Dispersion relations: recent and future applications

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FOR FUNDAMENTAL PHYSICS

High-precision QCD at low energy
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Outline

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Dispersion relations: basics I

- analyticity properties of Green's functions (and form factors and scattering amplitudes) can be rigorously established
- the absence of singularities for complex (unphysical) values of kinematic variables[1] follows from causality
- the presence of singularities is related to dynamical phenomena (exchange of particles) and can be understood in terms of the underlying dynamics
- analytic functions are determined by their singularities: dispersion relations provide an explicit representation of this mathematical property
- QFTs satisfy these properties automatically.
  Weinberg: QFT emerges by imposing analyticity and unitarity (and other properties)

[1] Exceptions known: anomalous thresholds.

Dispersion relations: basics II

- dispersion relations are exact
- their usefulness is directly related to our knowledge of the singularities of the function of interest
- depending on where one wants to calculate the function, some singularities (or regions thereof) may be more important than others: approximation schemes may be successfully applied
- singularities at infinity = subtraction constants, if present are essential input
- use of dispersion relations in combination with QFT calculations (whether perturbative or not) is always possible

Outline

Analytic properties of pion form factors

Mathematical problem:

1. $F(t)$ is an analytic function of t in the whole complex plane, with the exception of a cut for $4M_\pi^2 \leq t < \infty$;
2. approaching the real axis from above $e^{-i\delta(t)}F(t)$ is real on the real axis, where $\delta(t)$ is a known function.

Omnès ('58) found an exact solution to this problem:

$$F(t) = P(t)\Omega(t) = P(t)\exp\left\{\frac{t}{\pi}\int_{4M_\pi^2}^\infty \frac{dt'}{t'}\frac{\delta(t')}{t'-t}\right\},$$

where $P(t)$ is a polynomial which can only be constrained by the behaviour of $F(t)$ for $t \to \infty$, or by the presence of zeros.

$\Omega(t)$ is called the Omnès function

Scalar form factor of the pion

$$\Gamma_\pi(s) := \langle \pi(p')| m_u \bar{u}u + m_d \bar{d}d |\pi(p)\rangle \qquad s = (p'-p)^2$$

Value at $s = 0$ is the pion σ-term:

$$\Gamma_\pi(0) = m_u \frac{\partial M_\pi^2}{\partial m_u} + m_d \frac{\partial M_\pi^2}{\partial m_d} = M_\pi^2 + \mathcal{O}(m_q^2)$$

Omnès representation (assuming the absence of zeros):

$$\Gamma_\pi(s) = \Gamma_\pi(0)\Omega_\Gamma(s) \qquad \ln \Omega_\Gamma(s) = \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_\Gamma(s')}{s'(s'-s)}$$

Watson's theorem:

$$\delta_\Gamma(s) = \delta_0^0(s) \ \text{ for } s < 4M_K^2$$

negligible inelasticity due to 4π's

Scalar form factor of the pion

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Watson's theorem:

$$\Rightarrow \Omega_\Gamma(s) = \Omega_0^0(s)\cdot\exp\left[\frac{s}{\pi}\int_{4M_K^2}^\infty ds' \frac{\delta_\Gamma(s') - \delta_0^0(s')}{s'(s'-s)}\right] \simeq \Omega_0^0(s)\left(1 + c_1\frac{s}{4M_K^2} + \ldots\right)$$

Scalar form factor of the pion

Conclusions:

- the low-energy behaviour of $\Gamma(s)$ is determined to a large extent by the $\pi\pi$ phase shift $\delta_0^0(s)$
- the normalization of the form factor is fixed by the subtraction constant $\Gamma_\pi(0)$, the σ-term of the pion
- inelastic effects ($\bar{K}K$ channel) may be sizeable, but are well described by a smooth function at low energy

Scalar form factor of the pion

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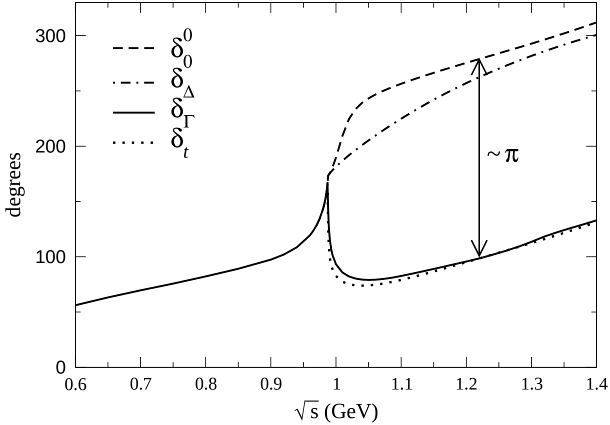
- inelastic effects ($\bar{K}K$ channel) may be sizeable, but are well described by a smooth function at low energy
- to have the latter under control a coupled-channel analysis is necessary
  Donoghue, Gasser, Leutwyler, 1990
- this leads to an accurate prediction for the scalar radius of the pion
  GC, Gasser, Leutwyler, 2001

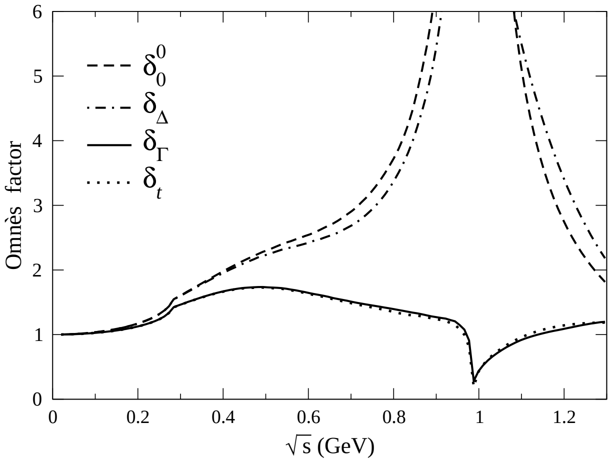
$$\langle r^2\rangle_s^\pi = \frac{6}{\pi}\int_{4M_\pi^2}^\infty \frac{ds\,\delta_\Gamma(s)}{s^2} = 0.61 \pm 0.04\mathrm{fm}^2$$

Scalar form factor: dispersive representation



Ananthanarayan, Caprini, GC, Gasser, Leutwyler, 2004

Scalar form factor: dispersive representation



Vector form factor of the pion

A similar discussion can be made for the vector form factor

$$\langle \pi^i(p')|V_\mu^k(0)|\pi^l(p)\rangle = i\epsilon^{ikl}(p'+p)_\mu F_V^\pi(s) \qquad s=(p'-p)^2$$

