Parabolic quasiminimizers

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Partial differential equations, optimal design and numerics
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Plan of the talk:

- Introduction: Parabolic quasiminimizers
- Stability and Higher Integrability
- Extensions to metric measure spaces

Papers:

- Fujishima, H., Kinnunen, Masson: *Stability for parabolic quasiminimizers*, Potential Analysis, 41, 983-1004, 2014.
- H.: *Global gradient estimates for vector-valued parabolic quasiminimizers*, Nonlinear Analysis TMA, 114, 43-73, 2015.
- H.: *Higher integrability for for vector-valued parabolic quasiminimizers on metric spaces*, Arkiv för Matematik, to appear.

Introduction: Parabolic quasiminimizers

Parabolic quasiminimizers: the model case

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and $\Omega_T := \Omega \times (0, T)$, $T > 0$, and let $u \in L^p(0, T; W^{1,p}(\Omega))$ satisfy

$$-\int_{\operatorname{spt}\varphi} u\,\partial_t \varphi \,\mathrm{d}z + \frac{1}{p}\int_{\operatorname{spt}\varphi} |Du|^p \,\mathrm{d}z \leq \frac{\mathcal{Q}}{p}\int_{\operatorname{spt}\varphi} |Du - D\varphi|^p \,\mathrm{d}z,$$

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for all $\varphi \in C_c^\infty(\Omega_T)$, $\mathcal{Q} \geq 1$,

under suitable initial and boundary conditions on

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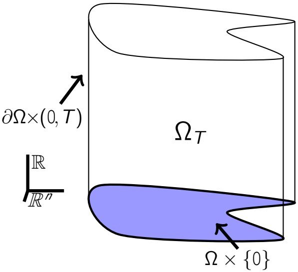

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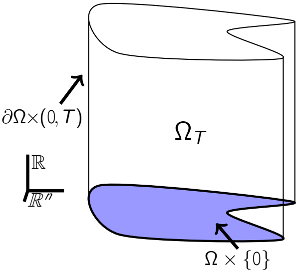

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Parabolic equations and parabolic quasiminimizers

- Weak solutions of the parabolic $\boldsymbol{p}$-Laplace equation

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0, \qquad \text{on } \Omega_T$$

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$$\partial_t u - \operatorname{div} A(x,t,Du) = 0, \qquad \text{on } \Omega_T$$

  with polynomial growth

$$A(x,t,\xi)\cdot\xi \geq \lambda|\xi|^p, \qquad |A(x,t,\xi)| \leq L|\xi|^{p-1},$$

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- Quasiminimizers $\longleftrightarrow$ Obstacle problems
- Other examples by Wieser ('87, Manus. Math.), Zhou ('93/'94, J. PDE)

Stability and Higher Integrability

Stability of Parabolic quasiminimizers

Problem: Take sequences $p_i \to p$ and $\mathcal{Q}_i \to \mathcal{Q}$ and consider a sequence u_i of parabolic $\mathcal{Q}_i$-minimizers of the p_i-energy with fixed boundary data $u_i = \eta$ on $\partial_{\mathrm{par}}\Omega_T$, for which holds

$$u_i(x,t) \to u(x,t) \quad \text{a.e. on} \quad \Omega_T.$$

Is u a parabolic $\mathcal{Q}$-minimizer of the p-energy with $u = \eta$ on $\partial_{\mathrm{par}}\Omega_T$?

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Answer: In general NO!
Counterexamples by Lindqvist, Kilpeläinen & Koskela, Kinnunen & Parviainen for the parabolic p-Laplace

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Answer is YES for regular domains [Hedberg, Kilpeläinen,'99].

The stability problem

Theorem (Fujishima,H.,Kinnunen,Masson, Potential Anal., '14):

Let $\Omega \subset \mathbb{R}^n$ be a domain such that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick, $p \geq 2$ and $\eta \in C^1(\Omega_T)$ be fixed.

Let $\{p_i\}_i$ and $\{\mathcal{Q}_i\}_i$ be two sequences with $p_i \to p \geq 2$ and $\mathcal{Q}_i \to \mathcal{Q} \geq 1$ as $i \to \infty$.

