τ hadronic spectral function moments in a nonpower QCD perturbation theory

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Plan

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Introduction

- The inclusive hadronic decay width of the τ lepton provides a very clean way to determine α_s at low energies.
- The perturbative QCD contribution is known to $O(\alpha_s^4)$.
- The nonperturbative corrections are predicted to be small.
- An important ambiguity is related to the prescription chosen for implementing renormalization-group invariance.
- Another serious problem is related to the fact that the coefficients of the perturbative series of the Adler function in QCD display a factorial growth.

Introduction

- The nonperturbative power corrections and the effects of quark-hadron duality violation (DV) generate additional uncertainties.
- The τ hadronic width receives small contributions from the power corrections and DV.
- Some moments may receive larger contributions from the nonperturbative condensates and terms involving DV.
- The most comprehensive analysis to date, attempted to include DV in a combined fit of several moments, which leads to a substantial increase in the error of the nonperturbative contributions.
  Boito, Golterman, Jamin, Mahdavi, Maltman, Osborne, and Peris 2012, 2015

Introduction

- To improve such analyses, however, also the properties of the perturbative expansions of the moments must be carefully examined.
- The perturbative expansions of a large class of spectral function moments have been studied within two standard QCD perturbative expansions, the fixed-order and the contour-improved perturbation theories (FOPT and CIPT).
  Beneke, Boito, and Jamin 2013
- Some moments that are commonly employed in the determinations of strong coupling from τ decays should be avoided because of their perturbative instability.

Introduction

- However this conclusion refers to the standard expansions, FOPT and CIPT.
- We shall show in this talk that improved expansions with no perturbative instability can be defined.
- We consider the perturbative behavior of these moments in the framework of a QCD nonpower perturbation theory.
- This is defined by the technique of series acceleration by conformal mappings, which simultaneously implements renormalization-group summation and has a tame large-order behavior.

QCD description

- The R ratio for the τ decays is defined as:

$$R_{\tau,V/A} \equiv \frac{\Gamma[\tau^- \to \mathrm{hadrons}\,\nu_\tau]}{\Gamma[\tau^- \to e^- \overline{\nu}_e \nu_\tau]}. \tag{1}$$

- We are interested in the τ decay rate into light u and d quarks, which proceeds either through a vector or an axialvector current.
- R_τ can also be expressed in the form

$$R_{\tau,V/A} = \frac{N_c}{2} S_{\rm EW} \, |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta'_{\rm EW} + \sum_{D\geq 2} \delta^{(D)}_{ud} \right]. \tag{2}$$

Braaten-Narison-Pich

- $S_{\rm EW} = 1.0198 \pm 0.0006$ Marciano and Sirlin 1988
  $\delta'_{\rm EW} = 0.0010 \pm 0.0010$ Braaten and Li 1990

QCD description

- Our main interest is in the perturbative corrections $\delta^{(0)}$ which can be written

$$\delta^{(0)}_{w_i}(s_0) = \frac{1}{2\pi i} \oint\limits_{|s|=s_0} \frac{ds}{s} W_i(s/s_0) \widehat{D}_{\rm pert}(s),$$

where $W_i(x)$ are weights functions and $\widehat{D}_{\rm pert}(s)$ is the perturbative part of the reduced Adler function

$$\widehat{D}(s) \equiv -s\, \mathrm{d}\Pi^{(1+0)}(s)/\mathrm{d}s - 1.$$

- A natural expansion for the Adler function is called '**fixed-order perturbation theory**' (FOPT)

$$\widehat{D}_{\rm FOPT}(s) = \sum_{n\geq 1} (a_s(\mu^2))^n [c_{n,1} + \sum_{k=2}^{n} k c_{n,k} \left(\ln \frac{-s}{\mu^2} \right)^{k-1}],$$

where $a_s(\mu^2) = \alpha_s(\mu^2)/\pi$.

QCD description

- A different approach would be to keep the full solution of the RGE and perform a numerical integration and choose $\mu^2 = -s$. This is called '**Contour Improved Perturbation Theory**'.
  Pivovarov 1991, Le Diberder and Pich 1992

$$\hat{D}_{\mathrm{CIPT}}(\alpha_s(-s)/\pi, 0) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi} \right)^n . \tag{2}$$

- In the expansion above, the leading known coefficients $c_{n,1}$ are

$$c_{1,1} = 1,\ c_{2,1} = 1.640,\ c_{3,1} = 6.371,\ c_{4,1} = 49.076,$$

  Baikov, Chetyrkin and Kuhn 2008
  $c_{5,1} = 283$ estimeted, Beneke and Jamin 2008.

