Compressible fluid and its interaction with a structure

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The compressible Navier-Stokes equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \text{ in } (0, T) \times \Omega \\ \rho \dfrac{\partial u}{\partial t} + \rho (u \cdot \nabla) u - \nabla \cdot \sigma_F(u, \rho) = 0 \text{ in } (0, T) \times \Omega \\ u = 0 \text{ on } (0, T) \times \partial\Omega \end{cases}$$

ρ: density, u: eulerian velocity, $\sigma_F(u, \rho)$: Cauchy stress tensor

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ρ: density, u: eulerian velocity, $\sigma_F(u, \rho)$: Cauchy stress tensor

$$\sigma_F(u, \rho) = 2\mu\epsilon(u) + \mu'(\nabla \cdot u) Id - (P(\rho) - P(\overline{\rho}))\,\mathrm{Id}$$

$P \in C^\infty(\mathbb{R}_+^*)$ and $\overline{\rho} > 0$.

Existence and regularity results

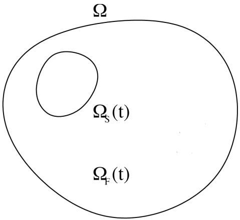
- Local in time existence of regular solution for general pressure law
  [Tani (1977)]
- Global in time existence of weak solution with small data for isentropic fluid:
  $$P(\rho) = \rho^{\gamma}$$
  with $\gamma \geq 1$
  [Hoff (1995)]
- Global in time existence of weak solution for large data
  [Lions (1993)] ($\gamma \geq 9/5$), [Feireisl (2001)] ($\gamma > 3/2$)
  $$\rho \in L^{\infty}(0, T; L^{\gamma}(\Omega)), \ \sqrt{\rho} u \in L^{\infty}(0, T; L^2(\Omega)), \ u \in L^2(0, T; H^1(\Omega)).$$
- Global in time existence of smooth solution for small data
  [Matsumura, Nishida (1982)]
  $$\rho \in C(0, T; H^3(\Omega)) \cap C^1(0, T; H^2(\Omega)), \ u \in C(0, T; H^3(\Omega)) \cap C^1(0, T; H^1(\Omega)).$$

Interaction between a compressible fluid and a rigid structure

- fluid and structure contained in $\Omega \subset \mathbb{R}^3$ a fixed bounded and connected set
- rigid structure immersed in the compressible fluid
- structure motion given by

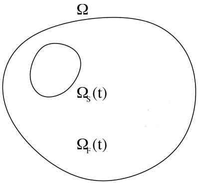
  a the translation vector
  Q the rotation matrix or ω the rotation velocity vector

$$\dot{Q}(t)Q(t)^T y = \omega(t) \wedge y \text{ for all } y \in \mathbb{R}^3.$$

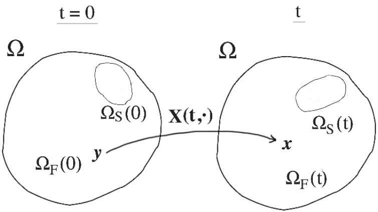


The modelling

$$
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho u) = 0 & \text{in } \Omega_F(t) \\
\rho \partial_t u + \rho (u \cdot \nabla) u - \nabla \cdot \sigma_F(u, \rho) = 0 & \text{in } \Omega_F(t) \\
u = 0 & \text{on } \partial\Omega \\
u(t,x) = \dot{a}(t) + \omega(t) \wedge (x - a(t)) & \text{on } \partial\Omega_S(t) \\
m\ddot{a} = \displaystyle\int_{\partial\Omega_S(t)} \sigma_F(u,p)\, n\, d\gamma & \\
J\dot{\omega} = (J\omega) \wedge \omega + \displaystyle\int_{\partial\Omega_S(t)} (x-a) \wedge \big(\sigma_F(u,p)\, n\big)\, d\gamma & \\
\dot{a}(0) = a_0,\ \omega(0) = \omega_0,\ \rho(0,\cdot) = \rho_0 \text{ in } \Omega_F(0),\ u(0,\cdot) = u_0 \text{ in } \Omega_F(0). &
\end{cases}
$$



The flow



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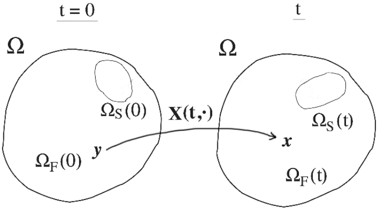


We need a change of variables to come back to the fixed domains.

The position at time t of a point located at $y \in \Omega_S(0)$ at time 0 is given by

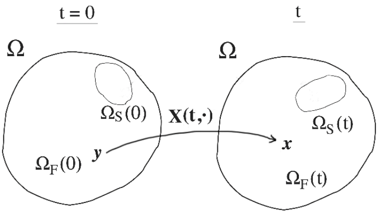
$$X(t, y) = a(t) + Q(t)y$$

This flow is extended on the whole domain Ω so that

- it is regular and invertible, t being fixed,
- the boundary of Ω is unchanged.

