# *On Neumann problems for nonlocal equations related to jump processes*

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Partial differential equations, optimal desing and numerics,
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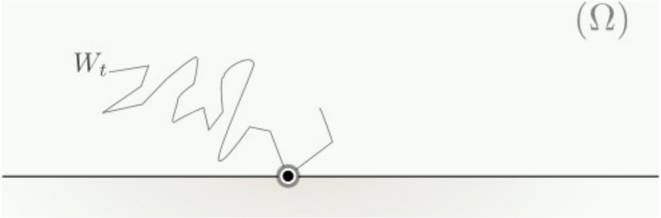
# *Probabilistic approach*

In the classical probabilistic approach, Neumann problems are associated to stochastic processes being reflected on the boundary.




Linked to the Skorohod problem: solving a stochastic differential equation with a reflecting boundary condition

# *Classical (local) Neumann problem*

KEY RESULT:
For a PDE with Neumann or oblique boundary conditions, there is a unique underlying reflection process

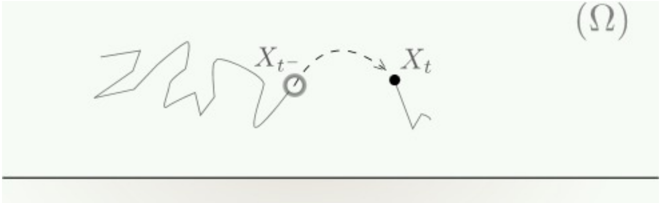
Any consistent approximation converge to it in the limit (Lions-Snitzman, Barles-Lions).

This relies on the underlying stochastic processes being continuous (at least in the case of normal reflections)

# *More general: Lévy processes*

Càdlàg:

- Right continuous, limits on the left
- Stationary independent increments



# *Generator of a stochastic process*

## Infinitesimal generator

of a stochastic process is a partial differential operator that encodes a great amount of information about the process

Example:
Brownian motion $W_t \longrightarrow$ Laplacian:

$$\mathcal{L}[u](x) = \triangle u(x) = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}(x)$$

We adopt an analytical approach: we keep in mind the idea of having a reflecting process but we don't define it precisely or prove its existence.

We work with the generators of the processes.

# *Generators*

- Lévy process jumping from $x$ to $x+z$ with a certain intensity, $\longrightarrow$ nonlocal operator

$$\mathcal{I}[u] = P.V. \int_{\mathbb{R}^N} [u(x+z) - u(x)] d\mu(z),$$

where $\mu$ is a Lévy measure, i.e. a positive Borel measures (under some integrability condition)

- $\mathcal{I}$ is a P.V. (principal value) integral, i.e.

$$\mathcal{I}[u] = \lim_{\delta \to 0^+} \int_{|z|>\delta} [u(x+z) - u(x)] d\mu(z)$$

## *Lévy measures*

The measure $\mu$ satisfies a quite general integrability condition at $z = 0$ and at infinity:

$$\int (|z|^2 \wedge 1) d\mu(z) < \infty \tag{1}$$

- Take for example $\mathcal{I}[\phi]$ with $\phi$ bounded and $C^1$ and

$$d\mu(z) = \frac{dz}{|z|^{N+\sigma}} \quad \sigma \in (0,1) \tag{2}$$

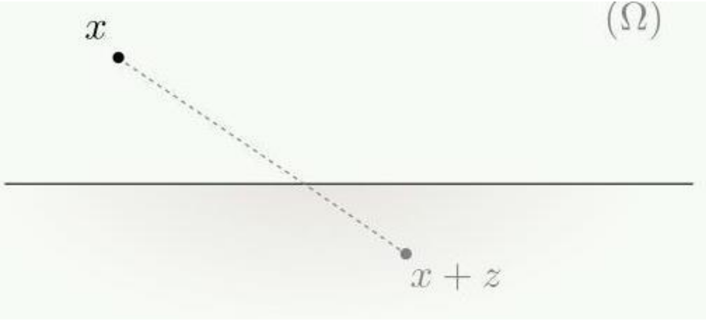
  In particular

$$\int (|z| \wedge 1) d\mu(z) < \infty$$

- If $\mu$ is as in (2) with $\sigma \in (0,2)$ and satisfies (1), we have to consider $\phi \in C^2$ and add and substract a compensator term.