- The β-function was calculated to four loops in the $\overline{\mathrm{MS}}$-renormalization scheme, the known coefficients are

$$\beta_0 = 9/4,\ \beta_1 = 4,\ \beta_2 = 10.0599,\ \beta_3 = 47.228.$$

  Larin, Ritbergen and Vermaseren 1997 and Czakon 2005

Renormalization Group Summed Perturbation Theory

- We use a method based on the explicit summation of all renormalization-group accessible logarithms.

$$\hat{D}_{RGSPT}(aL) = a(c_{1,1} + 2c_{2,2}aL + 3c_{3,3}a^2L^2 + \cdots) + a^2(c_{2,1} + 2c_{3,2}aL + 3c_{4,3}a^2L^2 + \cdots)$$
$$+ a^3(c_{3,1} + 2c_{4,2}aL + 3c_{5,3}a^2L^2 + \cdots) + \cdots = \sum_{n=1}^{\infty} a^n D_n(aL). \qquad (3)$$

where $L \equiv \ln \frac{-s}{\mu^2}$.
Maxwell and A. Mirjalili 2000
Ahmady, Chishtie, Elias, Fariborz, Fattahi, McKeon, Sherry, Steele 2002, 03

$$D_n(aL) \equiv \sum_{k=n}^{\infty}(k-n+1)c_{k,k-n+1}(aL)^{k-n}. \qquad (4)$$

- The Adler function is scale independent

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\left\{\hat{D}_{\mathrm{FOPT}}(aL)\right\} = 0. \qquad (5)$$

$$\beta(a)\frac{\partial \hat{D}_{\mathrm{FOPT}}}{\partial a} - \frac{\partial \hat{D}_{\mathrm{FOPT}}}{\partial L} = 0. \qquad (6)$$

- We derive following RGE equation

$$0 = -\sum_{n=1}^{\infty}\sum_{k=2}^{n} k(k-1)c_{n,k}a^n L^{k-2}$$
$$-\left(\beta_0 a^2 + \beta_1 a^3 + \beta_2 a^4 + \ldots + \beta_l a^{l+2} + \ldots\right) \times \sum_{n=1}^{\infty}\sum_{k=1}^{n} nkc_{n,k}a^{n-1}L^{k-1}. \quad (7)$$

- By extracting the aggregate coefficient of $a^n L^{n-p}$ one obtains the recursion formula ($n \geq p$)

$$0 = (n-p+2)c_{n,n-p+2} + \sum_{\ell=0}^{p-2}(n-\ell-1)\beta_\ell c_{n-\ell-1,n-p+1}. \quad (8)$$

- Multiplying both sides of (8) by $(n-p+1)(aL)^{n-p}$ and summing from $n = p$ to ∞, we obtain a set of first-order linear differential equation for the functions defined in (4), written as

$$\frac{\mathrm{d}D_n}{\mathrm{d}(aL)} + \sum_{\ell=0}^{n-1}\beta_\ell\left((aL)\frac{\mathrm{d}}{\mathrm{d}(aL)} + n - \ell\right)D_{n-\ell} = 0, \quad (9)$$

for $n \geq 1$, with the initial conditions $D_n(0) = c_{n,1}$ which follow from (4). The solution of the above Eq (9) can be found iteratively in an analytical closed form.

- The first two solutions are

$$D_1(aL) = \frac{c_{1,1}}{y}, \; D_2(aL) = \frac{c_{2,1}}{y^2} - \frac{\beta_1 c_{1,1} \ln y}{\beta_0 w^2}, \; y = 1 + \beta_0 aL. \tag{10}$$

- The RGSPT expansion of the Adler function is

$$\hat{D}_{\mathrm{RGSPT}}(aL) = \sum_{n=1}^{N} a^n D_n(aL), \tag{11}$$

- It can be written as

$$\widehat{D}_{\mathrm{RGSPT}}(s) = \sum_{n\geq 1} (\widetilde{a}_s(-s))^n [c_{n,1} + \sum_{j=1}^{n-1} c_{j,1} d_{n,j}(y)],$$

where

$$\widetilde{a}_s(-s) = \frac{a_s(\mu^2)}{1 + \beta_0 a_s(\mu^2) \ln(-s/\mu^2)}$$

is the solution of the RG equation to one loop, $d_{n,j}(y)$ are calculable functions and $y \equiv 1 + \beta_0 a_s(\mu^2) \ln(-s/\mu^2)$.

