# An Overview of Hadronic Topics Concerning $g_\mu - 2$

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*High-precision QCD at low energy*

$a_\mu(\mathrm{E821-BNL}) = 116\,592\,089(54)_{\mathrm{stat}}(33)_{\mathrm{syst}} \times 10^{-11}[0.54\mathrm{ppm}]$

Future Experiments:
FNAL with $\pm 0.14$ ppm overall uncertainty (data expected in 2017)
JPARC with similar uncertainty but very different technique

Standard Model Contributions to $a_\mu = \frac{1}{2}(g_\mu - 2)$

*J.P. Miller, E. de Rafael, B.L. Roberts, D. Stöckinger, Annu. Rev. Part. Nucl. Phys. '12*

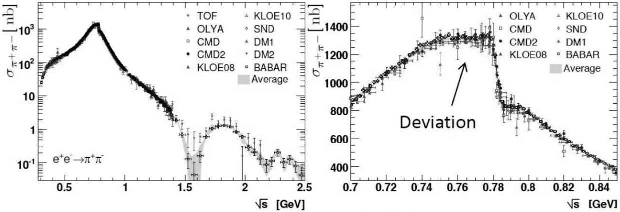
| Contribution | Result in $10^{-11}$ units |
|---|---|
| QED (leptons) | $116\,584\,718.85 \pm 0.04$ |
| HVP(lo)[$e^+e^-$] | $6\,923 \pm 42$ |
| HVP(ho) | $-98.4 \pm 0.7$ |
| HLbyL | $105 \pm 26$ |
| EW | $153 \pm 1$ |
| Total SM | $116\,591\,801 \pm 49$ |

**Persistent $3.6\sigma$ discrepancy between SM theory and Experiment**

$e^+e^- \to \pi^+\pi^-$

Babar most significant $(g\text{-}2)_\mu$ result

[BaBar: PRD 86 (2012) 032013, PRL 103 (2009) 231801]




Systematic uncertainties
at the ρ region
BaBar: 0.5%
CMD2: 0.8%
SND: 1.5%
KLOE: 0.8%

0.28–1.8 (GeV)

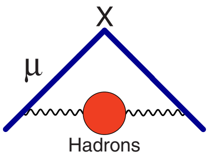
| | |
|---|---|
| BABAR | $(514.1 \pm 3.8) \times 10^{-10}$ |
| previous $e^+e^-$ combined | $(503.5 \pm 3.5) \times 10^{-10}$ * |
| τ combined | $(515.2 \pm 3.5) \times 10^{-10}$ * |

Deviation between BNL measurement and theory prediction reduced using BaBar $\pi^+\pi^-$ data
$a_\mu[\text{exp}] - a_\mu[\text{SM}] = (19.8 \pm 8.4) \times 10^{-10}$ (2.4σ)

* arXiv:0906.5443 M. Davier et al. 9

We shall have to wait and see how BaBar, Kloe and BESIII solve their discrepancies

Good opportunity for Lattice QCD !




## Muon Anomaly from HVP

Standard Formulation in terms of the **Hadronic Spectral Function**

$$\frac{1}{2}(g_\mu - 2)_{\rm Hadrons} \equiv a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt}{t}\int_0^1 dx \frac{x^2(1-x)}{x^2+\frac{t}{m_\mu^2}(1-x)}\frac{1}{\pi}{\rm Im}\Pi(t)$$

where

$$\sigma(t)_{[e^+e^- \to (\gamma) \to {\rm Hadrons}]} = \frac{4\pi^2\alpha}{t}\frac{1}{\pi}{\rm Im}\Pi(t)$$