- the normalization (subtraction constant) is fixed by gauge invariance:
$$F_V^\pi(0)=1$$
- for this form factor there are data coming from $e^+e^- \to \pi^+\pi^-$ which allow one to pin down the free parameters in the Omnès representation

Omnès representation including isospin breaking

- Omnès representation

$$F_V^\pi(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta(s')}{s'(s'-s)}\right] \equiv \Omega(s)$$

- Split elastic from inelastic contributions

$$\delta = \delta_1^1 + \delta_{\text{in}} \quad \Rightarrow \quad F_V^\pi(s) = \Omega_1^1(s)\Omega_{\text{in}}(s)$$

Eidelman-Lukaszuk: unitarity bound on δ_{in}

$$\sin^2\delta_{\text{in}} \leq \frac{1}{2}\left(1-\sqrt{1-r^2}\right) \quad r = \frac{\sigma^{I=1}_{e^+e^-\to\neq 2\pi}}{\sigma_{e^+e^-\to 2\pi}}$$

$$\Rightarrow \quad \text{Im}\Omega_{\text{in}}(s) \simeq 0 \qquad s \leq (M_\pi + M_\omega)^2$$

- $\rho-\omega$–mixing must also be explicitly taken into account

$$F_V(s) = \Omega_{\pi\pi}(s)\Omega_{\text{in}}(s)G_\omega(s)$$

Free parameters

$$\Omega_1^1(s) \Rightarrow \begin{cases} \phi_0 = \delta_{\pi\pi}((0.8\ \mathrm{GeV})^2) \\ \phi_1 = \delta_{\pi\pi}(68 M_\pi^2) \end{cases}$$

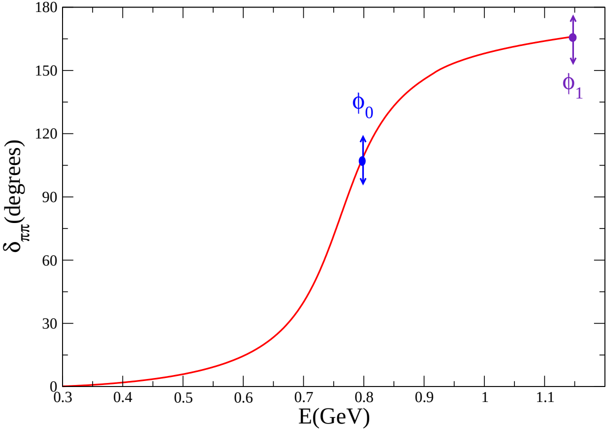
$$G_\omega(s) \Rightarrow \begin{cases} \epsilon & \omega - \rho \text{ mixing} \\ M_\omega & \end{cases}$$

$$\Omega_{\mathrm{in}}(s) \Rightarrow \begin{cases} c_1 \\ \vdots \\ c_P \end{cases} \qquad \mathrm{Im}\Omega_{\mathrm{in}}(s) = 0 \quad s \leq s_{\mathrm{in}}$$

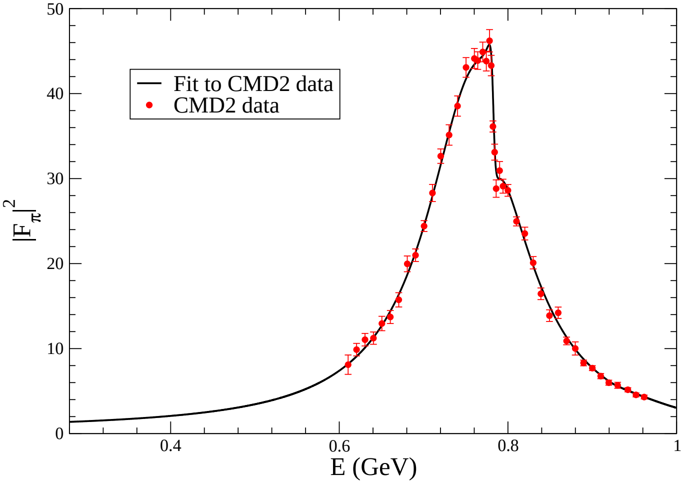
$$G_\omega(s) = 1 + \epsilon \frac{s}{s_\omega - s} \qquad \text{where} \qquad s_\omega = (M_\omega - i\Gamma_\omega/2)^2$$

$$\Omega_{\mathrm{in}}(s) = 1 + \sum_{k=1}^{n} c_k (z(s)^k - z(0)^k) \qquad z = \frac{\sqrt{s_{\pi\omega} - s_1} - \sqrt{s_{\pi\omega} - s}}{\sqrt{s_{\pi\omega} - s_1} + \sqrt{s_{\pi\omega} - s}}$$

Free parameters



Outcome of the fit



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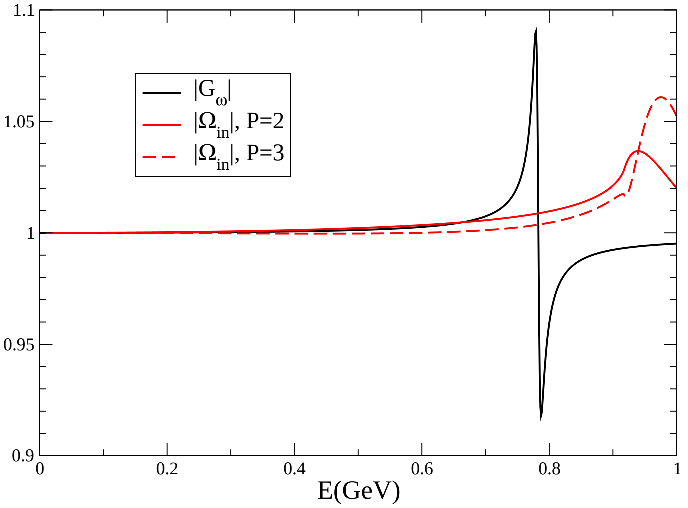
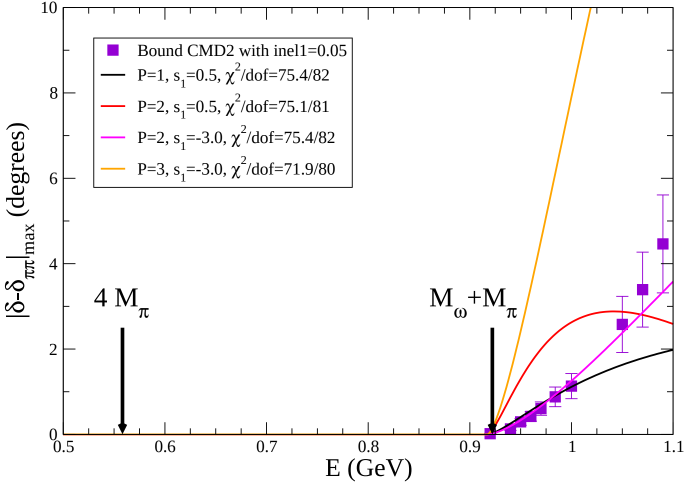