Consider a sequence $u_i \in L^{p_i}(0, T; W^{1,p_i}(\Omega))$ of parabolic Q_i-minimizers of the p_i-energy with $u_i = \eta$ on $\partial_{\mathrm{par}}\Omega_T$ and

$$u_i(x,t) \to u(x,t) \ \text{ almost everywhere in } \ \Omega_T.$$

Then

$$u \in L^p(0, T; W^{1,p}(\Omega)),$$

and u is a parabolic $\mathcal{Q}$-minimizer of the p-energy with $u = \eta$ on $\partial_{\mathrm{par}}\Omega_T$.

A very rough sketch of the proof

- Note: If p changes, then the space $L^p(0,T;W^{1,p}(\Omega))$ changes!
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Lemma (H., Nonlin. Anal., 2014): There exists a universal $\varepsilon > 0$, such that

$$\int_{\Omega_T} |Du|^{p+\varepsilon} \, \mathrm{d}z < \infty.$$

- **Parab. p-Laplace, $p \geq 2$:** Parviainen ('08)
- **Parab. $\mathcal{Q}$-min, $p = 2$:** Masson & Parviainen ('14)

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Remarks on the Proof:

- Global result. It uses the regularity of the boundary.
- Self-improving property of uniform p-thickness [Lewis, '88]
- Intrinsic geometry [DiBenedetto, Friedman, '85]

Some remarks on the stability proof

- **Establish convergence $u_i \to u$.**
  - Higher integrability $\longrightarrow$ **Uniform energy bound**

$$\sup_{i\in\mathbb{N}} \int_{\Omega_T} |u_i|^{p+\delta} + |Du_i|^{p+\delta}\,\mathrm{d}z < \infty,$$

  - Compactness argument (J. Simon, '87) provides that the limit function

$$u \in L^{p+\delta}(0, T; W^{1,p+\delta}(\Omega)),$$

    and for a subsequence we get $u_i \to u$ strongly in $L^{p+\delta}(\Omega_T)$ and $Du_i \to Du$ weakly in $L^{p+\delta}(\Omega_T)$.
- **u attains the initial and lateral boundary data η**
  - Use uniform energy bounds, uniform p-thickness of $\mathbb{R}^n \setminus \Omega$.
  - Self-improving property of uniform p-thickness [Lewis, '88].
  - Use characterization of boundary values for Sobolev functions
- **u is a parabolic $\mathcal{Q}$-minimizer of the p-energy**
  - Delicate argument, using again the uniform energy bounds

Some remarks on the stability proof

- Simple, direct proof, using merely
  - Energy estimates for the functions u_i and their gradients Du_i;
  - General properties and embeddings for Sobolev functions;
  - Characterizations of Sobolev functions at the boundary;
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Literature:

- Stationary case:
  - Lindqvist ('87, J. Math. Anal. Appl., '93, Potential Anal.)
  - Kilpelaïnen & Koskela ('94, Nonlin. Anal.)
  - Li & Martio ('98, Indiana Univ. Math. J.)
  - Zhikov ('97, Russian J. Math. Phys.)
- Time-dependent case:
  - Kinnunen & Parviainen ('10, Adv. Calc. Var.)

Extension to metric measure spaces

Parabolic quasiminimizers on metric measure spaces

Now: Replace $\mathbb{R}^n$ by a metric measure space $(\mathcal{X}, d, \mu)$.

How can we define a 'gradient' ∇u for a function $u\colon \mathcal{X} \to \mathbb{R}$?

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How can we define a 'gradient' ∇u for a function $u\colon \mathcal{X} \to \mathbb{R}$?

Recall: Characterization of Sobolev functions by means of path integrals: For $u \in W^{1,p}(\Omega)$ there holds

$$\big|u(x) - u(y)\big| \leq \int_\gamma |\nabla u| \,\mathrm{d}s,$$

for p-almost all rectifiable curves γ (parametrized by arclenth) connecting x and y.

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Definiton: A Borel measurable function $g \colon \mathcal{X} \to [0, \infty]$ is called an upper gradient of $u \colon \mathcal{X} \to \mathbb{R}$, if

$$\left|u(\gamma(\ell_\gamma)) - u(\gamma(0))\right| \leq \int_\gamma g \, \mathrm{d}s,$$

for all rectifiable curves $\gamma \colon [0, \ell_\gamma] \to \mathcal{X}$.