*The Largest Contribution ($\sim$ 75%) comes from $e^+e^- \to \pi^+\pi^-$*

*The Underlying Physics is well understood:*

**Large-$\mathrm{N_c}$ $\rho$ plus pQCD continuum agrees with data at the $\sim$ 10% level**

# Euclidean Formulation

Lattice QCD wants to go Euclidean

$$-\Pi(Q^2) = \int_0^\infty \frac{dt}{t} \underbrace{\frac{Q^2}{t+Q^2}} \frac{1}{\pi} \mathrm{Im}\Pi(t)\,, \quad \text{with euclidean} \quad Q^2 = \frac{x^2}{1-x} m_\mu^2 \geq 0\,.$$

## How to go Euclidean (*Lautrup- de Rafael '69*)

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi} \int_0^1 dx\, (1-x) \int_0^\infty \frac{dt}{t} \underbrace{\frac{\frac{x^2}{1-x} m_\mu^2}{t + \frac{x^2}{1-x} m_\mu^2}} \frac{1}{\pi} \mathrm{Im}\Pi(t)\,,$$

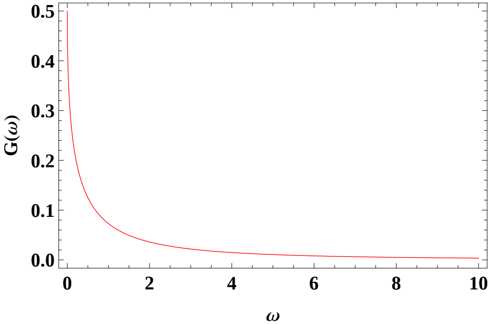
$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \left[ -\Pi\left( \frac{x^2}{1-x} m_\mu^2 \right) \right].$$

*Lattice QCD likes to use:* $\omega = \frac{Q^2}{m_\mu^2} = \frac{x^2}{1-x}\,,$

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi} \int_0^\infty \frac{d\omega}{\omega} \frac{1}{4} \left[ (2+\omega)\left(2+\omega-\sqrt{\omega}\sqrt{4+\omega}\right) - 2 \right] \left( -\omega \frac{d}{d\omega} \Pi\left(\omega m_\mu^2\right) \right)$$

# Comment on Lattice QCD Evaluations

$$a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_0^\infty \frac{d\omega}{\omega}\, \underbrace{\frac{1}{4}\left[(2+\omega)\left(2+\omega-\sqrt{\omega}\sqrt{4+\omega}\right)-2\right]}_{G(\omega)} \underbrace{\left(-\omega\frac{d}{d\omega}\Pi\left(\omega m_\mu^2\right)\right)}_{\text{Adler Function}}$$




Lattice QCD evaluations *-at a few $\omega$ points-* need extrapolations at very low $\omega$ values using *Models and/or Padé Approximants*

$$-\frac{d}{d\omega}\Pi\left(\omega m_\mu^2\right) = \int_{4m_\pi^2}^{\infty}\frac{dt}{t}\frac{m_\mu^2}{t}\underbrace{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{\omega m_\mu^2}{t}\right)^{-s}\Gamma(s)\Gamma(2-s)}_{\left(1+\frac{\omega m_\mu^2}{t}\right)^{-2}}\frac{1}{\pi}\mathrm{Im}\Pi(t)\,.$$

## Mellin-Barnes Integral Representation of $a_\mu^{\mathrm{HVP}}$

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} ds\, \mathcal{F}(s)\, \underbrace{\mathcal{M}(s)}\,, \qquad \mathrm{Re}\; c \in \left]0, +1\right[$$

$$\mathcal{F}(s) = -\Gamma(3-2s)\Gamma(-3+s)\Gamma(1+s)$$

$$\mathcal{M}(s) = \underbrace{\int_{4m_\pi^2}^{\infty}\frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1-s}\frac{1}{\pi}\mathrm{Im}\Pi(t)}_{}$$

*Mellin Transform of the Spectral Function*

*Useful representation to extract the asymptotic expansion for* $\frac{m_\mu^2}{t} < 1$.

