Localizable solutions to nonlinear evolution problems with irregular obstacles: Existence and regularity

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Partial differential equations, optimal design and numerics

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The obstacle-free case

Model case of equations:

$$\partial_t u - \operatorname{div}\left(|Du|^{p-2} Du\right) = f - \operatorname{div}\left(|F|^{p-2} F\right) \qquad \text{on } \Omega_T := \Omega \times (0, T)$$

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$$f - \operatorname{div}(|F|^{p-2}F) \in L^{p'}(0, T; W^{-1,p'}(\Omega)) = \left[L^p(0, T; W_0^{1,p}(\Omega))\right]'.$$

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More generally, we consider equations

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We now impose an obstacle constraint $u \geq \psi$ a. e. for a given obstacle with the same regularity, i.e.

$$\psi \in L^p\text{-}\,W^{1,p} \cap C^0\text{-}\,L^2 \quad \text{with} \quad \partial_t \psi \in L^{p'}\text{-}\,W^{-1,p'}.$$

Formulation of the obstacle problem

For given boundary values g, the solution space is

$$K_g(\Omega_T) := \{u \in g + L^p(0,T; W_0^{1,p}(\Omega)) \cap C^0([0,T]; L^2(\Omega)) \,:\, u \geq \psi\}$$

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A function $u \in K_g(\Omega_T)$ is called a weak solution to the obstacle problem $\mathrm{OP}(\psi; f, F)$ iff it solves the variational inequality

$$\int_0^T \langle \partial_t u, v - u \rangle \, dt + \int_{\Omega_T} a(z, Du) \cdot (Dv - Du) \, dz$$
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for all comparison maps v in the class

$$K_g'(\Omega_T) := \{v \in K_g(\Omega_T) \,:\, \partial_t v \in L^{p'}(0,T;W^{-1,p'}(\Omega))\}.$$

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- continuous obstacles: Brezis (1972), Struwe & Vivaldi (1985);
- bounded obstacles: Alt & Luckhaus (1983);
- obstacles with weak time derivative $\partial_t \psi \in L^{p'}(\Omega_T)$: Bögelein & Duzaar & Mingione (2011).
  (Construction via mollification in time and maximum construction)

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Problem I (Localizability of solutions):

For a solution u of an obstacle problem, is the restriction $u|_{\mathcal{O}_I}$ on a subset $\mathcal{O}_I := \mathcal{O} \times (t_1, t_2) \Subset \Omega_T$ again a solution?

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Since the answer was unknown so far, previous regularity results in the **interior** relied on unnatural assumptions on the **boundary** data.

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Existence of localizable solutions

For an arbitrary obstacle with $\partial_t\psi \in L^{p'}\text{-}\,W^{-1,p'}$ there holds

Theorem (S., 2011)

The obstacle problem $OP(\psi; f, F)$ *has a* **localizable solution** $u : \Omega_T \to \mathbb{R}$ *with* $u \geq \psi$ *a. e.*

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The obstacle problem $OP(\psi; f, F)$ *has a* **localizable solution** $u : \Omega_T \to \mathbb{R}$ *with* $u \geq \psi$ *a. e., in the following sense: for every Lipschitz regular domain* $\mathcal{O}_I := \mathcal{O} \times (t_1, t_2) \Subset \Omega_T$ *the restriction* $u|_{\mathcal{O}_I} \in K_u(\mathcal{O}_I)$ *solves the localized variational inequality*

$$\int_{t_1}^{t_2} \langle \partial_t v, v-u\rangle dt + \int_{\mathcal{O}_I} a(z, Du)\cdot(Dv-Du)dz + \frac{1}{2}\int_{\mathcal{O}\times\{t_1\}} |v-u|^2 dx$$
$$\geq \int_{\mathcal{O}_I} f(v-u) + |F|^{p-2}F\cdot(Dv-Du)dz$$

for all $v \in K_u'(\mathcal{O}_I)$.

Remarks

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- Under weak assumptions on $\partial\Omega$ (Lipschitz suffices), the solution u also solves on subdomains $\mathcal{O}_I$ touching the boundary.
- In the latter case, solutions are unique for given initial and boundary values.

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$$\Psi \in L^q_{\mathrm{loc}}(\Omega_T) \quad \Longrightarrow \quad |Du| \in L^q_{\mathrm{loc}}(\Omega_T)$$

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$$\Psi \in L^q_{\mathrm{loc}}(\Omega_T) \quad \Longrightarrow \quad |Du| \in L^q_{\mathrm{loc}}(\Omega_T)$$

for all $q > p$, *together with a local estimate of the form*

$$⨍_{C_R} |Du|^q \, dz \le c \left(\left[⨍_{C_{2R}} |Du|^p \, dz \right]^{\frac{1}{p}} + \left[⨍_{C_{2R}} \Psi^q \, dz \right]^{\frac{1}{q}} + 1 \right)^{p + d(q-p)},$$