P = number of parameters in Ω_{in}

Outcome of the fit



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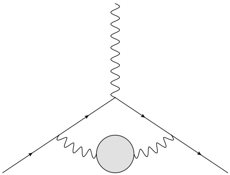
Hadronic vacuum polarization and $(g-2)_\mu$

The vector form factor of the pion represents the input in the dispersion relation for hadronic vacuum polarization



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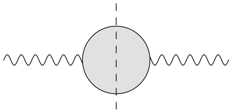


- in this case the dispersion relation is already the solution

$$\bar{\Pi}^{2\pi}(s) = \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{|F_V^\pi(s')|^2}{s'(s'-s)}$$

- input the vector form factor and calculate the integral
- the extension to other intermediate state is trivial:

$$|F_V^\pi(s')|^2 \to |\mathcal{M}(\gamma^* \to \text{hadrons})|^2$$

Outline

Roy equations

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

S.M. Roy (71)

$$
\begin{aligned}
\mathrm{Re}\, t_0^0(s) &= k_0^0(s) + \int_{4M_\pi^2}^{s_0} ds' K_{0\,0}^{0\,0}(s,s')\,\mathrm{Im}\, t_0^0(s') \\
&+ \int_{4M_\pi^2}^{s_0} ds' K_{0\,1}^{0\,1}(s,s')\,\mathrm{Im}\, t_1^1(s') \\
&+ \int_{4M_\pi^2}^{s_0} ds' K_{0\,0}^{0\,2}(s,s')\,\mathrm{Im}\, t_0^2(s') + f_0^0(s) + d_0^0(s) \\
k_0^0(s) &= a_0^0 + \frac{s-4M_\pi^2}{12M_\pi^2}(2a_0^0 - 5a_0^2) \\
f_0^0(s) &= \sum_{I'=0}^{2}\sum_{\ell'=0}^{1}\int_{s_0}^{s_3} ds' K_{0\ell'}^{0I'}(s,s')\,\mathrm{Im}\, t_{\ell'}^{I'}(s') \\
d_0^0(s) &= \text{all the rest}
\end{aligned}
$$

$[\sqrt{s_0} = 0.8\mathrm{GeV} \quad \sqrt{s_3} = 2\mathrm{GeV}]$

Roy equations

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

S.M. Roy (71)

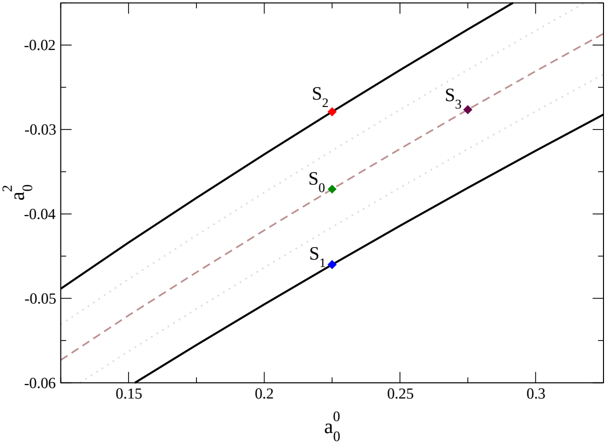
Numerical solutions of the Roy equations

Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
Ananthanarayan, GC, Gasser and Leutwyler (00)
Descotes-Genon, Fuchs, Girlanda and Stern (01)
Kamiński, Peláez and Ynduráin (08)
Garcia-Martin, Kamiński, Peláez, Ruiz de Elvira, Ynduráin (11)

Input: S- and P-wave imaginary parts above 0.8 GeV
imaginary parts of all higher waves
two subtraction constants, *e.g.* a_0^0 and a_0^2

Output: the full $\pi\pi$ scattering amplitude below 0.8 GeV
Note: a_0^0, a_0^2 inside the universal band $\Rightarrow$ the solution is unique

Numerical solutions



Ananthanarayan, GC, Gasser, Leutwyler (00)

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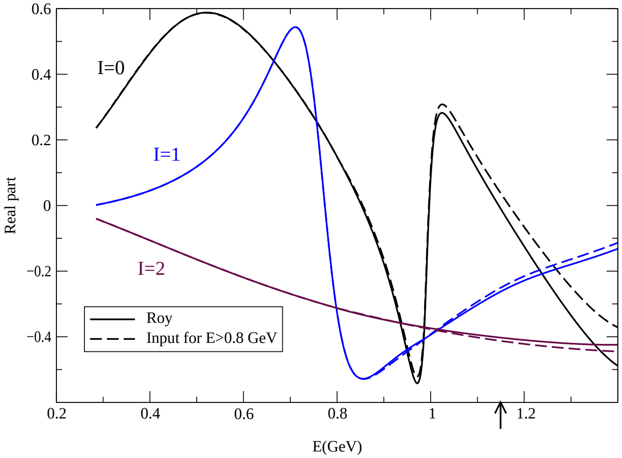

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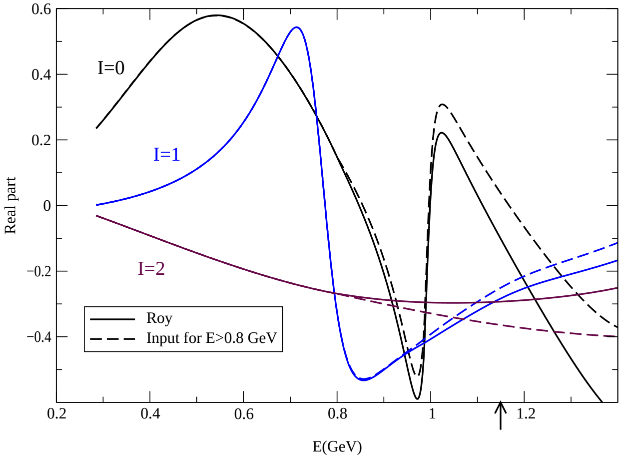

