 = \int_{B(t)} \rho(x, t) x \times \dot{X}\left(X(., t)^{-1}(x),\ t\right) \mathrm{d}x$$

The swimmer

Equations of motion

Newton's principle leads to:

$$\begin{aligned} m\ddot{\mathbf{h}} &= \int_{\Sigma^\dagger(t)} \sigma(\mathbf{u}^\dagger, p^\dagger)\mathbf{n}^\dagger \,\mathrm{d}\Gamma \\ \frac{\mathrm{d}\, J\boldsymbol{\omega}}{\mathrm{d}t} &= \int_{\Sigma^\dagger(t)} (x - \mathbf{h}) \times \sigma(\mathbf{u}^\dagger, p^\dagger)\mathbf{n}^\dagger \,\mathrm{d}\Gamma \end{aligned} \qquad \text{(PFD)}$$

The swimmer

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The coupled problem I

$$\begin{cases} 0 &= \nabla p^\dagger - \Delta \mathbf{u}^\dagger\,, & \text{in } F^\dagger(t) \\ 0 &= \operatorname{div} \mathbf{u}^\dagger\,, & \text{in } F^\dagger(t) \\ & \lim_{|x|\to\infty} \mathbf{u}^\dagger(x) = 0 & \end{cases}$$

$$\mathbf{u}^\dagger = \dot{\mathbf{h}} + R\omega \times (x - \mathbf{h}) + R\mathbf{w}\,, \quad \text{on } \Sigma^\dagger(t)$$

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The coupled problem II

Let us make the change of variables $\mathbf{u}(x) = R^\top \mathbf{u}^\dagger(Rx + \mathbf{h})$, $p(x) = p^\dagger(Rx + \mathbf{h})$,

$$\left\{\begin{array}{rcll} 0 & = & \nabla p - \Delta \mathbf{u}\,, & \text{in } F(t) \\ 0 & = & \operatorname{div} \mathbf{u}\,, & \text{in } F(t) \\ & & \lim_{|\mathbf{x}|\to\infty} \mathbf{u}(x) = 0 & \end{array}\right. \tag{S}$$

$$\mathbf{u} = R^\top \dot{\mathbf{h}} + \boldsymbol{\omega} \times x + \mathbf{w}\,, \quad \text{on } \Sigma(t) \tag{BC}$$

$$\left\{\begin{array}{rcl} 0 & = & \displaystyle\int_{\Sigma(t)} \sigma(\mathbf{u}, p)\mathbf{n}\,\mathrm{d}\Gamma \\ 0 & = & \displaystyle\int_{\Sigma(t)} x \times \sigma(\mathbf{u}, p)\mathbf{n}\,\mathrm{d}\Gamma \end{array}\right. \tag{CM}$$

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Drag and Momentum

Given $(\mathbf{u}, p)$ and $(\mathbf{v}, q)$ two solutions of the homogeneous Stokes problem.
By Green formula,

$$\int_{\Sigma} \sigma(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{v} \, \mathrm{d}\Gamma = 2 \int_{F} \mathrm{D}(\mathbf{u}) : \mathrm{D}(\mathbf{v}) \, \mathrm{d}x \,,$$

with $\mathrm{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\top}\right)$.

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with $\mathrm{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right)$.

Let us then define $(\mathbf{u}_i, p_i) \in W_0^1(F)^3 \times L^2(F)$, the solutions of the homogeneous Stokes problem with boundary condition:

$$\mathbf{u}_i = \begin{cases} \mathbf{e}_i & \text{if } i \in \{1,2,3\}\,, \\ x \times \mathbf{e}_{i-3} & \text{if } i \in \{4,5,6\} \end{cases} \qquad \text{on } \Sigma\,.$$

Then,

$$\begin{pmatrix} \displaystyle\int_\Sigma \sigma(\mathbf{u}, p)\mathbf{n} \, \mathrm{d}\Gamma \\ \displaystyle\int_\Sigma x \times \sigma(\mathbf{u}, p)\mathbf{n} \, \mathrm{d}\Gamma \end{pmatrix} = 2 \left(\int_F \mathrm{D}(\mathbf{u}) : \mathrm{D}(\mathbf{u}_i) \, \mathrm{d}x \right)_{i=1,\dots,6} .$$

