Nonlocal Conservation/Balance Laws

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The System of Nonlocal Conservation Laws

$$\begin{aligned} \dot{\rho}(t,x) &= -\boldsymbol{\lambda}(W(t,\rho)) \odot \rho_x(t,x) \\ \rho(0,x) &= \rho_0(x) \\ \boldsymbol{\lambda}(W(t,\rho)) \odot \rho(t,0) &= \boldsymbol{a}(t) \end{aligned}$$

with $(t,x) \in (0,T) \times (0,1)$ and $T \in \mathbb{R}_{>0}$, complemented by the WIP (Work in Progress)

$$W(t,\rho) := \sum_{i=1}^{N} \int_0^1 \rho_i(t,s)\,\mathrm{d}s$$

- $N > 1$: **Multi-Commodity Model**,
- ρ_0, $\boldsymbol{a}$ initial and boundary data, given,
- $\boldsymbol{\lambda} \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R}^N_{>0})$,
- $\odot$ “component wise” multiplication.

Comments

- For “single commodity” ($N = 1$) the proposed dynamics was introduced by

  Armbruster, Marthaler, Ringhofer, Kempf, Tae-Chang Jo. A continuum model for a re-entrant factory. *Operations Research*, 54(5):933–950.

- Amongst others studied by

  Coron, Kawski, Wang. (2010). Analysis of a Conservation Law Modeling a Highly Re-entrant Manufacturing System.*Discrete Contin. Dyn. Syst. Ser. B, 14(4):1337–1359.*

- Generalization to Multi-Commodity flow (Coupled System of Conservation Laws).

Example for two commodities

$$\boldsymbol{a}(\cdot) \longrightarrow \boldsymbol{y}(\cdot)$$

Let $T = 2$ and

$$\begin{aligned} \dot{\rho}^1(t,x) &= -\boldsymbol{\lambda}^1(W(t,\rho))\rho^1_x(t,x) & \dot{\rho}^2(t,x) &= -\boldsymbol{\lambda}^2(W(t,\rho))\rho^2_x(t,x) \\ \rho^1_0(x) &= 0 & \rho^2_0(x) &= 0 \\ \boldsymbol{a}^1(t) &= 1 & \boldsymbol{a}^2(t) &= 2 \\ \boldsymbol{\lambda}^1(W) &= \tfrac{1}{1+W} & \boldsymbol{\lambda}^2(W) &= 1 + W \end{aligned}$$

$$W(t,\rho) = \sum_{i=1}^{2} \int_0^1 \rho^i(t,s)\,\mathrm{d}s = \int_0^1 \rho^1(t,s) + \rho^2(t,s)\,\mathrm{d}s$$

be given. Then we obtain as solution

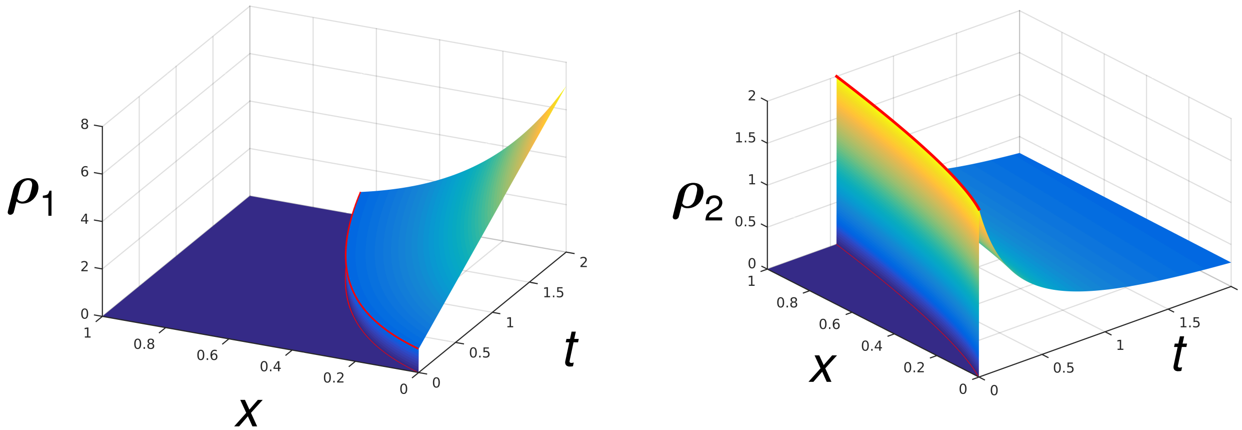


Figure: $\lambda^1(W) = \frac{1}{1+W}, \boldsymbol{a}^1 \equiv 1, \qquad \lambda^2(W) = 1 + W, \boldsymbol{a}^2 \equiv 2$