## Two types of Moments

Normal Power Moments:

$$\mathcal{M}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1+n} \frac{1}{\pi}\mathrm{Im}\Pi(t)\,, \quad n = 0, 1, 2, \ldots$$

Log Weighted Power Moments (first derivative of the Mellin transform at integer $n < 0$ values):

$$\tilde{\mathcal{M}}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^{1+n} \log\frac{m_\mu^2}{t}\, \frac{1}{\pi}\mathrm{Im}\Pi(t)\,, \quad n = 1, 2, 3, \cdots$$

## Expansion in Moment Approximants

$$\begin{aligned} a_\mu^{\mathrm{HVP}} &= \left(\frac{\alpha}{\pi}\right)\left\{\frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1)\right. \\ &+ \frac{97}{10}\mathcal{M}(-2) + 6\tilde{\mathcal{M}}(-2) \\ &+ \left.\frac{208}{5}\mathcal{M}(-3) + 28\tilde{\mathcal{M}}(-3) + \mathcal{O}\left[\tilde{\mathcal{M}}(-4)\right]\right\} \end{aligned}$$

*These moments are known phenomenologically from $e^+e^-$ data (M. Davier, private communication)*

# The Moment Approximants in a Phenomenological Toy Model

$$a_\mu^{\rm HVP}(e^+e^-) = (6.923 \pm 0.042) \times 10^{-8} \quad (0.6\%)$$

*M. Davier et al' 10*

$$a_\mu^{\rm HVP}(\text{toy model}) = 6.936 \times 10^{-8}$$

*D. Bernecker and H.B. Meyer, '11; L. Lellouch, '14*

## Numerical Values of the Moment Approximants (Toy Model)

$$\left(\frac{\alpha}{\pi}\right)\frac{1}{3}\mathcal{M}(0) = 8.071 \times 10^{-8} \quad (16\%)$$

$$\left(\frac{\alpha}{\pi}\right)\left[\frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1)\right] = 7.240 \times 10^{-8} \quad (4\%)$$

$$\left(\frac{\alpha}{\pi}\right)\left[\frac{1}{3}\mathcal{M}(0) + \frac{25}{12}\mathcal{M}(-1) + \tilde{\mathcal{M}}(-1) + \frac{97}{10}\mathcal{M}(-2) + 6\tilde{\mathcal{M}}(-2)\right] = 7.022 \times 10^{-8} \quad (1\%)$$

*Fourth Approximation is already within 0.4% of the toy model result*

## The Leading Moment provides a rigorous upper bound to $a_\mu^{\rm HVP}$

*J.S. Bell-de Rafael '69: the operator $\partial^\lambda F^{\mu\nu}\partial_\lambda F_{\mu\nu}$ governs low-energy hadronic QED observables*

$$a_\mu^{\rm HVP} < \left(\frac{\alpha}{\pi}\right)\frac{1}{3}\underbrace{\int_{4m_\pi^2}^\infty \frac{dt}{t}\frac{m_\mu^2}{t}\frac{1}{\pi}{\rm Im}\Pi(t)}_{\mathcal{M}(0)} = \left(\frac{\alpha}{\pi}\right)\frac{1}{3}\underbrace{\left(-m_\mu^2\frac{d}{dQ^2}\Pi(Q^2)\right)_{Q^2=0}}_{\text{Lattice QCD}}$$

- The bound overestimates $a_\mu^{\rm HVP}$ by less than 18% (not bad for a rigorous bound)
- The slope of $\Pi\left(Q^2\right)$ at the origin (r.h.s.) can be (*has been* ?) evaluated in lattice QCD
- *It is difficult to imagine that, unless lattice QCD does better than phenomenology in this simple case, it will ever reach a competitive accuracy of the full determination of $a_\mu^{\rm HVP}$.*

