Semilinear structurally damped wave equations: supercritical case

Well-posedness and existence of finite number of determining Fourier modes

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We consider the initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u + 2\beta(-\Delta)^{\alpha} u_t + f(u) = 0, \quad x \in \Omega, \ t > 0, \\ u|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1}$$

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where $\alpha \in [0,1]$, $\beta > 0$ are given numbers, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth enough boundary $\partial\Omega$.

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where $\alpha \in [0,1]$, $\beta > 0$ are given numbers, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth enough boundary $\partial\Omega$.

$$(-\Delta)^{\alpha} e_i = \lambda_i e_i, \quad i \geq 1,$$

where $\{\lambda_i\}_{i\geq 1}$, $\{e_i\}_{i\geq 1}$ are the eigenvalues and the eigenvectors of the Laplacian operator $-\Delta$ under Dirichlet boundary conditions.

The roots of the characteristic polynomial for the linear problem are:

$$\gamma_n^{\pm} := -\beta\lambda_n^{\alpha} \pm \left(\beta^2\lambda_n^{2\alpha} - \lambda_n\right)^{\frac{1}{2}} < 0, \quad n \geq 1.$$

The linear problem - The spectrum

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$$\gamma_n^{+}\gamma_n^{-} = \lambda_n,$$

$$\gamma_n^{+} = -\frac{\lambda_n}{\beta\lambda_n^{\alpha} + (\beta^2\lambda_n^{2\alpha} - \lambda_n)^{\frac{1}{2}}}.$$

The linear problem - Controllability properties

Gets worse as $\alpha \nearrow 1$. In fact, when $\alpha = 1$

$$\gamma_n \to \frac{1}{2\beta}, \quad \text{as} \quad n \nearrow \infty.$$

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Another fact we see for $\alpha \in [\frac{1}{2}, 1]$ and large n:

$$\gamma_n^+ \leq C\lambda_n^{1-\alpha} \leq C\lambda_n^{\frac{1}{2}}.$$

This and Weyl's theorem imply

$$\sum \frac{1}{\gamma_n^+} = \infty.$$

$\Rightarrow$ No spectral controllability via a scalar control through a profile (Müntz theorem).

The semilinear problem in $\mathbb{R}^3$ - Classical results

$$f(s) = |s|^q s - \lambda s, \quad \forall s \in \mathbb{R}.$$

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Uniqueness:
- if $q \in (0, 4]$ and $\alpha \in [0, \frac{1}{2}]$. (A. Savostianov, Adv.D.E., 2014)

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Uniqueness:
- if $q \in (0, 4]$ and $\alpha \in [0, \frac{1}{2}]$. (A. Savostianov, Adv.D.E., 2014)
- $q \in (0, \frac{8\alpha}{3-4\alpha}]$ if $\alpha \in [\frac{1}{2}, \frac{3}{4})$, and $q > 0$ if $\alpha \in [\frac{3}{4}, 1]$. (V. Kalantarov, S. Zelik, J.D.E., 2009)

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But, as a consequence

$$(u, u_t) \in L^\infty(0, T; \mathcal{E}),$$

that is the continuous evolution in the phase space $\mathcal{E}$ is not known for the supercritical exponents.

Finitely many determining modes

Definition

We say that the problem (1) has *finitely many determining modes in $\mathcal{E}$*, if for any two solutions u, v, that satisfy

$$\|P_N w(t)\|_{\mathcal{E}} \to 0 \text{ as } t \to +\infty,$$

we have

$$\|w(t)\|_{\mathcal{E}} \to 0 \text{ as } t \to +\infty,$$

where $w = u - v$, and P_N is the projection operator on to the space spanned by the first N eigenfunctions of the Laplacian under Dirichlet b.c.'s.

Finitely many weakly determining modes

Theorem

Suppose that $q \in (0, \frac{8\alpha}{3-4\alpha}]$ *if* $\alpha \in [\frac{1}{2}, \frac{3}{4})$, *and* $q > 0$ *if* $\alpha \in [\frac{3}{4}, 1]$. *Then, for any two solutions* u, v, *that satisfy*

$$\|P_N w(t)\|_{\mathcal{E}} \to 0 \text{ as } t \to +\infty,$$

we have

$$\|w(t)\|_{\mathcal{E}_\alpha} \to 0 \text{ as } t \to +\infty,$$

and

$$\int_t^{t+1} \|w(\tau)\|_{\mathcal{E}} d\tau \to 0 \text{ as } t \to +\infty.$$

Thank You!