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where the **scaling deficit** *is defined by*

$$d \equiv \begin{cases} \frac{p}{2} & \text{if } p \ge 2\,, \\ \frac{2p}{p(n+2)-2n} & \text{if } \frac{2n}{n+2} < p < 2\,. \end{cases}$$

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$$\mathbf{W}^{\mu}_{\beta,p}(x_0, r) := \int_0^R \left[\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-\beta p}}\right]^{\frac{1}{p-1}} \frac{d\varrho}{\varrho}\,.$$

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- Duzaar & Mingione (2010): analogous estimates on the level of the gradient:

$$|Du(x_0)| \leq c\|Du\|_{L^1} + c\mathbf{W}^{\mu}_{\frac{1}{p},p}(x_0, R).$$

Parabolic obstacle problems

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$$\le 1 + c\|Du\|_{L^2} + c\, I^{\mathcal{P}}_{2/3}\big([I^{\mathcal{P}}_{2/3}(\Psi^2)]^{1/2}\big)$$

where $\Psi := |f| + |\partial_t\psi| + |D^2\psi|$, and $I^{\mathcal{P}}_{2/3}$ denotes the classical parabolic Riesz potential.

Applications

The methods yield in particular criteria for

- **Lorentz regularity**:

$$f, \partial_t \psi, |D^2\psi| \in L(r,s) \quad \Longrightarrow \quad |Du| \in L_{\mathrm{loc}}(\tfrac{Nr}{N-r}, s),$$

for all $2 < r < N := n+2$ and $1 \leq s \leq \infty$.

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- **Continuity of the spatial gradient**:

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- $C^{1,\alpha}$-**Regularity**: the Morrey-type condition

$$\sup_{z_0 \in \mathcal{O}_I} \sup_{0<\varrho<1} \varrho^{2-2\gamma} \fint_{C_\varrho(z_0)\cap\Omega_T} |f|^2 + |\partial_t\psi|^2 + |D^2\psi|^2 \, dz < \infty$$

for every subset $\mathcal{O}_I \Subset \Omega_T$ and some $\gamma \in (0,1)$ implies Hölder continuity of Du.

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- Analogous results hold for the solution u itself instead of the gradient Du.
- In the elliptic case, the estimates hold much more generally with measure valued right-hand sides and arbitrary growth exponents $p > 2 - \frac{1}{n}$. This yields estimates by non-linear Wolff-potentials.

Construction of localizable solutions I: Regularization

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- Consider solutions $u_\varepsilon \in K'_{g_\varepsilon}(\Omega_T)$ to the regularized obstacle problems $\mathrm{OP}(\psi_\varepsilon; f, F_\varepsilon)$.

Construction of localizable solutions I: Regularization

Strategy of proof for regular $\Omega \subset \mathbb{R}^n$ (Lipschitz suffices)

- Approximate g, ψ, F by regularized data $g_\varepsilon, \psi_\varepsilon, F_\varepsilon$ with

$$\partial_t \psi_\varepsilon - \operatorname{div}[a(\cdot, D\psi_\varepsilon)] \in L^{p'}(\Omega_T),$$
$$\operatorname{div}(|F_\varepsilon|^{p-2}F_\varepsilon) \in L^{p'}(\Omega_T).$$

- Consider solutions $u_\varepsilon \in K'_{g_\varepsilon}(\Omega_T)$ to the regularized obstacle problems $\mathrm{OP}(\psi_\varepsilon; f, F_\varepsilon)$.
- For data as above, these solutions satisfy

$$\partial_t u_\varepsilon \in L^{p'}(0, T; W^{-1,p'}(\Omega)).$$

(Bögelein & Duzaar & Mingione (2011)).

Construction of localizable solutions II: Passing to the limit

- As $\varepsilon \searrow 0$, there holds subconvergence $u_\varepsilon \to u$ strongly in $L^p(0,T;W^{1,p}(\Omega))$, but in general $\partial_t u \notin L^{p'}(0,T;W^{-1,p'}(\Omega))$.

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- Define extensions v_ε of u_ε on a subdomain $\mathcal{O}_I \Subset \Omega_T$ as solutions to

$$\begin{cases} \partial_t v_\varepsilon - \operatorname{div} a(z, Dv_\varepsilon) = \partial_t \psi_\varepsilon - \operatorname{div} a(z, D\psi_\varepsilon) & \text{on } \mathcal{O}_I, \\ v_\varepsilon = u_\varepsilon & \text{on } \partial_{\mathcal{P}}\mathcal{O}_I. \end{cases}$$

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- These extensions subconverge in the sense

$$\begin{cases} v_\varepsilon \rightharpoonup v & \text{weakly in } L^p(t_1,t_2;W^{1,p}(\Omega)), \\ \partial_t v_\varepsilon \rightharpoonup \partial_t v & \text{weakly in } L^{p'}(t_1,t_2;W^{-1,p'}(\Omega)). \end{cases}$$

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- The limit v is the desired extension $v \in K'_u(\mathcal{O}_I)$, and u solves the obstacle problem $\mathrm{OP}(\psi; f, F)$ **locally**.