# *Neumann boundary condition*

In the case of a jump process, there are several ways to keep the process inside the domain. For example.. (take $\Omega$ ="halfspace")



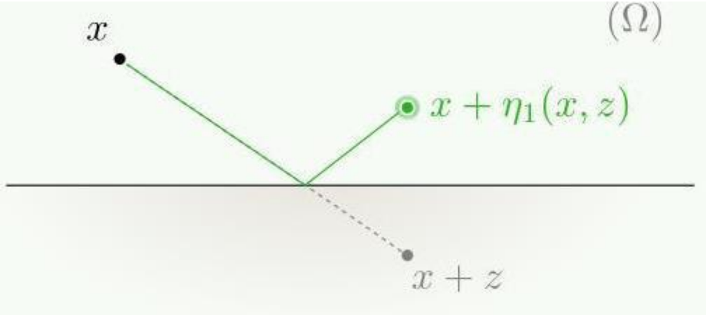
## *Linear PIDE-BCGJ*

G. Barles, E.Chasseigne, C. Georgeline, E. R. Jacobsen, 2014, *On neumann type problems for non-local equations set in a half space*

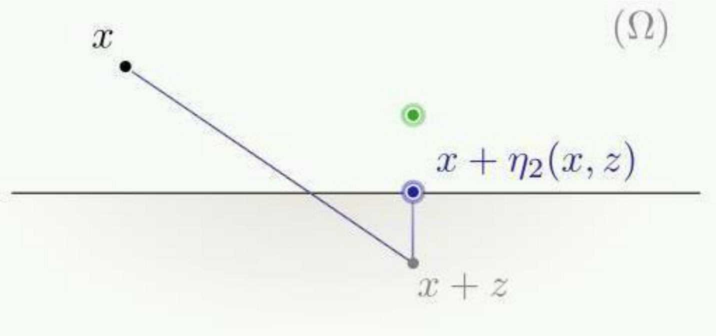
- simple linear equations: $u(x) - \mathcal{I}[u](x) + f(x) = 0$ + Neumann BC
- domains with flat boundary

They considered a nonlocal diffusion $\mathcal{I}$ which can be of different types: they proposed at least the previous 4 different and coherent (w.r.t the classical case) models.

# *Mirror reflection*

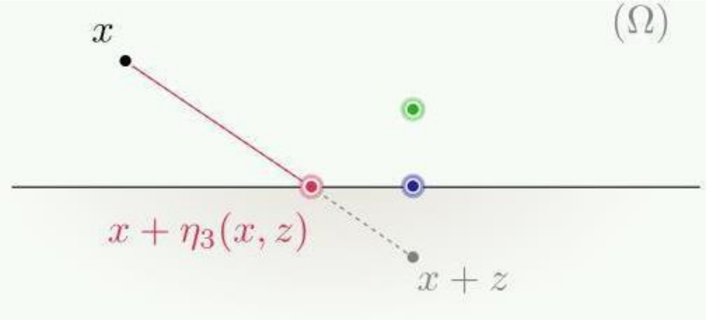


## *Normal projection, Lions-Sznitman*




Problem investigated by Barles, Georgelin, Jakobsen 2013 in the framework of fully non-linear equations in general domains.