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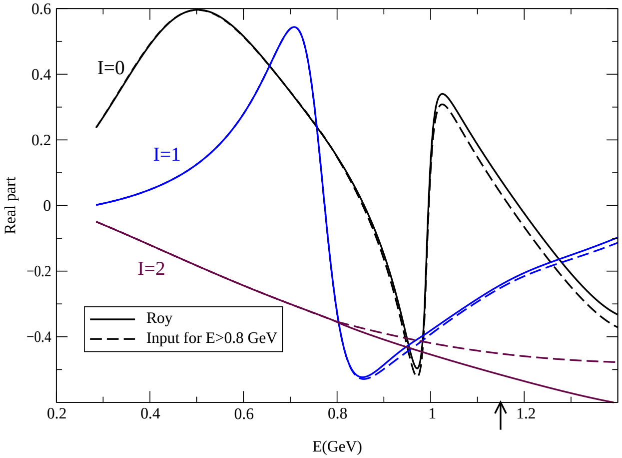
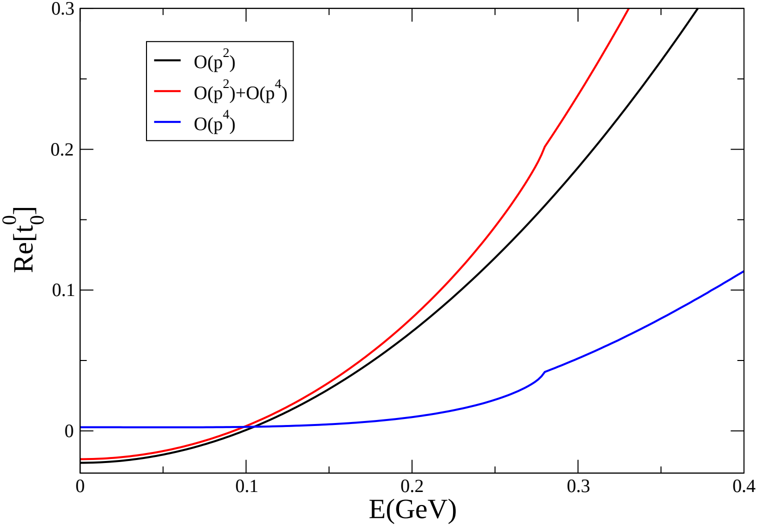


Ananthanarayan, GC, Gasser, Leutwyler (00)

Roy + ChPT

- at fixed input above 0.8 GeV, the only free parameters in the Roy equations are the two S-wave scattering lengths;
- chiral perturbation theory predicts these
- actually the most reliable prediction is for the $\pi\pi$ amplitude below threshold
- we have fixed the two subtraction constants in this way

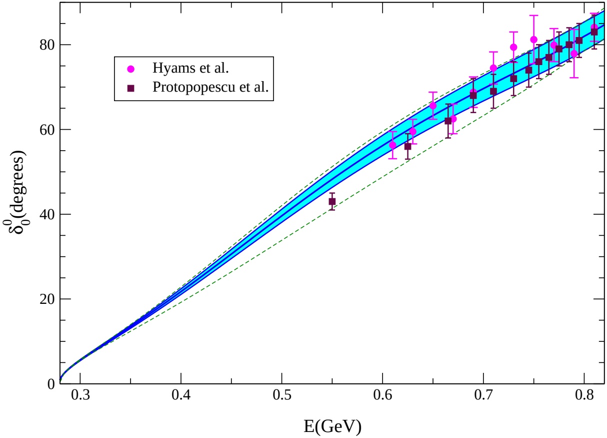
Roy + ChPT



Roy+ChPT: final results

GC, Gasser and Leutwyler (01)

Phase shifts:



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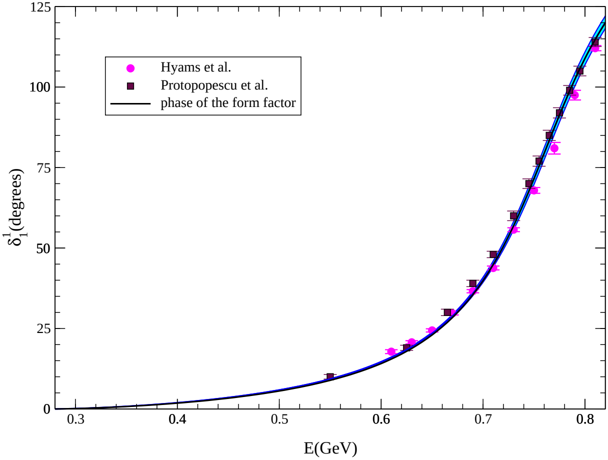


Roy+ChPT: final results

GC, Gasser and Leutwyler (01)

Scattering lengths

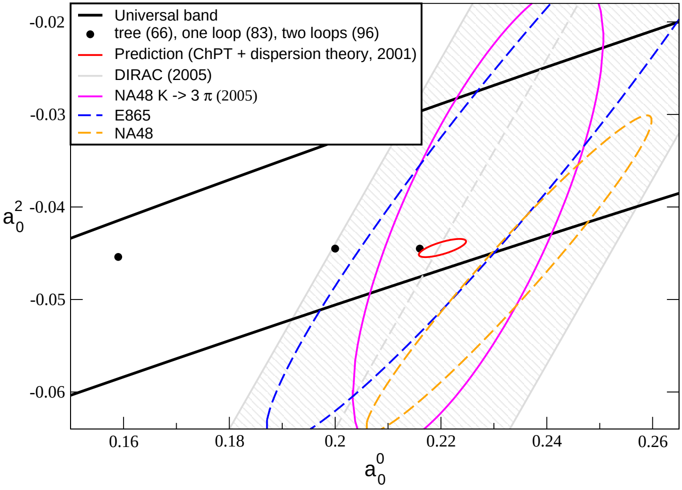
$$
\begin{aligned}
a_0^0 &= 0.220 \pm 0.001 + 0.009\Delta\ell_4 - 0.002\Delta\ell_3 \\
10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.01\Delta\ell_4 - 0.004\Delta\ell_3
\end{aligned}
$$

where $\bar{\ell}_4 = 4.4 + \Delta\ell_4$ $\quad\bar{\ell}_3 = 2.9 + \Delta\ell_3$

Adding errors in quadrature $\quad[\Delta\ell_4 = 0.2,\ \Delta\ell_3 = 2.4]$

$$
\begin{aligned}
a_0^0 &= 0.220 \pm 0.005 \\
10 \cdot a_0^2 &= -0.444 \pm 0.01 \\
a_0^0 - a_0^2 &= 0.265 \pm 0.004
\end{aligned}
$$