	$\delta^{(0)}_{\mathrm{FOPT}}$	$\delta^{(0)}_{\mathrm{CIPT}}$	$\delta^{(0)}_{\mathrm{RGSPT}}$
$n = 1$	0.1082	0.1479	0.1455
$n = 2$	0.1691	0.1776	0.1797
$n = 3$	0.2025	0.1898	0.1931
$n = 4$	0.2199	0.1984	0.2024
$n = 5$	0.2287	0.2022	0.2056

Table: Predictions of $\delta^{(0)}$ by the standard FOPT, CIPT and the RGSPT, for various truncation orders n using $\alpha_s = 0.34$.

For $n = 4$, the difference between the results of the RGSPT and the standard FOPT is 0.01754, and the difference from the RGSPT and CIPT is 0.0039, which confirms that the new expansion gives results close to those of the CIPT.

Large order behaviour

- The Borel-transform

$$B[\widehat{D}](t) \equiv \sum_{n=0}^{\infty} r_n \, \frac{t^n}{n!} \,. \tag{12}$$

- If $B[\widehat{D}](t)$ has no singularities for real positive t (which is not the case for the Adler function), one can define the Borel integral,

$$\widehat{D}(\alpha) \equiv \int_0^{\infty} dt \, \mathrm{e}^{-t/\alpha} \, B[\widehat{D}](t) \,, \tag{13}$$

  which has the same series expansion in α as $\widehat{D}(s)$ does in $\alpha_s(\sqrt{s})$.

- The integral $\widehat{D}(\alpha)$, if it exists, gives the Borel sum of the original divergent series.

Calculating so-called bubble-chain diagrams, it was found that the Borel-transformed Adler function $B[\widehat{D}](t)$ has ultraviolet (UV) and infrared (IR) renormalon poles . Beneke 1993, Broadhurst 1993

UV Renormalons

- Sign-alternating, singularity (factorial divergences) structure of the Borel transformed Adler function related to higher-dim operators in the cut-off QCD Lagrangian.
- They exist at $u = \beta_0 t/2\pi = -n$, $n \geq 1$.
- The leading UV renormalon, being close to $u = 0$, dictates the large-order behaviour of the perturbative expansion.

Beneke and Jamin 2008

IR Renormalons

- Fixed-sign, singularity structure related to power corrections in the OPE (dimension $D = 2n$).
- They exist at $u = \beta_0 t/2\pi = +n$, $n \geq 2$.
- Intermediate orders are governed by IR renormalons, $u = 2$ especially simple, only one operator (gluon condensate) and gives dominant contribution.

Beneke and Jamin 2008

Large order behaviour

- We consider the model where the Adler function is defined in terms of its Borel transform $B(u)$ by the principal value prescription

$$\widehat{D}(s) = \frac{1}{\beta_0} \, \mathrm{PV} \int_0^\infty e^{-\frac{u}{\beta_0 a(-s)}} \, B(u) \, \mathrm{d}u, \tag{14}$$

where the function $B(u)$ is expressed in terms of a few ultraviolet (UV) and infrared (IR) renormalons

$$B_{\mathrm{BJ}}(u) = B_1^{\mathrm{UV}}(u) + B_2^{\mathrm{IR}}(u) + B_3^{\mathrm{IR}}(u) + d_0^{\mathrm{PO}} + d_1^{\mathrm{PO}} u. \tag{15}$$

Beneke and Jamin 2008

- These terms were written as

$$B_p^{\mathrm{IR}}(u) = \frac{d_p^{\mathrm{IR}}}{(p-u)^{\gamma_p}} \left[1 + \tilde{b}_1 (p-u) + \dots \right],$$

$$B_p^{\mathrm{UV}}(u) = \frac{d_p^{\mathrm{UV}}}{(p+u)^{\bar{\gamma}_p}} \left[1 + \bar{b}_1 (p+u) + \dots \right],$$

Large order behaviour

- Finally, the free parameters of the model were fixed by the requirement of reproducing the perturbative coefficients $c_{n,1}$ for $n \leq 4$ and the estimate $c_{5,1} = 283$, and read:

$$d_1^{\mathrm{UV}} = -1.56\times 10^{-2}, \quad d_2^{\mathrm{IR}} = 3.16, \quad d_3^{\mathrm{IR}} = -13.5, d_0^{\mathrm{PO}} = 0.781, \quad d_1^{\mathrm{PO}} = 7.66\times 10^{-3}. \tag{16}$$

- The reference model (RM) is parameterized by the UV and first IR renormalons. **Beneke and Jamin,** 2008
- In the alternative model (AM), the first IR renormalon is removed by hand. **Boito, Beneke and Jamin,** 2013

RGS Non-Power Perturbation Theory

- We improve the convergence of the RGSPT expansion by the analytical continuation in the Borel plane.
  Caprini & Fischer 1999, 2000, 2009, 2011

- The method was applied to FOPT and CIPT by Caprini and Fischer in the past.

