The Obstacle Problem for the Total Variation

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The TV obstacle problem

Consider

- a bounded open set Ω in $\mathbb{R}^n$, n positive integer,
- an obstacle $\psi\colon \overline{\Omega} \to \mathbb{R}$ with $\psi \leq 0$ on $\partial\Omega$.

Obstacle problem: Minimize the total variation (TV)

$$\int_\Omega |\nabla u| \,\mathrm{d}x$$

among functions $u\colon \overline{\Omega} \to \mathbb{R}$ with

$$u \equiv 0 \text{ on } \partial\Omega \qquad \text{and} \qquad u \geq \psi \text{ on } \Omega\,.$$

The generalized TV obstacle problem

Natural space for existence results (thanks to weak$*$ compactness):

$$\mathrm{BV}_0(\overline{\Omega}) := \left\{ u \in \mathrm{L}^1(\mathbb{R}^n) \ : \begin{array}{c} \text{gradient } \mathrm{D}u \text{ is finite measure on } \mathbb{R}^n \\ \text{and } u \equiv 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega \end{array} \right\}$$

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Generalized obstacle problem: Minimize the total mass of $\mathrm{D}u$

$$|\mathrm{D}u|(\overline{\Omega}) = |\mathrm{D}u|(\Omega) + \underbrace{|\mathrm{D}u|(\partial\Omega)}_{\approx \|\mathrm{int\,trace}(u)\|_{\mathrm{L}^1(\partial\Omega)}}$$

among

$$u \in \mathrm{BV}_0(\overline{\Omega}) \qquad \text{with } u \geq \psi \text{ a.e. on } \Omega\,.$$

Existence and duality for $\mathrm{W}^{1,1}$ obstacles

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$$\min\{|\mathrm{D}u|(\overline{\Omega}) \,:\, u \in \mathrm{BV}_0(\overline{\Omega})\,,\, u \geq \psi \text{ a.e. on } \Omega\}$$
$$= \max\left\{ \int_\Omega \sigma\cdot\nabla\psi\,\mathrm{d}x \,:\, \sigma \in \underbrace{S^\infty(\Omega,\mathbb{R}^n)}_{\text{sub-unit vector fields}}\,,\, \operatorname{div}\sigma \leq 0 \text{ in } \mathscr{D}'(\Omega)\right\}.$$

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$\rightsquigarrow$ to say more, need products $\sigma \cdot \mathrm{D}\psi$ and $\sigma \cdot \mathrm{D}u$ if merely $\psi, u \in \mathrm{BV}$ (e.g. if ψ is a characteristic function).

The Anzellotti pairing

Consider:

- $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$,
- a vector field $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ (w.r.t. Lebesgue measure $\mathrm{d}x$).

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Can one define a product $[\![\sigma, \mathrm{D}u]\!]$? If $\mathrm{div}\,\sigma$ is suitably good, yes:

Definition (Kohn & Temam '82/'83, Anzellotti '83, ...)

For u, σ as above, the distribution

$$[\![\sigma, \mathrm{D}u]\!] := \mathrm{div}(\sigma u) - u\,\mathrm{div}\,\sigma \in \mathscr{D}'(\Omega)\,.$$

makes sense (and behaves reasonably) if ...

- *... either* $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, $\mathrm{div}\,\sigma \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$
- *... or* $\mathrm{div}\,\sigma \in \mathrm{L}^n_{\mathrm{loc}}(\Omega)$ (*then uses Sobolev's embedding*).

A pairing for divergence-measure fields

But even if $\operatorname{div}\sigma \notin \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, we still have:

Definition (a new Anzellotti type pairing, Scheven & S.)

For $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ *and* $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ *such that* $\operatorname{div}\sigma$ *is Radon measure (in particular if* $\operatorname{div}\sigma \le 0$ *in* $\mathscr{D}'(\Omega)$*), we define*

$$[\![\sigma, \mathrm{D}u^+]\!] := \operatorname{div}(\sigma u) - u^+ \operatorname{div}\sigma \in \mathscr{D}'(\Omega)\,.$$

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- makes sense because
  - $\operatorname{div}\sigma$ vanishes on $\mathcal{H}^{n-1}$-negligible sets (Chen & Frid '99),
  - u has $\mathcal{H}^{n-1}$-a.e. defined representatives $u^\pm$ s.t., for $\mathcal{H}^{n-1}$-a.e. x,

    either $u^+(x) = u^-(x)$ is the Lebesgue value of u at x

    or $u^-(x) < u^+(x)$ are the approximate jump values of u at x.
- pairing $[\![\sigma, \mathrm{D}u^*]\!]$ with representative $u^* := \frac{u^+ + u^-}{2}$ already used by Mercaldo & Segura de León & Trombetti '09.