Experimental tests



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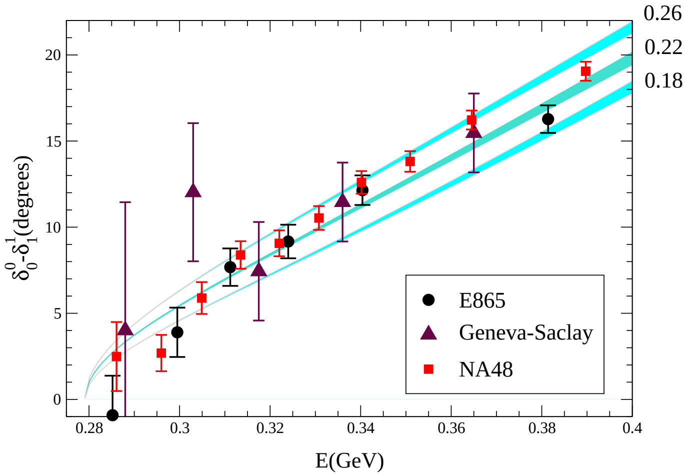

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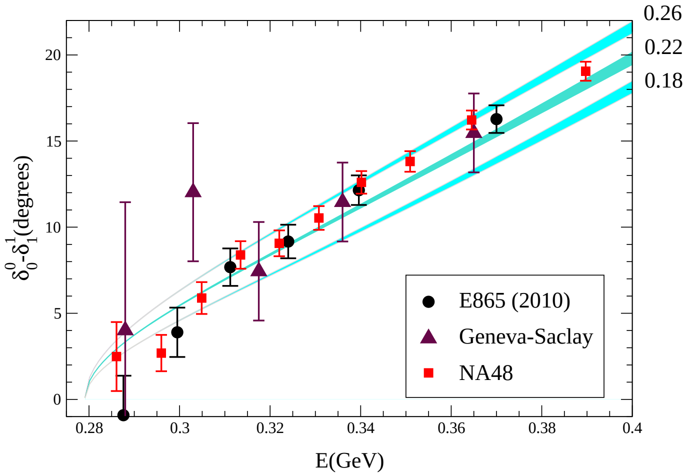
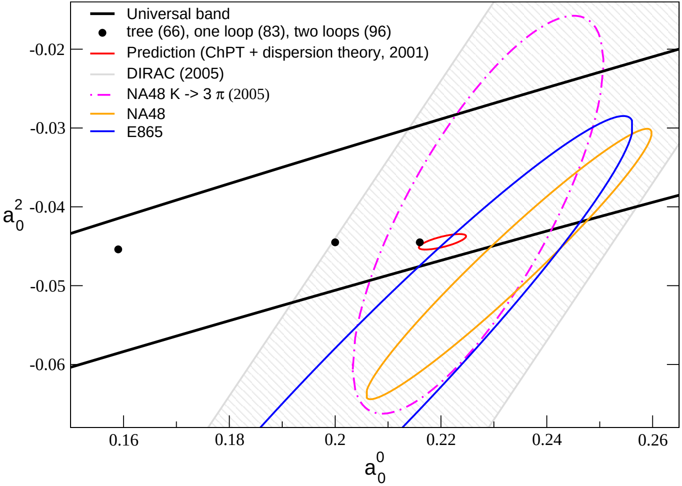


E865 corrected their data

Experimental tests



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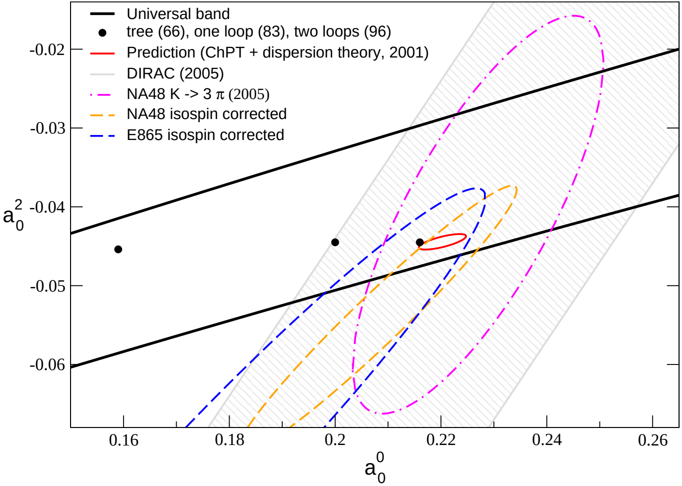
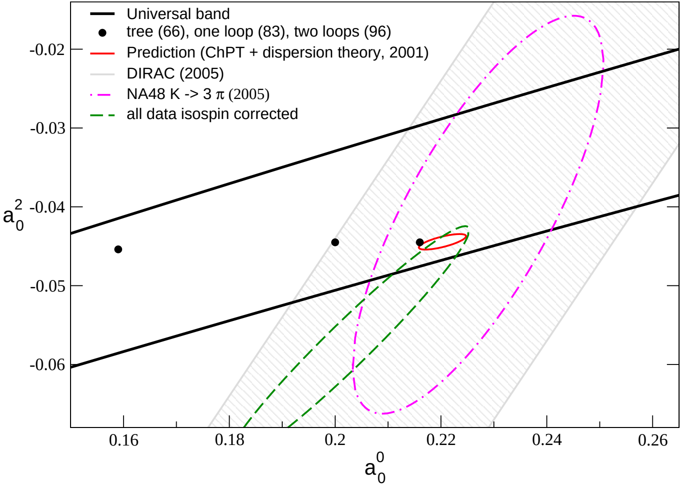


isospin breaking corrections recently calculated for K_{e4} are essential at this level of precision GC, Gasser, Rusetsky (09)

Experimental tests



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Experimental tests

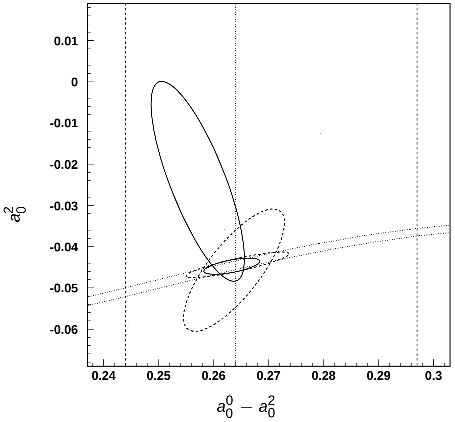


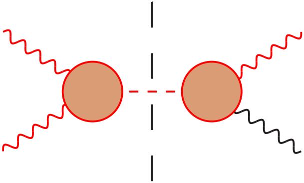
Figure from NA48/2 Eur.Phys.J.C64:589,2009

Outline

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: known

Projection on the BTT basis: done

Our master formula=explicit expressions in the literature

Setting up the dispersive calculation

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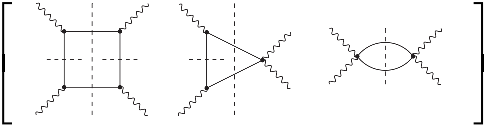
In JHEP '14:

$$F^V_\pi(q_1^2) F^V_\pi(q_2^2) F^V_\pi(q_3^2) \times$$



Contribution with two simultaneous cuts

– analytic properties like the box diagram in sQED

– triangle and bulb diagram required by gauge invariance

– multiplication with F^V_π gives the correct q^2 dependence

Claim: FsQED is not an approximation!