- The method cannot be applied in the α_s plane but can be applied to the Borel transform, $B(u)$ of the Adler function in the u plane.

RGS Non-Power Perturbation Theory

- The Taylor exapnsion of the Borel transform, $B(u)$ converges only in the disk $|u| < 1$.

$$B(u) = \sum_{n=0}^{\infty} c_{n+1,1} \, \frac{u^n}{\beta_0^n \, n!} \tag{17}$$

- The region of convergence can be enlarged if the series in powers of u is replaced by a series in powers of an "optimal" variable $\widetilde{w}(u)$ that conformally maps the holomorphy domain of $B(u)$, *i.e.* the u-plane with cut along $u \geq 2$ and $u \leq -1$, onto the unit disk $|w| < 1$.

- This also accelerates the convergence rate at all points in the holomorphy domain. Ciulli & Fischer 1961, Caprini & Fischer 2011

RGS Non-Power Perturbation Theory

- We introduce the Borel transform of the RGSPT expansion of the Adler function

$$B_{\mathrm{RGSPT}}(u,y) = B(u) + \sum_{n=0}^{\infty} \frac{u^n}{\beta_0^n\, n!} \sum_{j=1}^{n} c_{j,1} d_{n+1,j}(y), \tag{18}$$

where $y = 1 + \beta_0 a L$.

- We consider the functions

$$\widetilde{w}_{lm}(u) = \frac{\sqrt{1+u/l} - \sqrt{1-u/m}}{\sqrt{1+u/l} + \sqrt{1-u/m}}, \quad l \geq 1, m \geq 2 \tag{19}$$

where l, m are positive integers satisfying $l \geq 1$ and $m \geq 2$. The function $\widetilde{w}_{lm}(u)$ maps the u-plane cut along $u \leq -l$ and $u \geq m$ onto the disk $|w_{lm}| < 1$ in the plane $w_{lm} \equiv \widetilde{w}_{lm}(u)$.

- We define further the class of compensating factors of the simple form

$$S_{lm}(u) = \left(1 - \frac{\widetilde{w}_{lm}(u)}{\widetilde{w}_{lm}(-1)}\right)^{\gamma_1^{(l)}} \left(1 - \frac{\widetilde{w}_{lm}(u)}{\widetilde{w}_{lm}(2)}\right)^{\gamma_2^{(m)}}, \tag{20}$$

RGS Non-Power Perturbation Theory

- The exponents are

$$\gamma_1^{(l)} = \gamma_1(1+\delta_{l1}), \quad \gamma_2^{(m)} = \gamma_2(1+\delta_{m2}),$$
$$\gamma_1 = 1.21, \qquad \gamma_2 = 2.58\,, \tag{21}$$

  are chosen such that $S_{lm}(u)$ cancel the dominant singularities on the real axis in the u-plane.

- We further expand the product $S_{lm}(u)B_{\mathrm{RGSPT}}(u,y)$ in powers of the variable $\widetilde{w}_{lm}(u)$, as

$$S_{lm}(u)B_{\mathrm{RGSPT}}(u,y) = \sum_{n\geq 0} c_{n,\mathrm{RGSPT}}^{(lm)}(y)\,(\widetilde{w}_{lm}(u))^n. \tag{22}$$

- We are led to the class of RGSNPPT expansions

$$\widehat{D}_{\mathrm{RGSNPPT}}(s) = \sum_{n\geq 0} c_{n,\mathrm{RGSPT}}^{(lm)}(y)\,\mathcal{W}_{n,\mathrm{RGSPT}}^{(lm)}(s), \tag{23}$$

where

$$\mathcal{W}_{n,\mathrm{RGSPT}}^{(lm)}(s) = \frac{1}{\beta_0}\mathrm{PV}\int_0^\infty \exp\left(\frac{-u}{\beta_0\tilde{a}_s(-s)}\right)\frac{(\widetilde{w}_{lm}(u))^n}{S_{lm}(u)}\,\mathrm{d}u, \tag{24}$$

and the coefficients $c_{n,\mathrm{RGS}}^{(lm)}(y)$ are defined by the expansion (22).