An ODE system I

Assume now that X is given by:

$$X(t,x) = D_0(x) + \sum_{i=1}^{n} \mathbf{s}_i(t) D_i(x)\,.$$

Then the geometry of the problem can be only described by the parameter $\mathbf{s} = (\mathbf{s}_1, \cdots, \mathbf{s}_n)$ and the set of deformations $\mathfrak{D} = \big(D_0, (D_1, \ldots, D_n)\big)$. Thus, $X(t, \cdot)$ can be recast as $X_{\mathfrak{D}}(\mathbf{s}) = D_0 + \sum_{i=1}^{n} \mathbf{s}_i D_i$.

The boundary condition (BC) is then:

$$\mathbf{u} = R^\top \dot{\mathbf{h}} + \boldsymbol{\omega} \times x + \sum_{i=1}^{n} \dot{\mathbf{s}}_i D_i \circ X_{\mathfrak{D}}(\mathbf{s})^{-1}\,, \quad \text{on } \Sigma_{\mathfrak{D}}(\mathbf{s})\,.$$

Let us write $(\mathbf{v}_i, q_i)$ the solution of the homogeneous Stokes problem with the boundary condition $\mathbf{v}_i = D_i \circ X_{\mathfrak{D}}(\mathbf{s})^{-1}$ on $\Sigma_{\mathfrak{D}}(\mathbf{s})$.

An ODE system II

Let us define the matrices:

$$M_{\mathfrak{D}}(\mathbf{s}) = 2\left(\int_{F_{\mathfrak{D}}(\mathbf{s})} \mathrm{D}(\mathbf{u}_i) : \mathrm{D}(\mathbf{u}_j)\,\mathrm{d}x\right)_{i,j=1,\dots,6} \in M_6(\mathbb{R})$$

$$\text{and} \quad N_{\mathfrak{D}}(\mathbf{s}) = 2\left(\int_{F_{\mathfrak{D}}(\mathbf{s})} \mathrm{D}(\mathbf{u}_i) : \mathrm{D}(\mathbf{v}_j)\,\mathrm{d}x\right)_{\substack{i=1,\dots,6\\ j=1,\dots,n}} \in M_{6,n}(\mathbb{R})\,.$$

Using the linearity of the homogeneous Stokes problem with respect to the boundary condition, (CM) is:

$$M_{\mathfrak{D}}(\mathbf{s})\begin{pmatrix} R^\top \dot{\mathbf{h}} \\ \omega \end{pmatrix} = N_{\mathfrak{D}}(\mathbf{s})\dot{\mathbf{s}}\,.$$

An ODE system III

And hence, the full coupled system (S)-(BC)-(CM) can be written as:

$$\dot{\mathbf{h}} = R\boldsymbol{\ell} \tag{1a}$$

$$\dot{R} = R\hat{\boldsymbol{\omega}} \tag{1b}$$

$$\dot{\mathbf{s}} = \boldsymbol{\lambda} \tag{1c}$$

$$\begin{pmatrix} \boldsymbol{\ell} \\ \boldsymbol{\omega} \end{pmatrix} = M_{\mathfrak{D}}(\mathbf{s})^{-1} N_{\mathfrak{D}}(\mathbf{s}) \boldsymbol{\lambda} \tag{1d}$$

This fits the form of geometric control problems, with control variable $\boldsymbol{\lambda} \in \mathbb{R}^n$ and state variable $(\mathbf{h}, R, \mathbf{s}) \in \mathbb{R}^3 \times O^+(3) \times \mathbb{R}^n$,

$$(\dot{\mathbf{h}}, \dot{R}, \dot{\mathbf{s}}) = \sum_{i=1}^{n} f_i(R, \mathbf{s}) \lambda_i \,.$$

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A controllability result, Chow Theorem