## $\mathcal{M}(-n)$ Moments correspond to successive derivatives of $\Pi(Q^2)$ at $Q^2=0$

$$\underbrace{\mathcal{M}(-n)}_{n=0,1,2\dots} = \int_{4m_\pi^2}^\infty \frac{dt}{t}\left(\frac{m_\mu^2}{t}\right)^{1+n}\frac{1}{\pi}{\rm Im}\Pi(t) = \frac{(-1)^{n+1}}{(n+1)!}(m_\mu^2)^{n+1}\left(\frac{\partial^{n+1}}{(\partial Q^2)^{n+1}}\Pi(Q^2)\right)_{Q^2=0}$$

These derivatives can (*should*) be determined in Lattice QCD

$$\tilde{\mathcal{M}}(-n) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left(\frac{m_\mu^2}{t}\right)^n \log \frac{m_\mu^2}{t} \frac{1}{\pi} \mathrm{Im}\Pi(t)$$

They require the evaluation of integrals of the type

## Integrals in the Euclidean to be evaluated in lattice QCD

$$\Sigma(-n) \equiv \int_{4m_\pi^2}^{\infty} dQ^2 \left(\frac{m_\mu^2}{Q^2}\right)^{n+1} \left(-\frac{\Pi(Q^2)}{Q^2}\right) \quad n = 1, 2, 3 \ldots$$

Example:

$$\tilde{\mathcal{M}}(-1) = -\log \frac{4m_\pi^2}{m_\mu^2} \underbrace{\mathcal{M}(-1)}_{\text{Latt. QCD}} + \underbrace{\Sigma(-1)}_{\text{Latt. QCD}} - \frac{m_\mu^2}{4m_\pi^2} \underbrace{\mathcal{M}(0)}_{\text{Latt. QCD}} + \mathcal{O}\left[\mathcal{M}(-2)\right]$$

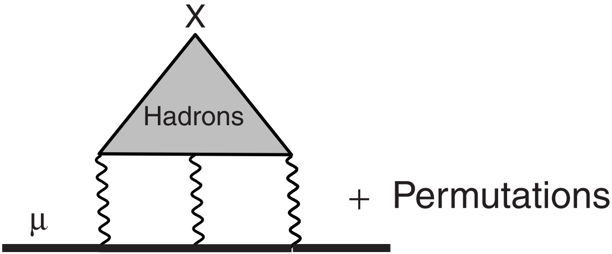
- Contrary to the evaluation of $a_\mu^{\mathrm{HVP}}$, the Euclidean moments $\Sigma(-1)$, $\Sigma(-2)$, ... are not weighted by a heavily peaked kernel at small $Q^2$.
- The threshold of integration is at a rather large value $Q^2 = 4m_\pi^2$ instead of zero.
- The determination of these Euclidean moments in lattice QCD and their comparison with the corresponding phenomenological expressions in terms of the hadronic spectral function, provide *valuable further tests.*

## Conclusions about the HVP Contribution

- Present Lattice QCD determinations of HVP in the Euclidean *need to be complemented* by approximation methods in order to get $a_\mu^{\rm HVP}$.

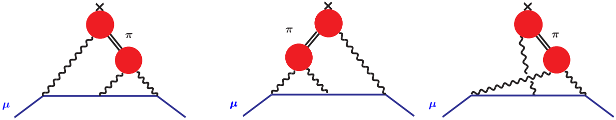
- The *moment analysis* approach may gradually lead to an accurate determination of $a_\mu^{\rm HVP}$, providing at the same time many tests of *lattice QCD evaluations* to be confronted with phenomenological determinations using experimental data.

- This workshop is a good place to discus *optimal lattice strategies* and *optimal approximation methods* to obtain, eventually, a *robust determination* of $a_\mu^{\rm HVP}$ which can be confronted with the determinations from $e^+e^-$ data.