We still denote by X this extension.

The flow



Definition of the spaces on the moving domains

$$H^r_T(H^p) = \{u/u \circ X \in H^r(0,T;H^p(\Omega_F(0)))\}$$

$$C^r_T(H^p) = \{u/u \circ X \in C^r(0,T;H^p(\Omega_F(0)))\}$$

Existence and regularity results

- Global in time existence of weak solution [Feireisl (2003)]

$$\rho \in L^\infty_T(L^\gamma),\ \sqrt{\rho} u \in L^\infty_T(L^2),\ u \in L^2_T(H^1),$$
$$\dot{a} \in L^\infty(0,T),\ \omega \in L^\infty(0,T).$$

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- Global in time existence of regular solution for small data [M.B., S. Guerrero (2009)]
  There exists $\delta > 0$ such that if

$$\|\rho_0 - \overline{\rho}\|_{H^3(\Omega_F(0))} + \|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |\omega_0| < \delta,$$

we have a unique solution defined on $(0,T)$ in the space:

$$\rho \in C^0_T(H^3) \cap C^1_T(H^2) \cap H^2_T(L^2),$$
$$u \in L^2_T(H^4) \cap C^0_T(H^3) \cap C^1_T(H^1) \cap H^2_T(L^2),$$
$$\dot{a} \in H^2(0,T),\ \omega \in H^2(0,T).$$

Moreover, either $T = +\infty$ or $\lim_{t\to T} d(\partial\Omega, \overline{\Omega}_S(t)) = 0$.

Sketch of the proof for the global in time existence of regular solution for small data

[M.B., S. Guerrero (2009)]

Two main steps:

- Local existence of solution

  Change of variables to come back to fixed domains

  Linearization of the problem

  Fixed-point argument

- Global estimate for a solution of the problem

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$$\begin{cases} \partial_t \tilde{\rho} + ((\nabla X)^{-1}(\tilde{u} - \partial_t X)) \cdot \nabla \tilde{\rho} + \overline{\rho}\, \nabla \cdot \tilde{u} = g_0(\tilde{\rho}, \tilde{u}, a, \omega) & \text{in } \Omega_F(0), \\ \partial_t \tilde{u} - \nabla \cdot (2\mu\epsilon(\tilde{u}) + \mu'(\nabla \cdot \tilde{u}) Id) + \frac{P'(\overline{\rho})}{\overline{\rho}} \nabla \tilde{\rho} = g_1(\tilde{\rho}, \tilde{u}, a, \omega) & \text{in } \Omega_F(0), \\ m\ddot{a} = \int_{\partial\Omega_S(0)} \sigma(\tilde{u}, P(\tilde{\rho})) n \, d\gamma + g_2(\tilde{\rho}, \tilde{u}, a, \omega) & \\ J\dot{\omega} = \int_{\partial\Omega_S(0)} (Qy) \wedge (\sigma(\tilde{u}, P(\tilde{\rho}))n) \, d\gamma + g_3(\tilde{\rho}, \tilde{u}, a, \omega) & \\ \tilde{u} = 0 & \text{on } \partial\Omega, \\ \tilde{u} = \dot{a} + \omega \wedge (Qy) & \text{on } \partial\Omega_S(0) \end{cases}$$

Linearization of the problem

Fixed-point argument

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$$\begin{cases} \partial_t \tilde{\rho} + ((\nabla \hat{X})^{-1}(\hat{u} - \partial_t \hat{X})) \cdot \nabla \tilde{\rho} + \overline{\rho}\, \nabla \cdot \tilde{u} = g_0(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) & \text{in } \Omega_F(0), \\ \partial_t \tilde{u} - \nabla \cdot (2\mu\epsilon(\tilde{u}) + \mu'(\nabla \cdot \tilde{u}) Id) + \frac{P'(\overline{\rho})}{\overline{\rho}} \nabla \tilde{\rho} = g_1(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) & \text{in } \Omega_F(0), \\ m\ddot{a} = \int_{\partial\Omega_S(0)} \sigma(\tilde{u}, P(\tilde{\rho}))n \, d\gamma + g_2(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) & \\ \hat{J}\dot{\omega} = \int_{\partial\Omega_S(0)} (\hat{Q}y) \wedge (\sigma(\tilde{u}, P(\tilde{\rho}))n) \, d\gamma + g_3(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) & \\ \tilde{u} = 0 & \text{on } \partial\Omega, \\ \tilde{u} = \dot{a} + \omega \wedge (\hat{Q}y) & \text{on } \partial\Omega_S(0) \end{cases}$$

Fixed-point argument

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- Global estimate for a solution of the problem

  [Matsumura, Nishida (1982)]

Bibliography on incompressible fluid-rigid structure interaction

- Weak solution

  [Desjardins, Esteban (1999)]
  [Conca, San Martin, Tucsnak (2000)]
  [Gunzburger, Lee, Seregin (2000)]...

- Strong solution

  [Takahashi (2003)], [Cumsille, Takahashi (2008)]...