- The coupling, $\widetilde{a}_s(-s)$, entering in the Laplace-Borel integral is the one-loop solution of the RGE, a novel feature given by RGSPT.

Results

The moments employed in the extraction of α_s are

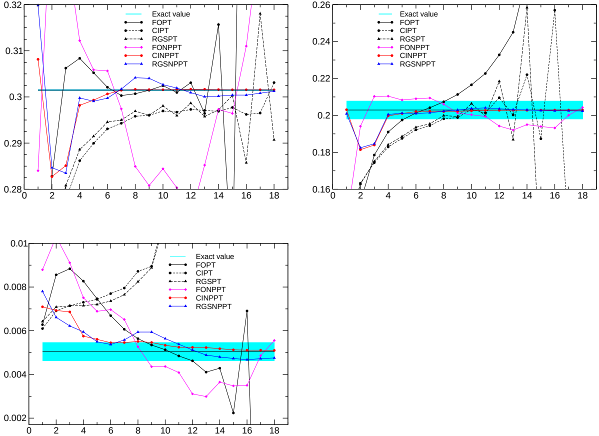
i	$W_i(x)$
1	$2(1-x)$
2	$1-x^2$
3	$\frac{2}{3}(1-x^3)$
4	$\frac{1}{2}(1-x^4)$
5	$\frac{2}{5}(1-x^5)$
6	$(1-x)^2$
7	$\frac{2}{3}(1-x)^2(2+x)$
8	$\frac{1}{2}(3-4x+x^4)$
9	$\frac{1}{4}(1-x)^3(3+x)$
10	$\frac{2}{3}(1-x)^3$
11	$\frac{1}{2}(1-x)^4$
12	$(1-x)^3(1+x)$
13	$\frac{1}{10}(1-x)^4(7+8x)$
14	$\frac{1}{6}(1-x)^3(1+3x)$
15	$\frac{1}{6}(1-x)^4(1+2x)^2$
16	$\frac{1}{210}(1-x)^4(13+52x+130x^2+120x^3)$
17	$\frac{1}{70}(1-x)^4(2+8x+20x^2+40x^3+35x^4)$

Table: Functions $W_i(x)$ for corresponding moments..

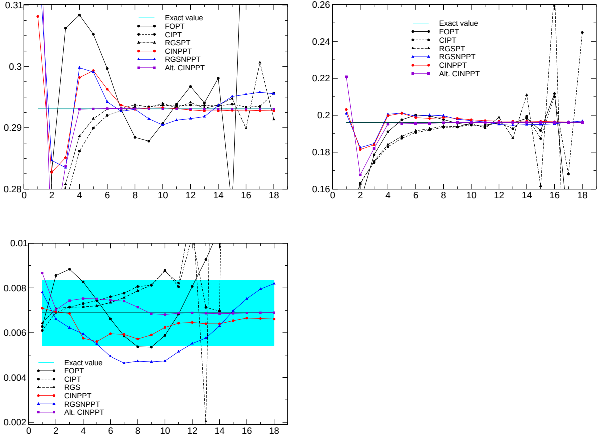
Results

- The weights shown in the figures are $W_1(x) = 2(1-x)$, $W_6(x) = (1-x)^2$, $W_{16}(x) = \frac{1}{210}(1-x)^4(13 + 52x + 130x^2 + 120x^3)$.

Results



$\delta^{(0)}_{w_i}$ for the weights W_1, W_6, and W_{16} calculated for the RM, as functions of the perturbative order up to which the series was summed. The horizontal bands give the uncertainties of the exact values. We use $\alpha_s(M_\tau^2) = 0.3186$.



$\delta_{w_i}^{(0)}$ for the weights W_1, W_6, and W_{16} calculated for the AM, as functions of the perturbative order up to which the series was summed.

Summary

- The moments of the hadronic spectral functions are of interest for the extraction of the strong coupling and other QCD parameters from the hadronic decays of the τ lepton.
- We consider the perturbative behavior of these moments in the framework of a QCD nonpower perturbation theory, defined by the technique of series acceleration by conformal mappings, which simultaneously implements renormalization-group summation and has a tame large-order behavior.
- Two recently proposed models of the Adler function are employed to generate the higher order coefficients of the perturbation series and to predict the exact values of the moments, required for testing the properties of the perturbative expansions.
- The CINPPT and RGSNPPT expansions provide a good perturbative description of a large class of τ hadronic spectral function moments, including some for which all the standard expansions fail.
- A program that employs these expansions for the simultaneous determination of the strong coupling and other parameters of QCD from hadronic τ decays is of interest for future investigations.