On a manifold $\mathcal{M}$, we consider the dynamical system:

$$\dot{z} = \sum_{i=1}^{n} f_i(z) u_i \,, \tag{2}$$

We associate to this system the Lie algebra $\mathrm{Lie}\{f_1, \dots, f_n\}$ which is the smallest algebra stable for the Lie bracket:

$$\begin{array}{rcl} [f,g] \,:\, \mathcal{M} & \to & \mathrm{T}\mathcal{M} \\ z & \mapsto & \mathrm{D}_z g \cdot f(z) - \mathrm{D}_z f \cdot g(z) \,. \end{array}$$

Theorem (Chow)

If for every $z_0 \in \mathcal{M}$ we have $\dim \mathrm{Lie}_{z_0}\{f_1, \dots, f_m\} = \dim \mathrm{T}_{z_0}\mathcal{M}$, then the system is controllable.

Corollary

For any trajectory $\bar{z} : [0, T] \to \mathcal{M}$ and any $\varepsilon > 0$, there exists a control u such that the solution z of (2) with $z(0) = \bar{z}(0)$ satisfies:

$$\sup_{t \in [0,T]} |z(t) - \bar{z}(t)| \leqslant \varepsilon \,.$$

Self-propelling conditions

For $X_{\mathfrak{D}}(\mathbf{s}) = D_0 + \sum \mathbf{s}_i D_i$, the conditions:

$$\int_B D \,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_B D \times D' \,\mathrm{d}x = 0 \qquad (D, D' \in \{D_0, \dots, D_n\}),$$

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ensure the self-propelling conditions.

We define $\mathcal{C}(n)$ the set of $\mathfrak{D} = \big(D_0, (D_1, \dots, D_n)\big) \in \mathcal{D}_0^1(\mathbb{R}^3) \times C_0^1(\mathbb{R}^3)^n$ satisfying those conditions.
And for $\mathfrak{D} \in \mathcal{C}(n)$, given, we set $\mathcal{S}(\mathfrak{D})$ the connected component of $\left\{ \mathbf{s} \in \mathbb{R}^n ,\; D_0 + \sum_{i=1}^{n} \mathbf{s_i} D_i \in \mathcal{D}_0^1(\mathbb{R}^3) \right\}$ containing 0.
Finally, we define:

$$\mathfrak{S}(n) = \{(\mathfrak{D}, \mathbf{s}) ,\; \mathfrak{D} \in \mathcal{C}(n) ,\; \mathbf{s} \in \mathcal{S}(\mathfrak{D})\} \; .$$

Lemma

$\mathfrak{S}(n)$ is a connected and analytic sub-manifold of $C_0^1(\mathbb{R}^3) \times C_0^1(\mathbb{R}^3)^n \times \mathbb{R}^n$.

Analyticity of $M_{\mathfrak{D}}$ and $N_{\mathfrak{D}}$

Lemma

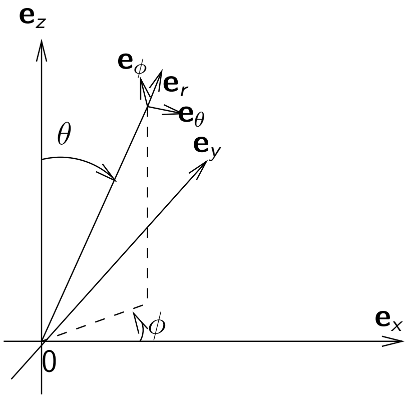
The maps

$$\begin{array}{rcl} \mathfrak{S}(n) & \to & M_6(\mathbb{R}) \\ (\mathfrak{D}, \mathbf{s}) & \mapsto & M_{\mathfrak{D}}(\mathbf{s}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathfrak{S}(n) & \to & M_{6,n}(\mathbb{R}) \\ (\mathfrak{D}, \mathbf{s}) & \mapsto & N_{\mathfrak{D}}(\mathbf{s}) \end{array}$$

are analytic.