Properties of the pairing

Theorem (properties of $[\![\sigma, \mathrm{D}u^+]\!]$, Scheven & S.)

For $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div}\sigma \leq 0$ in $\mathscr{D}'(\Omega)$,

- *$[\![\sigma, \mathrm{D}u^+]\!]$ is a Radon measure with product estimate*
$$|[\![\sigma, \mathrm{D}u^+]\!]| \leq \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}|\mathrm{D}u| \qquad \text{on } \Omega\,,$$
- *and its absolutely continuous part is the pointwise product, i.e.*
$$[\![\sigma, \mathrm{D}u^+]\!]^{\mathrm{a}} = (\sigma \cdot \nabla^{\mathrm{a}} u)\mathrm{d}x \qquad \text{on } \Omega\,.$$

- In particular, $[\![\sigma, \mathrm{D}u^+]\!] = (\sigma \cdot \nabla u)\mathrm{d}x$ trivializes for $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$.

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- Proofs based on fine (semi)continuity and capacity methods (e.g., since u^+ is *not* the limit of standard mollifications, need one-sided approximations of Carriero-Dal Maso-Leaci-Pascali '88).
- Up-to-the-boundary pairing $[\![\sigma, \mathrm{D}u^+]\!]_0$ on $\overline{\Omega}$ accounts for zero Dirichlet datum (on mildly regular $\partial\Omega$; cf. S. '15, Beck & S. '15).

Duality for BV obstacles

Theorem (duality for the TV obstacle problem, Scheven & S.)

For mildly regular $\partial\Omega$, $\psi \in \mathrm{BV}_0(\overline{\Omega}) \cap \mathrm{L}^\infty(\Omega)$ *with* $|\mathrm{D}\psi|(\partial\Omega) = 0$:

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Two methods of proof (both rely on the properties of the pairing):

- Either look at obstacle problems for the p-Laplace in $\mathrm{W}^{1,p}_0$ and pass $p \searrow 1$ (this way, if $\psi \in \mathrm{W}^{1,1+\varepsilon}_0$, also get a convergence result for minimizers when $p \searrow 1$),

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- or deduce it from (abstract) convex duality.

BV optimality conditions

Heuristically, minimizers u should satisfy

$$\operatorname{div} \frac{\nabla u}{|\nabla u|} \leq 0\,,$$

and we can now make this precise:

Corollary (optimality conditions for the TV obstacle problem)

Every minimizer $u \in \mathrm{BV}_0(\overline{\Omega})$ is super-1-harmonic on Ω in the sense that there exists some $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with

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Moreover, u is 1-harmonic away from the obstacle in the sense of

$$\operatorname{div} \sigma \equiv 0 \text{ on } \Omega \cap \{u^+ > \psi^+\}\,.$$

Extensions

We can also treat ...

- (much) more general obstacles:
  - thin and, most generally, quasi upper semicontinuous obstacles (then need additional tools: relaxation, De Giorgi's measure, ...),
  - obstacles which are positive up to $\partial\Omega$ (then need modified pairing),

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- the non-parametric area $\int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x$ and similar functionals,
- general boundary values.

Related topics

Related work in progress concerns . . .

- BV *super*solutions to 1-Laplace and minimal surface equations, in particular:
  - compactness results,
  - the question if simultaneous super- and sub-solutions are solutions (for the 1-Laplace surprisingly non-trivial, since σ is not unique $\rightsquigarrow$ duality argument of possible interest; cf. Yan '11),

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- variational existence results for measure data problems to the 1-Laplace equation and the prescribed mean curvature equation (parametric or non-parametric; in the last case yields an alternative to the approach of Dai & Trudinger & Wang '12).