Setting up the dispersive calculation

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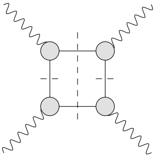
Now, with BTT:



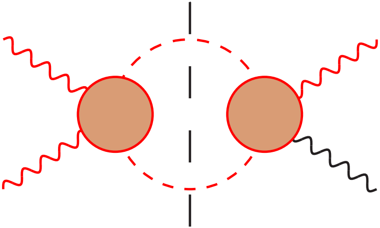
– we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole
– we have explicitly checked that this is identical to FsQED

Proven: FsQED is not an approximation!

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



The “rest” with 2π intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

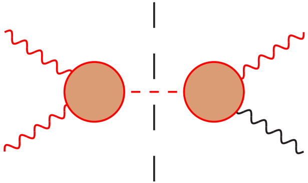
We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states will be neglected for the time being

Dispersive analysis of the pion transition form factor

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



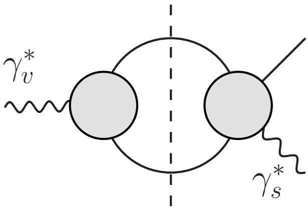
Pion pole: $\Pi_i^{\pi^0\text{-pole}}(s,t,u) = \frac{\rho_{i;s}}{s-M_\pi^2} + \frac{\rho_{i;t}}{t-M_\pi^2} + \frac{\rho_{i;u}}{u-M_\pi^2}$

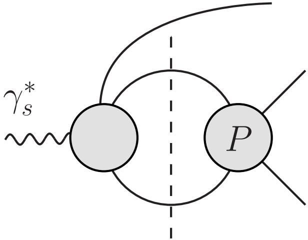
$$\rho_{i,s} = \delta_{i1}\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2),$$
$$\rho_{i,t} = \delta_{i2}\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,q_4^2),$$
$$\rho_{i,u} = \delta_{i3}\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_4^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,q_3^2),$$

Dispersive analysis of the pion transition form factor

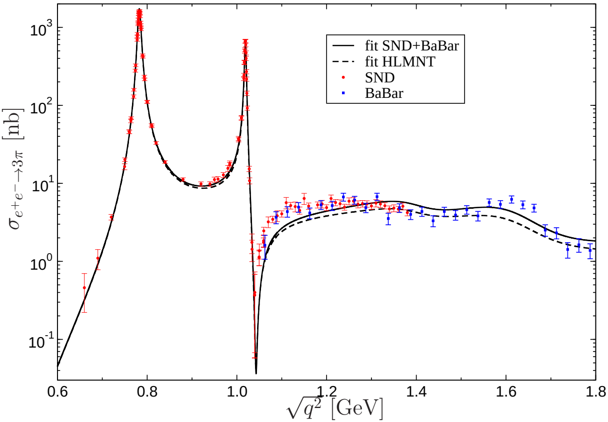
Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

- To calculate the pion-pole contribution the crucial ingredient is the pion transition form factor
- a dispersive representation thereof requires as input:
  - the pion vector form factor [dispersive repr. well known]
  - the $\gamma^* \to 3\pi$ amplitude [analyzed dispersively in this work]
  - the $\pi\pi$ scattering amplitude [dispersive repr. well known]



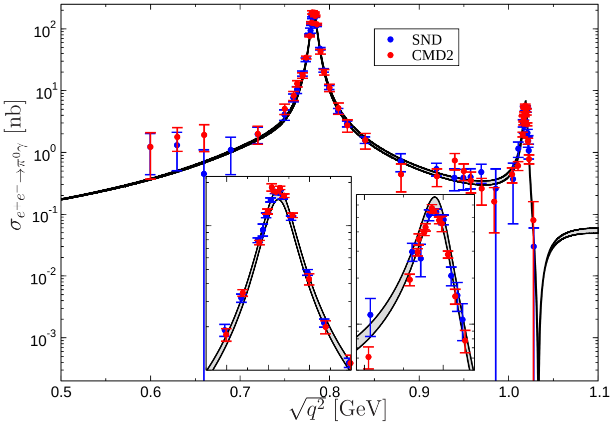


Results for $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$



fit to $\sigma(e^+e^- \to 3\pi)$ Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

Results for $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$



prediction for $\sigma(e^+e^- \to \pi^0\gamma)$ Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

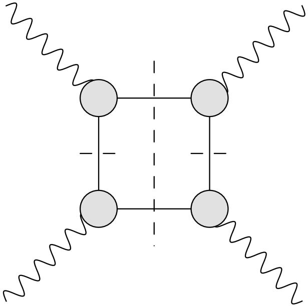
Results for $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$

Results for the doubly-virtual pion transition form factor not yet available – data from *e.g.* KLOE on $\phi \to \pi^0 e^+e^-$, or the old, puzzling ones on $\omega \to \pi^0 e^+e^-$ represent useful input

η transition form factor: Hanhart, Kupsc, Meißner, Stollenwerk, Wirzba (2013)

Pion box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion box contribution

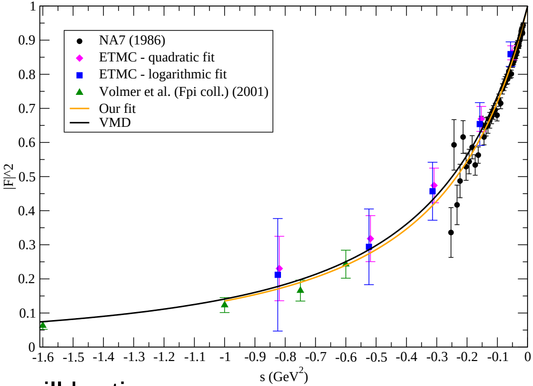
The only ingredient needed for the pion-box contribution is the vector form factor

$$\Pi_i^{\text{FsQED}} = F_V^\pi(q_1^2)F_V^\pi(q_2^2)F_V^\pi(q_3^2)\bar{\Pi}_i^{\text{sQED}}(s,t,u)$$