Stokes solution in exterior domains I

We use spherical coordinates,



Stokes solution in exterior domains II

Solutions of the homogeneous Stokes system take the form (c.f. Lamb, 1993):

$$\mathbf{u} = \sum_{n=0}^{\infty} \Big(\mathrm{rot}(\chi_{-(n+1)} r\mathbf{e}_r) + \nabla \varphi_{-(n+1)} - \frac{n-2}{2n(2n-1)} r^2 \nabla \pi_{-(n+1)} + \frac{n+1}{n(2n-1)} \pi_{-(n+1)} r\mathbf{e}_r \Big) \,,$$

$$p = \sum_{n=0}^{\infty} \pi_{-(n+1)} \,,$$

with $\pi_{-(n+1)}$, $\chi_{-(n+1)}$ and $\varphi_{-(n+1)}$ rigid spherical harmonics,

$$(r, \theta, \phi) \mapsto r^{-(n+1)} \sum_{m=-n}^{n} \gamma_m Y_{n,m}(\cos\theta, \phi) \,.$$

We have:

$$\int_\Sigma \sigma(\mathbf{u}, p)\mathbf{n} \,\mathrm{d}\Gamma = -4\pi \nabla(r^3 \pi_{-2}) \qquad \text{and} \qquad \int_\Sigma x \times \sigma(\mathbf{u}, p)\mathbf{n} \,\mathrm{d}\Gamma = -8\pi \nabla(r^3 \chi_{-2}) \,.$$

Lie algebra evaluated at a particular point

Lemma

The dimension of the Lie Algebra at the point $(\mathbf{h}, R, \mathbf{s})$ *is independent of* $\mathbf{h}$ *and* R.

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Let us chose the deformations $\mathfrak{D} = \big(\mathrm{Id}, (D_1, \dots, D_4)\big)$,

$$\begin{aligned} D_1(r,\theta,phi) &= r^{-4}\Re Y_{3,1}(\cos\theta,\phi)\mathbf{e}_r\,, \\ D_2(r,\theta,phi) &= r^{-4}\Im Y_{3,1}(\cos\theta,\phi)\mathbf{e}_r\,, \\ D_3(r,\theta,phi) &= r^{-4}\Re Y_{3,2}(\cos\theta,\phi)\mathbf{e}_r\,, \\ D_4(r,\theta,phi) &= r^{-5}\Re Y_{4,2}(\cos\theta,\phi)\mathbf{e}_r\,. \end{aligned}$$

and compute the evaluation of the Lie algebra at point $\mathbf{s} = 0 \in \mathbb{R}^4$, $R = I_3$ and $\mathbf{h} = 0$.

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Using `maxima`, we obtain that the Lie algebra evaluated at this point is of dimension $10 = 3 + 3 + 4$.

Result

Using analyticity together with Chow theorem,

Theorem

Set $\bar{D}_0 \in D_0^1(\mathbb{R}^3)$ *such that* $\int_B \bar{D}_0 \,\mathrm{d}x = 0$ *and set an absolutely continuous function* $t \in [0, T] \to (\bar{\mathbf{h}}(t), \bar{R}(t)) \in \mathbb{R}^3 \times S0(3)$.
Then for every $\varepsilon > 0$, *there exists* $D_0 \in \mathcal{D}_0^1(\mathbb{R}^3)$ *such that:*

1. $\|\bar{D}_0 - D_0\|_{C_0^1(\mathbb{R}^3)^3} \leqslant \varepsilon$;
2. *for almost every* $(D_0, (D_1, \dots, D_4)) \in \mathcal{C}(4)$, *there exists a function:* $t \in [0, T] \mapsto \mathbf{s}(t) \in \mathbb{R}^4$ *such that the solution* $(\mathbf{h}, R)$ *of the dynamical system satisfies:*

$$\sup_{t\in[0,T]} \left(\|\bar{R}(t) - R(t)\|_{M_3(\mathbb{R})} + \|\bar{\mathbf{h}}(t) - \mathbf{h}(t)\|_{\mathbb{R}^3}\right) \leqslant \varepsilon\,.$$

Remark

It is also possible approximatively follow a prescribed non rigid deformation, $t \in [0, T] \mapsto \bar{X}(t, \cdot) \in \mathcal{D}_0^1(\mathbb{R}^3)$.

Conclusion

- What is the minimal number of controls?
- Swimming in a bounded domain? (work in progress with T. Takahashi)
- Collective swimming?
- Controllability in the presence of inertia?

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Thank you for your attention.