Interaction between a compressible fluid and an elastic structure

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 & \text{in } \Omega_F(t) \\ \rho \partial_t u + \rho (u \cdot \nabla) u - \nabla \cdot \sigma_F(u, \rho) = 0 & \text{in } \Omega_F(t) \\ u = 0 & \text{on } \partial\Omega \\ \partial_{tt} \xi - \nabla \cdot \Sigma_S(\xi) = 0 & \text{in } \Omega_S(0) \\ u \circ X = \partial_t \xi & \text{on } \partial\Omega_S(0) \\ \sigma_F(u, \rho) \circ X \, \mathrm{cof} \nabla X \, \mathrm{n} = \Sigma_S(\xi) \, \mathrm{n} & \text{on } \partial\Omega_S(0) \end{cases}$$

ξ: elastic displacement, $\Sigma_S(\xi) = 2\lambda \epsilon(\xi) + \lambda' (\nabla \cdot \xi) Id$

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Definition of the flow

$$X(t, y) = y + \xi(t, y), \forall\, y \in \Omega_S(0)$$

which can be extended to Ω.

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Initial conditions

$$\rho(0, \cdot) = \rho_0 \text{ in } \Omega_F(0),\ u(0, \cdot) = u_0 \text{ in } \Omega_F(0),\ \xi(0, \cdot) = 0 \text{ in } \Omega_S(0),\ \partial_t \xi(0, \cdot) = \xi_1 \text{ in } \Omega_S(0).$$

A priori energy estimate

Energy-level solution:

$$\rho \in L^\infty_T(L^\gamma),\ \sqrt{\rho}\, u \in L^\infty_T(L^2),\ u \in L^2_T(H^1),$$

$$\xi \in W^{1,\infty}(0,T; L^2(\Omega_S(0))) \cap L^\infty(0,T; H^1(\Omega_S(0))).$$

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This regularity is insufficient for three main reasons:

- The set $\Omega_S(t) = (Id + \xi)(\Omega_S(0))$ is not Lipschitz.
- The flow $X(t, \cdot) = Id + \xi(t, \cdot)$ is a priori not invertible.
- We can have instantaneously collision, interpenetration and loss of orientation.

Remedies

- Add of a regularizing term in the elastic equation

  $\sqrt{\rho}\, u \in L^\infty_T(L^2)$, $u \in L^2_T(H^1)$,

  $\xi \in W^{1,\infty}(0,T;L^2(\Omega_S(0))) \cap L^\infty(0,T;H^1(\Omega_S(0)))$ energy-level solution

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  $\xi \in H^1(0, T; H^3(\Omega_S(0)))$

  [M.B. (2005)]

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- Obtaining smooth solution

  The solution will be defined locally in time.

Existence and uniqueness of smooth solution

[M.B., S. Guerrero (2010)]

Hypotheses:

- $\mathrm{dist}(\Omega_S(0), \partial\Omega) > 0$, $\Omega_S(0)$ and Ω regular
- $\rho_0 \in H^3(\Omega_F(0))$, $\rho_0 \geq \rho_{\min} > 0$ in $\Omega_F(0)$, $u_0 \in H^4(\Omega_F(0))$, $\xi_1 \in H^2(\Omega_S(0))$
- compatibility conditions.

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  $\xi_1 \in H^2(\Omega_S(0))$
- compatibility conditions.

Result: There exists $T^* > 0$ such that our problem admits a unique solution (ρ, u, ξ) defined on $(0, T^*)$ and belonging to:

$$\rho \in L^2_{T^*}(H^2) \cap H^2_{T^*}(L^2)$$

$$u \in L^2_{T^*}(H^3) \cap C^0_{T^*}(H^{11/4}) \cap C^2_{T^*}(L^2)$$

$$\xi \in C^0([0, T^*]; H^3(\Omega_S(0))) \cap C^3([0, T^*]; L^2(\Omega_S(0)))$$

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$$\xi \in C^0([0, T^*]; H^3(\Omega_S(0))) \cap C^3([0, T^*]; L^2(\Omega_S(0)))$$

Remarks:

- We need more regularity on the data than for a compressible fluid alone
- Loss of regularity due to the coupling
- Later result with less regular data [Kukavica, Tuffaha (2012)]

Bibliography on incompressible fluid-elastic structure interaction

- Regularization of the elasticity equation

  [Desjardins, Esteban, Grandmont, Le Tallec (2001)]
  [Beirao da Veiga (2004)]
  [Chambolle, Desjardins, Esteban, Grandmont (2005)]
  [M.B. (2007)], [M.B., E. Schwindt, T. Takahashi (2012)]

- Smooth solution for the original equations

  [Coutand, Shkoller (2005, 2006)], [Kukavica, Tuffaha (2012)],
  [Raymond, Vanninathan (2014)]

Open questions

- Global in time smooth solution (even with small data)
- Less regular solutions
    - for a compressible fluid alone
    - in another framework
- Use of a global eulerian framework ?