X
Hadrons
μ
+ Permutations

## *Sponteneous Chiral Symmetry Breaking in QCD*

- Implies a spectrum with *GOLDSTONE PARTICLES (pions)* and a *MASS GAP M* to the other hadronic states.
- The HLbyL contribution to $a_\mu$ in Large-$\mathrm{N_c}$QCD and in the limit where $m_{u,d,s} \to 0$ and $M$ Large is known from $\chi$PT with the point-like WZW couplings:




### HLbyL Contribution to the Muon Anomaly in Chiral Limit with $M$ Large

$$a_\mu^{(\mathrm{HLbyL})} = \underbrace{\left(\frac{\alpha}{\pi}\right)^3 \mathrm{N_c}^2 \frac{m_\mu^2}{16\pi^2 f_\pi^2}\left[\frac{1}{3}\log^2\frac{M}{m_\pi} + \mathcal{O}\left(\log\frac{M}{m_\pi}\right) + \mathcal{O}(1)\right]}_{Knecht-Nyffeler-Perrottet-de\ Rafael'02} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3 \mathrm{N_c}\frac{m_\mu^2}{M^2}\right)$$

## HLbyL Contribution to the Muon Anomaly in Chiral Limit with $M \to \infty$

$$a_\mu^{(\mathrm{HLbyL})} = \underbrace{\left(\frac{\alpha}{\pi}\right)^3 \mathrm{N_c}^2 \frac{m_\mu^2}{16\pi^2 f_\pi^2}\left[\frac{1}{3}\log^2\frac{M}{m_\pi}\right.}_{95\times 10^{-11}\ \text{for}\ M=M_\rho} \left. + \mathcal{O}\left(\log\frac{M}{m_\pi}\right) + \mathcal{O}(1)\right] + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3 \mathrm{N_c}\frac{m_\mu^2}{M^2}\right)$$

- Clearly, *in the M-Large limit*, the $\log^2 \frac{M}{m_\pi}$ term dominates.
- Once $m_\mu^2$ factored out, the pion mass is the *infrared cut-off*.

However, in our World

- The mass gap of the hadronic spectrum $M = M_\rho$ (*is not that large*) and $m_\pi$ is *bigger* than $m_\mu$.
- Therefore, in practice one has to worry about $\mathcal{O}\left(\log\frac{M}{m_\pi}\right)$, $\mathcal{O}(1)$, $\mathcal{O}\left(\mathrm{N_c}\frac{m_\mu^2}{M^2}\right)$ corrections and $\frac{m_\mu}{m_\pi}$ dependence.
- Furthermore, subleading corrections in $1/\mathrm{N_c}$ (*pion-loop contribution*), will likely become relevant at the wanted level of accuracy.

## Comments on the term $\left(\frac{\alpha}{\pi}\right)^3 N_c{}^2 \frac{m_\mu^2}{16\pi^2 f_\pi^2}\mathcal{O}\left(\log\frac{M}{m_\pi}\right)$

There are two contributions to this term

- i) from the point-like contribution (two loops) *Ramsey-Musolf and Wise' 02*
- ii) from the effective couplings (one loop) *Savage, Luke and Wise' 92*:

$$\frac{3i}{32}\left(\frac{\alpha}{\pi}\right)^2 \bar{l}\gamma^\mu\gamma_5 l\left[\chi_1 \mathrm{tr}(Q_R Q_R D_\mu U U^\dagger - Q_L Q_L D_\mu U^\dagger U) + \chi_2 \mathrm{tr}(U^\dagger Q_R D_\mu U Q_L - U Q_L D_\mu U^\dagger Q_R)\right]$$

which contribute to $\pi^0 \to e^+e^-$ *Knecht-Peris-Perrottet-de Rafael'99*

- The relevant combination: $\chi = -\frac{1}{4}(\chi_1 + \chi_2)$ has been recently extracted from KTeV data on $\pi^0 \to e^+e^-$ decays *in the presence of radiative corrections* *Vasko, Novotny '11, Husek, Kampf and Novotny '14* (In $\frac{N_c}{3}$ units): $\chi_{(\text{phen.})}(M_\rho) = 4.5 \pm 1.0$

  As a reference: $\chi_{(\text{Large } N_c \text{ estimate})}(M_\rho) = 2.2 \pm 0.9$; $\chi(M \text{ Large}) = \frac{11}{4}$.