Upper gradients in metric spaces

Minimal upper gradient: Defined by the property that

$$\|g_u\|_{L^p(\Omega)} = \inf_G \|g\|_{L^p(\Omega)}$$

where G denotes the set of all upper gradients $g \in L^p(\Omega)$.

$\longrightarrow$ Analog concept to the one of Sobolev spaces: Newtonian space

$$\mathcal{N}^{1,p}(\Omega).$$

[Cheeger, Hajlasz, Shanmugalingam]

Parabolic quasi minimizers on $(\mathcal{X}, d, \mu)$

Given a metric measure space $(\mathcal{X}, d, \mu)$, $\Omega \subset \mathcal{X}$ open and $\Omega_T \equiv \Omega \times (0, T) \subset \mathcal{X} \times \mathbb{R}$.

Consider $u \colon \Omega_T \to \mathbb{R}^N$, $N \geq 1$, satisfying

$$\iint_{\operatorname{spt}\varphi} u\, \partial_t \varphi \,\mathrm{d}\mu\,\mathrm{d}t + \iint_{\operatorname{spt}\varphi} g_u^p \,\mathrm{d}\mu\,\mathrm{d}t \leq \mathcal{Q} \iint_{\operatorname{spt}\varphi} g_{u-\varphi}^p \,\mathrm{d}\mu\,\mathrm{d}t,$$

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Assumptions on the metric measure space:

- $(\mathcal{X}, d, \mu)$ is doubling, i.e.

$$\frac{\mu(B_{2R}(x))}{\mu(B_R(x))} \leq C, \quad \text{for all } B_{2R}(x) \subset \mathcal{X}.$$

- $(\mathcal{X}, d, \mu)$ supports a $(1, p)$-Poincaré inequality

$$⨍_{B_\varrho(x_o)} |u - u_{x_o,\varrho}| \,\mathrm{d}\mu \leq c_p\, \varrho \left[⨍_{B_{\Gamma\varrho}(x_o)} g^p \,\mathrm{d}\mu \right]^{1/p}$$

Motivation and Examples

Examples:

- Some weighted Euclidean spaces [Heinonen, Kilpeläinen, Martio, 1993]
- Classes of Riemannian manifolds [Saloff-Coste, 2002]
- Some Ahlfors Q-regular spaces [Laakso, 2000]

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Goals:

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Obstacles:

- No PDEs, fundamental solution, comparison arguments,...
- Upper gradients are not linear
- No standard approximation procedures

Towards stability in metric measure spaces

Global higher integrability of Gehring type:

Theorem (Fujishima,H.,Preprint):
Let $\Omega \subset \mathcal{X}$ be a 'regular' domain, $u \in L^p(0, T; \mathcal{N}^{1,p}(\Omega))$ be a parabolic quasi minimizer with boundary values $u = \eta$ on $\partial_{\mathrm{par}}\Omega_T$. Then there exists $\varepsilon > 0$, depending only on the structure parameters of the problem, such that

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- Euclidean setting ($\mathcal{X} = \mathbb{R}^n$):
  - *Wieser '87, Kinnunen & Lewis '00, Bögelein '08, Bögelein & Parviainen '10, Bögelein & Duzaar '11, H. '14*
- Metric spaces $(\mathcal{X}, d, \mu)$:
  - *Masson, Miranda, Paronetto, Parviainen '13* ($p = 2$, local)
  - *Masson, Parviainen '15* ($p = 2$, up-to-the-boundary)
  - *H. '14* ($p \neq 2$, local)

Literature on quasiminimizers on metric measure spaces:

- Metric measure spaces, properties: *Cheeger, Saloff-Coste, Lewis, Keith, Zhong, Koskela,...*
- Elliptic problems studied in the past 10-15 years: *Kinnunen, Shanmugalingam, Björn, Marola, Koskela, MacManus, Maasalo, Lindqvist, Zatorska-Goldstein,...*
- Parabolic problems on metric measure spaces: *Saloff-Coste, Grigoryan, Kinnunen, Kilpeläinen, Koskela, Marola, Miranda, Paronetto, Masson, Parviainen, Siljander,...*

Many techniques also come from the study of parabolic problems in the Euclidean setting: *DeGiorgi, Nash, Giusti, DiBenedetto, Gianazza, Vespri, Wieser, Duzaar, Bögelein, Zhou, ...*

Thank you for your attention.