# *Stick on the wall, fleas on the window*

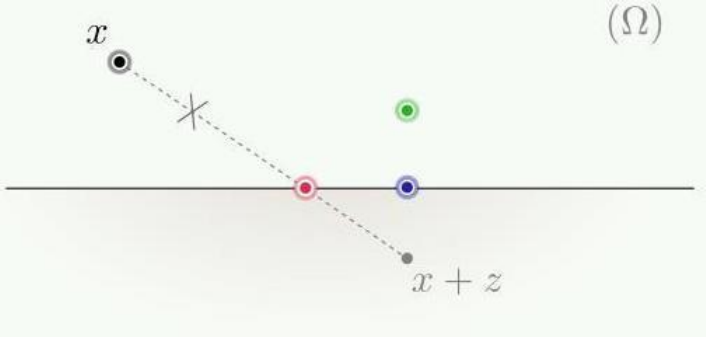



Outwards jumps are stopped where the jump path hits the boundary, and then the process is restarted there.

- To the best of our knowledge, processes with generators of "Mirror Reflection" and "Fleas on the Window" have not been considered yet.
- Natural ways to define "reflections" (in particular mirror reflection).
- It could be problematic to work with in general domains due to the possibility of multiple reflections

# *Don't jump, censored*

We focus on the so-called censored process




Any outwards jump is cancelled (censored) and the process is restarted (resurrected) at the origin of that jump.

# *The censored process*

Actually we don't jump outside: we never reach the point $x+z$:

$$\mathcal{I}[u](x) = P.V. \int_{x+z\in\bar{\Omega}} [u(x+z)) - u(x)]d\mu(z)$$

- Notice that (part of the) Neumann condition is included in the definition of the nonlocal operator. The Neumann condition influences the equation inside the domain.
- Boundary value problem inside $\bar{\Omega}$. No need for conditions on $\Omega^c$.
- Notice the bad dependence on $x$ in the domain of integration, this gives the main problems in the proof of comparison

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# *Underlying process-probabilistic references*

Question: is it realized by a concrete Markov process, i.e., does it corresponds to the generator of such a process?
The process can be constructed via some probabilistic methods: K. Bogdan, K. Burdzy and Z. Chen. 2003
Q.Y. Guan. and Z.M. Ma. 2006
M. Fukushima, Y. Oshima and M. Takeda. 1994
N. Jacob 2005
In particular, the underlying processes are related to the censored stable processes of Bogdan et al. and the reflected $\alpha$-stable process of Guan and Ma.

We consider

$$d\mu(z) = \frac{dz}{|z|^{N+\sigma}}, \quad \sigma \in (0,2). \tag{3}$$

Two main cases:

(I) not too singular measures $\sigma < 1$

(II) strong singularity $\sigma \in (0,2)$

(I) Strong singularity:
Interesting (but difficult) case...
We don't know! Even in the simple case of linear equations the results are not completely satisfactory.

(II) $\sigma \in (0,1)$: What happens in the case of linear equation considered by BCGJ?

$$u(x) - \mathcal{I}[u](x) + f(x) = 0 \text{ in } \Omega.$$

The nonlocal terms that are the principal terms.
Simpler problem: The process does not reach the boundary!

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## Remark

This is not true anymore in the presence of a drift term! The drift can push the process to hit the boundary

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The flatness of the boundary in BCGJ eliminates some technical difficulties which arise when dealing with Neumann boundary conditions in general domains and are mainly due to the estimation of the nonlocal terms.

Questions: Nonlinear equations? More general domains?

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*"On Neumann problems for nonlocal Hamilton-Jacobi equations with dominating gradient terms"*, 2015

$$\begin{cases} u(x) - \mathcal{I}[u](x) + H(x, Du) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

$\mathcal{I}$ is of censored type $\sigma < 1$ (previous slides)

$H$ is a nonlinear Hamiltonian (typically relatd to optimal control problems)
the growth of $H$ in the gradient strictly dominates the nonlocal diffusion

- $H$ either in coercive form in the gradient term,
- either in Bellman form, not necessarely coercive

$\Omega \subset \mathbb{R}^N$ is an open domain possibly unbounded smooth enough ($W^{2,\infty}$).