$$\begin{aligned}
\bar{\Pi}_i^{\text{sQED}} = {} & p_i + a_i A_0(M_\pi^2) \\
& + b_i^1 B_0(q_1^2, M_\pi^2, M_\pi^2) + b_i^2 B_0(q_2^2, M_\pi^2, M_\pi^2) + b_i^3 B_0(q_3^2, M_\pi^2, M_\pi^2) + b_i^4 B_0(q_4^2, M_\pi^2, M_\pi^2) \\
& + b_i^s B_0(s, M_\pi^2, M_\pi^2) + b_i^t B_0(t, M_\pi^2, M_\pi^2) + b_i^u B_0(u, M_\pi^2, M_\pi^2) \\
& + c_i^{12} C_0(q_1^2, q_2^2, s, M_\pi^2, M_\pi^2, M_\pi^2) + c_i^{13} C_0(q_1^2, q_3^2, t, M_\pi^2, M_\pi^2, M_\pi^2) + c_i^{14} C_0(q_1^2, q_4^2, u, M_\pi^2, M_\pi^2, M_\pi^2) \\
& + c_i^{34} C_0(q_3^2, q_4^2, s, M_\pi^2, M_\pi^2, M_\pi^2) + c_i^{24} C_0(q_2^2, q_4^2, t, M_\pi^2, M_\pi^2, M_\pi^2) + c_i^{23} C_0(q_2^2, q_3^2, u, M_\pi^2, M_\pi^2, M_\pi^2) \\
& + d_i^{st} D_0(q_1^2, q_2^2, q_4^2, q_3^2, s, t, M_\pi^2, M_\pi^2, M_\pi^2, M_\pi^2) \\
& + d_i^{su} D_0(q_1^2, q_2^2, q_3^2, q_4^2, s, u, M_\pi^2, M_\pi^2, M_\pi^2, M_\pi^2) \\
& + d_i^{tu} D_0(q_1^2, q_3^2, q_2^2, q_4^2, t, u, M_\pi^2, M_\pi^2, M_\pi^2, M_\pi^2),
\end{aligned}$$

where B_0, C_0 and D_0 are Passarino-Veltman functions.

Pion box contribution



Uncertainties will be tiny

Preliminary! numbers:

$$a_\mu^{\text{FsQED}} = -15.9 \cdot 10^{-11} \qquad a_\mu^{\text{FsQED,VMD}} = -16.4 \cdot 10^{-11}$$

Pion box contribution

Table 13

Summary of the most recent results for the various contributions to $a_\mu^{\text{LbL;had}} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3\pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Uncertainties will be tiny

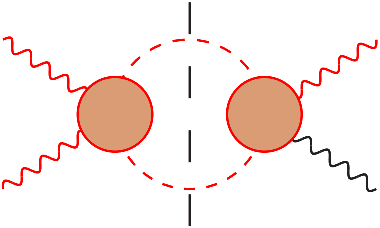
Preliminary! numbers:

$$a_\mu^{\text{FsQED}} = -15.9 \cdot 10^{-11} \qquad a_\mu^{\text{FsQED,VMD}} = -16.4 \cdot 10^{-11}$$

Our dispersive representation of the $\bar{\Pi}^{\mu\nu\lambda\sigma}$ tensor

GC, Hoferichter, Procura, Stoffer (2014)

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Our dispersive representation of the $\bar{\Pi}^{\mu\nu\lambda\sigma}$ tensor

GC, Hoferichter, Procura, Stoffer (2014)

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_{i=1}^{15} \left(A^{\mu\nu\lambda\sigma}_{i,s} \Pi_i(s) + A^{\mu\nu\lambda\sigma}_{i,t} \Pi_i(t) + A^{\mu\nu\lambda\sigma}_{i,u} \Pi_i(u) \right)$$

- the $\Pi_i(s)$ are single-variable functions having only a right-hand cut
- for the discontinuity we keep only the lowest partial wave
- the dispersive integral that gives the $\Pi_i(s)$ in terms of its discontinuity has the required soft-photon zeros
- soft-photon zeros constrain
  the subtraction polynomial to vanish
  (unless one wanted to subtract more, which is unnecessary)

Dispersion relations for the $\Pi_i(s)$

Requiring that the BTT functions be free of singularities determines the kernels, including non-diagonal terms. S-waves:

$$\Pi_1^s = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left[K_1\,\mathrm{Im}\bar{h}^0_{++,++}(s') + \frac{2\xi_1\xi_2}{\lambda'_{12}}\,\mathrm{Im}\bar{h}^0_{00,++}(s')\right]$$

$$y\Pi_2^s = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left[K_1\,\mathrm{Im}\bar{h}^0_{00,++}(s') + \frac{2q_1^2q_2^2}{\xi_1\xi_2\lambda'_{12}}\,\mathrm{Im}\bar{h}^0_{++,++}(s')\right]$$

$$K_1 := \frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda'_{12}}$$

Remark: $\mathrm{Im}h^0_{++,++}(s)$ and $\mathrm{Im}h^0_{00,++}(s)$ given by S-wave helicity amplitudes of $\gamma^*\gamma^*\to\pi\pi$

Once the projection on the BTT basis is done
$\Rightarrow$ use the master formula to calculate the contribution to a_μ

Dispersion relations for the $\Pi_i(s)$

Requiring that the BTT functions be free of singularities determines the kernels, including non-diagonal terms. S-waves:

$$\Pi_1^s = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left[K_1\,\mathrm{Im}\bar{h}^0_{++,++}(s') + \frac{2\xi_1\xi_2}{\lambda'_{12}}\,\mathrm{Im}\bar{h}^0_{00,++}(s')\right]$$

$$y\Pi_2^s = \frac{s-q_3^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-q_3^2}\left[K_1\,\mathrm{Im}\bar{h}^0_{00,++}(s') + \frac{2q_1^2q_2^2}{\xi_1\xi_2\lambda'_{12}}\,\mathrm{Im}\bar{h}^0_{++,++}(s')\right]$$

$$K_1 := \frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda'_{12}}$$

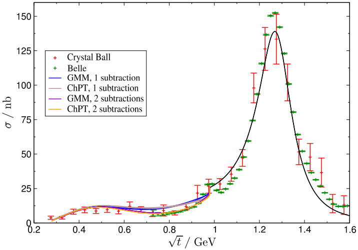
Remark: $\mathrm{Im}h^0_{++,++}(s)$ and $\mathrm{Im}h^0_{00,++}(s)$ given by S-wave helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$

Extension to D waves is in progress
(diagonal kernels already given explicitly in JHEP (14))

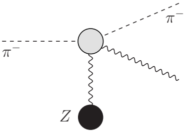
Dispersion relations for $\gamma^*\gamma^* \to \pi\pi$