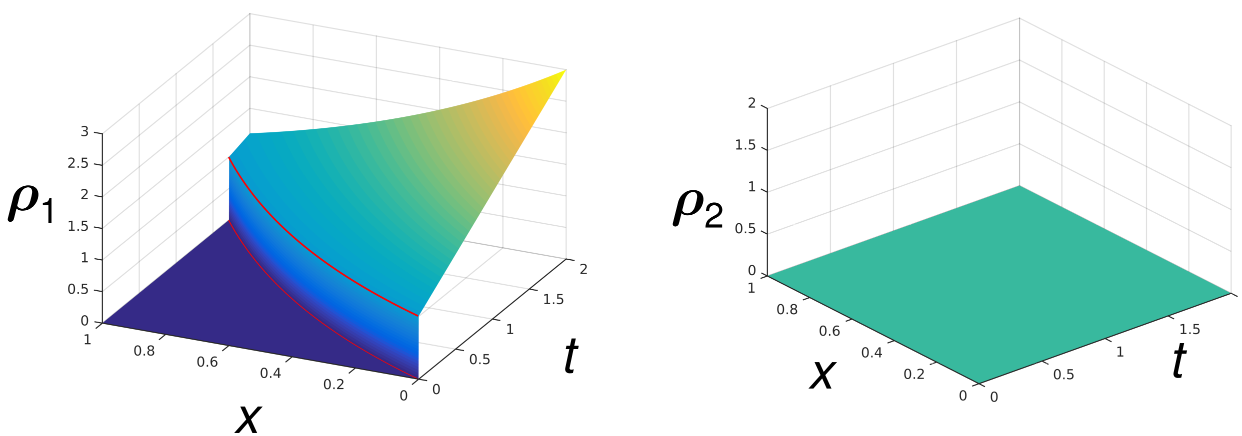


Figure: $\lambda^1(W) = \frac{1}{1+W}, \boldsymbol{a}^1 \equiv 1, \qquad \lambda^2(W) = 1 + W, \boldsymbol{a}^2 \equiv 0$

Existence, weak solutions

Theorem: Existence and Uniqueness of a weak solution

Let $N \in \mathbb{N}$ (Number of Commodities), $p \in [1, \infty)$ and $\boldsymbol{\rho}_0 \in L^p((0,1); \mathbb{R}^N_{\geq 0})$, $\boldsymbol{a} \in L^p((0,T); \mathbb{R}^N_{\geq 0})$ be given. Then, there exists a unique weak and nonnegative solution $\boldsymbol{\rho}$ with

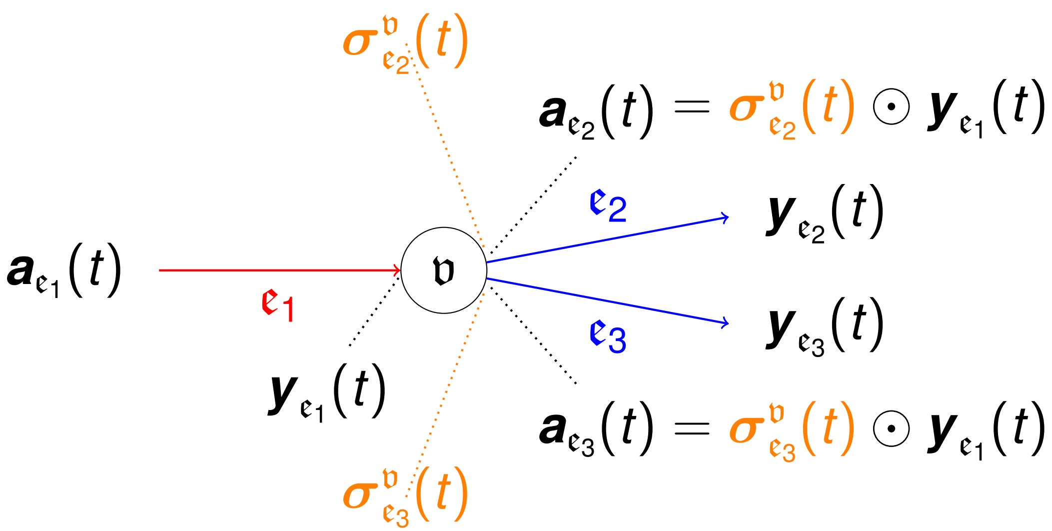
$$\boldsymbol{\rho} \in C([0,T]; L^p((0,1); \mathbb{R}^N)) \cap C([0,1]; L^p((0,T); \mathbb{R}^N)).$$

□

Main idea of the proof: Method of Characteristics $\Longrightarrow$ Fixed-point equation in the characteristics due to the nonlocalness of the system, Construction of the solution via the characteristics for sufficiently small time, Extension to arbitrary time.

Simply generalizable to BV data and $W^{1,p}$ data.

Distributing edges and functions on the network



Conservation of Mass at node $\mathfrak{v}$:

$$\mathbf{0} \leq \boldsymbol{\sigma}^{\mathfrak{v}}_{\mathfrak{e}_2}(t), \quad \mathbf{0} \leq \boldsymbol{\sigma}^{\mathfrak{v}}_{\mathfrak{e}_3}(t), \quad \mathbf{1} = \boldsymbol{\sigma}^{\mathfrak{v}}_{\mathfrak{e}_2}(t) + \boldsymbol{\sigma}^{\mathfrak{v}}_{\mathfrak{e}_3}(t), \qquad t \in [0, T] \text{ a. e.}$$

We call the set of distributing functions Σ.