# *Coercive Hamiltonian*

Coercive Hamiltonians of the type

$$H(x,p) = b(x)|p|^m + a_1(x)|p|^l + (a_2(x),p) - f(x),$$

for $m > \sigma$, where $x, p \in \mathbb{R}^n, 0 < l < m$, $a_1, a_2, f, b$ are continuous and bounded functions, $b$ is Lipschitz continuous and satisfies

$$b(x) \geq b_0 > 0 \quad \forall x \in \Omega.$$

# *Hamiltonian in Bellman form*

Non-coercive Hamiltonians in Bellman form
related to optimal control problems

$$H(x,p) = \sup_{\alpha \in \mathcal{A}} \{-b(x,\alpha) \cdot p - l(x,\alpha)\};$$

where $\mathcal{A}$ is a compact metric space, $b : \bar{\Omega} \times \mathcal{A} \to \mathbb{R}^n$ and $f : \bar{\Omega} \times \mathcal{A} \to \mathbb{R}$ are continuous and bounded functions.

(I) Uniform continuity of the cost $l$;

(II) Uniform Lipschitz continuity of the drift $b$.

# *Main results*

Framework of viscosity solutions

COMPARISON [G. 2015]

*Under the assumptions of the previous slides, let $u$ be a bounded usc subsolution of (4) and $v$ a bounded lsc supersolution of (4). Then*

$$u \leq v \text{ in } \bar{\Omega}.$$

EXISTENCE, UNIQUENESS
Once the comparison holds, we use Perron's method for integro-differential equations to get as a corollary existence and uniqueness for the problem (4) in the class of continuous functions.

EVOLUTIVE PROBLEMS
behaviour as $t \to +\infty$ of the solutions, convergence to the associated stationary problem.

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## Main difficulties

Bad $x$-dependence+general geometry of the boundary!
We aim at estimating

$$\mathcal{I}[u](x) - \mathcal{I}[v](y) =$$

$$\int_{x+z\in\bar\Omega}[u(x+z)-u(x)]\frac{dz}{|z|^{N+\sigma}} - \int_{y+z\in\bar\Omega}[v(y+z)-v(y)]\frac{dz}{|z|^{N+\sigma}}$$

How to compare the two domains of integration?

We write

$$\{x+z\in\bar\Omega\} =$$

$$\{x+z\in\bar\Omega, y+z\in\bar\Omega\}\cup\{x+z\in\bar\Omega, y+z\notin\bar\Omega\}$$

- $\{x+z\in\bar\Omega, y+z\in\bar\Omega\}$ : plus simple
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*Thank you for the attention!*

## *A significant example: the halfspace* $\Omega = \{(x_1, \cdots, x_N) \in \mathbb{R}^N \,|\, x_N > 0\}$

In this case

$$\mathcal{I}[u](x) - \mathcal{I}[v](y) =$$

$$\int_{x_N+z_N \geq 0} [u(x+z) - u(x)] \frac{dz}{|z|^{N+\sigma}} \int_{y_N+z_N \geq 0} [v(y+z) - v(y)] \frac{dz}{|z|^{N+\sigma}}$$

Suppose $x_N \geq y_N$ (the other being symmetric):

$$\{x_N + z_N \geq 0\} =$$

$$\{x_N + z_N \geq 0, y_N + z_N \geq 0\} \cup \{x_N + z_N \geq 0, y_N + z_N < 0\}$$

Note that

$$\{x_N + z_N \geq 0, y_N + z_N \leq 0\} = \{-x_N \leq z_N \leq -y_N.\}$$

and its measure goes to zero if $x_N \to y_N$!

USEFUL REMARK!

## *Proof*

- Underlying idea: prove that we are in the case of the previous remark, $x_N = y_N$. We use the superfractional growth in the gradient of $H$ to control the nonlocal diffusion.