### HLbyL Contribution with next to leading term and $M = M_\rho$

$$a_\mu^{(\text{HLbyL})} = \underbrace{\underbrace{\left(\frac{\alpha}{\pi}\right)^3 N_c{}^2 \frac{m_\mu^2}{48\pi^2 f_\pi^2}}_{95\times 10^{-11}} \left\{\log^2\frac{M_\rho}{m_\pi} + \left[\underbrace{-f\left(\frac{m_\pi^2}{m_\mu^2}\right) + \frac{1}{2}}_{RM-W} \underbrace{-\frac{2}{3}\chi_{(\text{phen.})}(M_\rho)}_{KNPdeR}\right]\log\frac{M_\rho}{m_\pi}\right.}_{7.7\times 10^{-11} \quad (71\times 10^{-11} \text{ with } \chi = \frac{11}{4})} \left. + \mathcal{O}(1)\right\}$$

## Comments on Terms Contributing at $\left(\frac{\alpha}{\pi}\right)^3 N_c \mathcal{O}\left(\frac{m_\mu^2}{M^2}\right)$

When the photon momenta are small with respect to the *mass gap scale M* , the HLbyL tensor must be reproduced by an effective Lagrangian of the Euler-Heisenberg type:

$$\mathcal{L}_{\rm eff}(E,H) = \frac{1}{16\pi^2 M^4} e^4 \left\{ \lambda_1 \left[F_{\mu\nu}(x)F^{\mu\nu}\right]^2 + \lambda_2 \left[\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right]^2 \right\}$$

with $\lambda_1$, $\lambda_2$ dimensionless couplings of $\mathcal{O}(N_c)$.
As a guidance, in the C$\chi$QM

$$\frac{\lambda_1}{M^4} = N_c \frac{2}{9}\frac{1}{90}\frac{1}{M_Q^4} \quad \text{and} \quad \frac{\lambda_2}{M^4} = N_c \frac{2}{9}\frac{7}{360}\frac{1}{M_Q^4}$$

For $M_Q = (240 \pm 10)$ MeV one finds

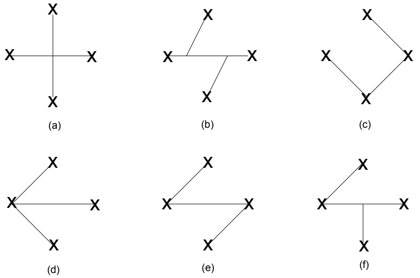
$$\frac{\lambda_1}{M^4} = (2.23 \pm 0.74)\ \text{GeV}^{-4} \quad \text{and} \quad \frac{\lambda_2}{M^4} = [3.91 \pm 1.30)\ \text{GeV}^{-4}$$

of the same order of magnitude as expected in a Large–$N_c$ lowest pole saturation:

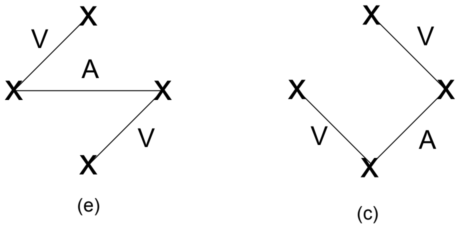
$$\mathcal{O}\left(\frac{\lambda_i}{M^4}\right) \sim \mathcal{O}\left(\frac{N_c}{M_\rho^4}\right) = 8.3\ \text{GeV}^{-4}$$

**Insertion** of the Euler-Heisenberg effective coupling in the HLbyLS contribution to $a_\mu$ leads to a **quadratic divergent integral** which, when **cut-off** at $M^2$, gives the claimed order of magnitude: $\left(\frac{\alpha}{\pi}\right)^3 N_c \mathcal{O}\left(\frac{m_\mu^2}{M^2}\right)$.