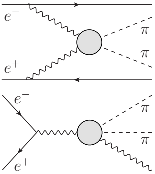
Roy-Steiner eqs. $=$ Dispersion relations $+$ partial-wave expansion
$+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- On-shell $\gamma\gamma \to \pi\pi$: prominent *D*-wave reson. $f_2(1270)$ Moussallam (10) Hoferichter, Phillips, Schat (11)
- $\gamma^*\gamma \to \pi\pi$ Moussallam (13)
- $\gamma^*\gamma^* \to \pi\pi$, new feature: anomalous thresholds Hoferichter, GC, Procura, Stoffer (13)



- Constraints
  - Low energy: pion polar., ChPT
  - Primakoff: $\gamma\pi \to \gamma\pi$ at COMPASS, JLAB
  - Scattering: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - Decays: $\omega, \phi \to \pi\pi\gamma$



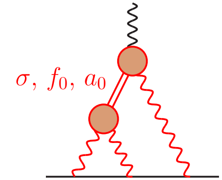


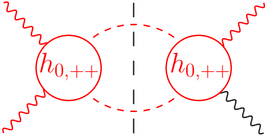
Physics of $\gamma^*\gamma^* \to \pi\pi$

- $\pi\pi$ rescattering $\Leftrightarrow$ resonances, e.g. $f_2(1270)$
- S-wave provides model-independent implementation of the σ
- Analytic continuation with dispersion theory: resonance properties
  - Precise determination of σ-pole from $\pi\pi$ scattering Caprini, GC, Leutwyler 2006

$$M_\sigma = 441^{+16}_{-8}\,\mathrm{MeV} \qquad \Gamma_\sigma = 544^{+18}_{-25}\,\mathrm{MeV}$$

  - Coupling $\sigma \to \gamma\gamma$ from $\gamma\gamma \to \pi\pi$ Hoferichter, Phillips, Schat 2011





$f_0(500)$ PARTIAL WIDTHS

$\Gamma(\gamma\gamma)$ **Γ_2**

VALUE (keV)	DOCUMENT ID		TECN	COMMENT
• • • We do not use the following data for averages, fits, limits, etc. • • •				
1.7 ± 0.4	[54] HOFERICHTER	11	RVUE	Compilation
3.08 ± 0.82	[55] MENNESSIER	11	RVUE	Compilation
$2.08 \pm 0.2\ ^{+0.07}_{-0.04}$	[56] MOUSSALLAM	11	RVUE	Compilation
2.08	[57] MAO	09	RVUE	Compilation
1.2 ± 0.4	[58] BERNABEU	08	RVUE	

$f_0(500)$ or σ
was $f_0(600)$

$I^G(J^{PC}) = 0^+(0^{++})$

A REVIEW GOES HERE – Check our WWW List of Reviews

$f_0(500)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\,\mathrm{Im}(\sqrt{s_{\mathrm{pole}}})$.

VALUE (MeV)	DOCUMENT ID		TECN	COMMENT
(400–550)$-i$(200–350) OUR ESTIMATE				
• • • We do not use the following data for averages, fits, limits, etc. • • •				
$(445 \pm 25)-i(278^{+22}_{-18})$	[1,2] GARCIA-MAR..	11	RVUE	Compilation
$(457^{+14}_{-13})-i(279^{+11}_{-7})$	[1,3] GARCIA-MAR..	11	RVUE	Compilation
$(442^{+5}_{-8})-i(274^{+6}_{-5})$	[4] MOUSSALLAM	11	RVUE	Compilation
$(452 \pm 13)-i(259 \pm 16)$	[5] MENNESSIER	10	RVUE	Compilation
$(448 \pm 43)-i(266 \pm 43)$	[6] MENNESSIER	10	RVUE	Compilation

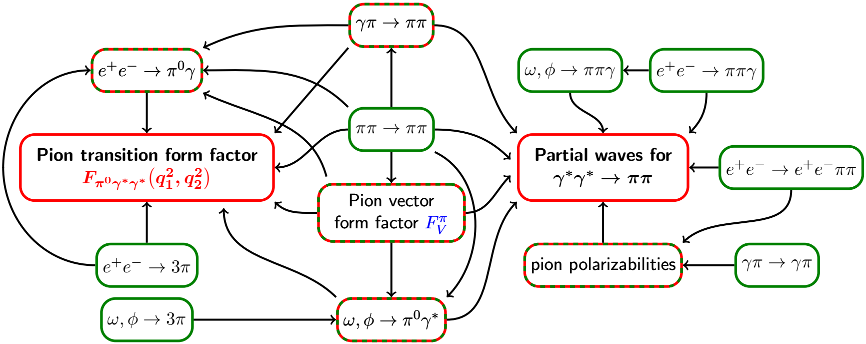
Hadronic light-by-light: Outlook

Path to a numerical evaluation of HLbL contributions to a_μ:

- take into account experimental constraints on the pion transition form factor to evaluate the pion pole contribution
- using as input a dispersive description of the pion em form factor $\Rightarrow$ evaluate the FsQED contribution
- take into account all experimental constraints on $\gamma^{(*)}\gamma \to \pi\pi$
- estimate the dependence on the q^2 of the second photon (theoretically, there are no data yet on $\gamma^*\gamma^* \to \pi\pi$)
- $\Rightarrow$ solve the dispersion relation for the helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$
- input the outcome into the master formula

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer `arXiv:1408.2517` (PLB '14)



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists

Outline

Conclusions

- dispersion relations are a very useful tool in low-energy hadronic physics
- they are complementary to both ChPT and lattice:
  - dispersion relations are the best tool for dealing with unitarity effects
  - ChPT provides symmetry relations with explicit quark mass dependence
  - lattice provides numerical input on nonperturbative dynamical effects determining quark mass dependences and subtraction constants
- their use becomes challenging as soon as more than two channels are open and similarly important
- there are plenty of applications where dispersion relations could lead to improvements in the calculations – HLbL is probably the most urgent one

Other recent applications

- Form factors
  - πK Bernard, Boito, Passemar and Moussallam
  - $\pi\eta$ Albaladejo, Moussallam
  - nucleon form factors (proton radius) Lorenz, Hammer, Meißner
- scattering amplitudes
  - πK Büttiker, Descotes-Genon, Moussallam
  - $K_{\ell 4}$ GC, Passemar, Stoffer
  - $\eta \to 3\pi$ Kampf, Knecht, Novotny, Zdrahal and Guo *et al.* (JLAB) and GC, Lanz, Leutwyler, Passemar
  - πN Ruiz de Elvira, Ditsche, Hoferichter, Kubis, Meißner
  - $\gamma^{(*)}\gamma^{(*)} \to \pi\pi$ Hoferichter, Phillips, Schat, and Garcia Martin, Moussallam