- The flat case is far more simple. In case of ageneral domain, the description of the set of integrations in the nonlocal terms is more complicated due to the geometry of $\partial\Omega$ and there are some left terms which we have to estimate.

# *Evolutive problem, large time behaviour*

We provide large time behaviour for the problem

$$\begin{cases} \partial_t u(x) - \mathcal{I}[u(\cdot,t)](x) + H(x,Du) = 0 & \text{in } \Omega \times (0,+\infty) \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \bar{\Omega}. \end{cases} \tag{5}$$

- $\Omega$ is a bounded open subset of $\mathbb{R}^n$ of class $W^{2,\infty}$
- $H$ is an Hamiltonian in coercive form
- $$\mathcal{I}[u(\cdot,t)](x) = P.V. \int_{x+z\in\bar{\Omega}} [u(x+z,t) - u(x,t)] \frac{dz}{|z|^{N+\sigma}}$$

  with $\sigma \in (0,1)$.

## *Large time behaviour*

Let $u(x,t)$ be the unique solution to the previous evolution problem.
Question: What is the behaviour of $u(x,t)$ as $t \to +\infty$?
Does it converge to a/the solution of a suitably associated problem?
And how?
Ergodic behaviour

- $$u(\cdot,t)/t \to c$$

  where $c$ is the so-called ergodic constant.
- We search for an asymptotic development of $u$ of the type

  $$u(x,t) = ct + w(x) + o(1)$$

  where $o(1)$ tends to zero as $t \to \infty$ .

# *Large time behaviour*

(I) Prove that there exists a unique constant $c \in \mathbb{R}$ for which the stationary ergodic problem

$$\begin{cases} -\mathcal{I}[w(\cdot)](x) + H(x, Dw) = -c & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

has a solution $w \in C^{\frac{m-\sigma}{m}}(\bar{\Omega})$ (unique up to an additive constant).

(II) Convergence as $t \to +\infty$:
There exists a pair $(w, c)$ solution to (6) such that

$$u(x,t) - ct - w(x) \to 0 \text{ as } t \to +\infty$$

uniformly on $\bar{\Omega}$.

# *Main steps*

We follow closely the arguments given in
G. Barles, O. Ley, S. Koike, E. Topp, *Regularity results and large time behavior for integro-differential equations with coercive hamiltonians* (see also G. Barles, E. Chasseigne, A. Ciomaga, C. Imbert, *Lipschitz regularity of solutions for mixed integro-differential equations* in the local framework).

- Key result: Hölder regularity for the solutions of (5): G. Barles, O. Ley, S. Koike and E. Topp (see above)
- We solve the ergodic problem, by means of the approximant problems and uniform estimates given by regularity results.
- We prove a strong maximum principle for the evolutive problem considered (5).
- We conclude the asymptotic convergence as $t \to +\infty$.

## *Some other references*

Menaldi, Robin, 1985: reflection problems solved in the case of diffusion processes with jumps only inside $\Omega$.
Garroni, Menaldi, 2002: large class of uniformly elliptic integro-differential equations, where the principal part is a non-degenerate 2nd order term. Dirichlet type problems.
Barles, Topp, Koike, Ley, 2014: same kind of nonlinearity in $H$ for Dirichlet type problems.

We recall the "natural" Neumann boundary condition for the reflected $\alpha$-stable process of Guan and Ma is different from the one we consider here. They show that the boundary condition arising through the variational formulation and Green type formulas is

$$\lim_{t\to 0} t^{2-\alpha}\frac{\partial u}{\partial x_N}(x+teN)=0.$$

This formula allows the normal derivative $\frac{\partial u}{\partial x_N}$ to explode less rapidly than $|x_N|^{\sigma-2}$ and justifies the use of boundary conditions in the viscosity sense since $u_{x_N}$ is not necessarily equal to 0 on the boundary for $\sigma < 2$.

# *Other references on censored processes*

Widely used in physics (Levy flights), operation research, queuing theory, mathematical finance, risk estimation.


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