## Topologies of the HLbyL Four-Point Function




- (a) The central vertex has to be a four resonance coupling $\mathcal{RRRR}$ of $\mathcal{O}\left(\frac{1}{N_c}\right)$.
- (b) Each vertex has to be a $\mathcal{RRR}$ coupling of $\mathcal{O}\left(\frac{1}{\sqrt{N_c}}\right)$.
- (c) The $\mathcal{RR}\gamma$ vertices are each of $\mathcal{O}(1)$.
- (d) The $\mathcal{RRR}\gamma$ vertex is of $\mathcal{O}\left(\frac{1}{\sqrt{N_c}}\right)$
- (e) The $\mathcal{RR}\gamma$ vertices here are like those of (c) i.e. $\mathcal{O}(1)$
- (f) There is a vetex $\mathcal{RR}\gamma$ of $\mathcal{O}(1)$ and a vertex $\mathcal{RRR}$ of $\mathcal{O}\left(\frac{1}{\sqrt{N_c}}\right)$.




Relevant Large-$N_c$ Chiral Lagrangian (*V and A tensor-like formulation*)

$$\mathcal{L}_{\text{int.}} = \underbrace{\frac{F_V}{2\sqrt{2}} \text{tr}\, V_{\mu\nu} f_+^{\mu\nu}}_{Ecker-Gasser-Pich-de\ Rafael\ '89} + \underbrace{H_{V,A}\, g_{\rho\sigma} \epsilon_{\mu\nu\alpha\beta} \text{tr}\, \{V^{\mu\nu}, A^{\alpha\rho}\} f_+^{\beta\sigma}}_{Kampf-Novotny\ '11}$$

$$F_V \simeq \sqrt{2} f_\pi\,,$$

$$|H_{V,A}| \simeq 0.80 \quad \text{from} \quad f_1(1285) \to \rho^0 + \gamma\,.$$

# Very Preliminary Results

- From the *Leading Topologies* one gets an Effective Lagrangian, Euler-Heisenberg like:

$$\mathcal{L}_{\rm LbyL} = \frac{4}{3} e^4 \, \frac{F_V^2 |H_{(V,A)}|^2}{M_V^4 M_A^2} \left(\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}\right)^2 \,,$$

with $\lambda_1 = 0$ and $\frac{\lambda_2}{M^4} = \frac{16\pi^2 f_\pi^2}{M_V^2} \frac{|H_{(V,A)}|^2}{M_V^2 M_A^2} \frac{8}{3} \simeq 4.0 \, {\rm GeV}^{-4}$ (very close to the $\lambda_2$ coupling of the C$\chi$QM).

- *Quick Qualitative Estimate:*

$$a_\mu^{\rm (HLbyL)}({\rm quark\ loop}) \sim \left(\frac{\alpha}{\pi}\right)^3 \frac{m_\mu^2}{M_V^2} \frac{16 f_\pi^2}{M_V^2} |H_{(V,A)}|^2 4\pi \frac{M^2}{M_A^2} \,,$$

which, for a cut-off $M = M_\rho$, gives

$$a_\mu^{\rm (HLbyL)}({\rm quark\ loop}) \sim 16 \times 10^{-11} \,.$$

# Conclusions about the HLbyL Contribution

- The main underlying Physics of the HLbyL contribution is *Qualitatively* well understood.

- Elaborated phenomenological approaches may gradually lead to a *More Accurate* determination of $a_\mu^{\mathrm{HLbyL}}$. They should be checked, however, at all possible stages with the simple expectations from the underlying Physics.

- Lattice QCD should consider computing the relevant effective couplings:

  $\chi(M_\rho)$ from $\pi^0 \to e^+e^-$

  $\lambda_1$ and $\lambda_2$ of the *Hadronic Euler-Heisenberg* effective Lagrangian

- This workshop should be a good place to discus *Possible QCD Lattice Strategies* to compute $a_\mu^{\mathrm{HLbyL}}$ and to *Suggest Tests*.