Some Comments and Results

Wellposedness and uniqueness of a solution on the whole network (directed acyclic network) for given

- $\boldsymbol{\sigma} \in \Sigma$ and for initial and inflow data of L^p regularity ($p \in [1, \infty)$),
- $\boldsymbol{\sigma} \in \Sigma \cap BV$ and for initial and inflow data of BV regularity,
- $\lightning$ for $W^{1,p}$ data.

Optimal Control, Existence

$$J(\boldsymbol{a},\boldsymbol{\sigma},\boldsymbol{y}) := \|\boldsymbol{a}-\boldsymbol{a}_{\mathrm{d}}\|_{L^2}^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{\mathrm{d}}\|_{L^2}^2 + \|\boldsymbol{y}-\boldsymbol{y}_{\mathrm{d}}\|_{L^2}^2$$

Theorem: Existence of a minimizers

For any $K \in \mathbb{R}_{>0}$ there exists $\boldsymbol{a}^* \in L^2$ and $\boldsymbol{\sigma}^* \in \Sigma \cap BV_K$ with $BV_K := \{f \in BV : \|f\|_{BV} \leq K\}$:

$$\inf_{\substack{\boldsymbol{a}\in L^2\\ \boldsymbol{\sigma}\in\Sigma\cap BV_K}} J(\boldsymbol{a},\boldsymbol{\sigma},\boldsymbol{y}) = J(\boldsymbol{a}^*,\boldsymbol{\sigma}^*,\boldsymbol{y}^*)$$

□

For the named results (and more) see

Gugat, K. ,Leugering, Wang. Analysis of a system of nonlocal conservation laws for multi-commodity flow on networks (to appear in NHM).

For the optimality conditions on the network see

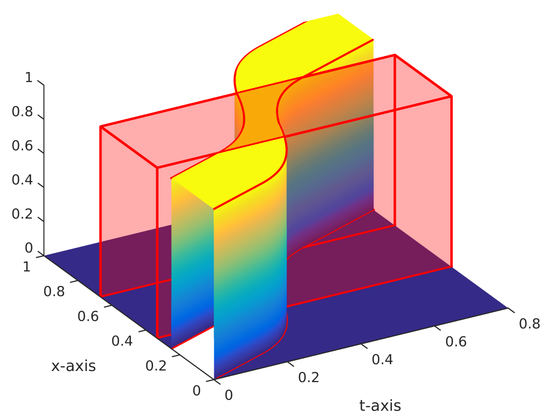
K. , Leugering, Gröschel, Wang. Regularity Theory and Adjoint-Based Optimality Conditions for a Nonlinear Transport Equation with Nonlocal Velocity, 2014 (SICON)

Work in Progress, Generalizations

- Generalizing WIP to a Weighted WIP (WWIP),

$$W(t, \boldsymbol{\rho}, \mathbf{f}) := \sum_{i=1}^{N} \int_0^1 \rho_i(t, s)\mathbf{f}_i(t, s)\,\mathrm{d}s$$

with **f** of certain regularity.



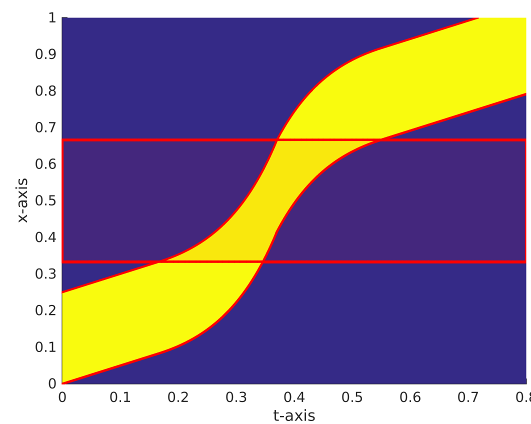


Figure: $\lambda(W) = \frac{1}{2} + 10W,\ \rho_0(x) = \chi_{[0,0.25]}(x),\ f(t,x) = \chi_{[\frac{1}{3},\frac{2}{3}]}(x)$

Work in Progress, Generalizations

- Generalizing the conservation law to a balance law

$$\dot{\rho}(t,x) + \boldsymbol{\lambda}(W(t,\boldsymbol{\rho})) \odot \boldsymbol{\rho}_x(t,x) = \mathbf{h}(t,x)$$

with **h** of certain regularity.

- Generalizing the conservation law for $i \in \{1, \dots, N\}$ to

$$\dot{\rho}_i(t,x) + \boldsymbol{\lambda}_i(W(t,\boldsymbol{\rho}))\boldsymbol{\rho}_{i,x}(t,x) = \mathbf{h}_i(t,x,\boldsymbol{\rho}_i),$$

"semi-linear" problems.

- Thank you very much for your attention!

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