

# Dispersion Relation for Hadronic Light-by-Light Scattering

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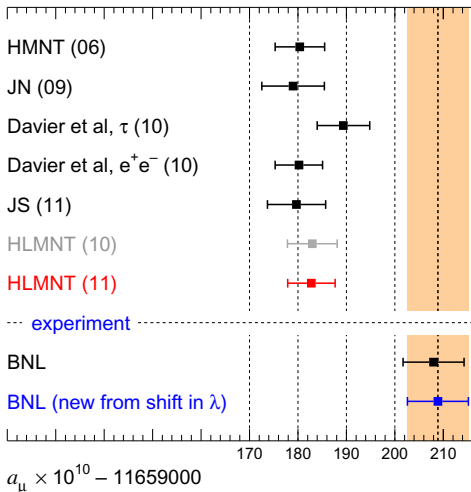
5th August 2015

High-precision QCD at low energy, Benasque, Spain

- 1 Introduction
- 2 Lorentz Structure of the HLbL Tensor
- 3 Master Formula for  $(g - 2)_\mu$
- 4 Mandelstam Representation
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# $(g - 2)_\mu$ : comparison of theory and experiment

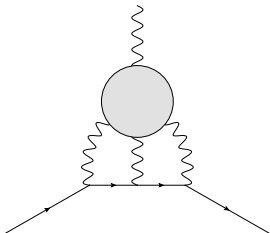


→ Hagiwara et al. 2012

## $(g - 2)_\mu$ : theory vs. experiment

- discrepancy between SM and experiment  $\sim 3\sigma$
- hint to new physics?
- new experiments (FNAL, J-PARC) aim at reducing the experimental error by a factor of 4
- theory error completely dominated by hadronic effects
- hadronic vacuum polarisation responsible for largest uncertainty, but will be systematically improved with better data input

## Hadronic light-by-light (HLbL) scattering



- up to now only model calculations
- uncertainty estimate based rather on consensus than on a systematic method
- lattice QCD making progress, but not yet competitive
- will dominate theory error in a few years

# Model calculations of HLbL

Table 13

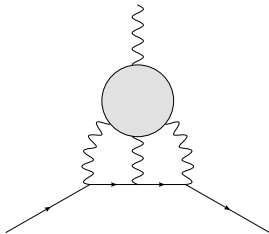
Summary of the most recent results for the various contributions to  $a_\mu^{\text{LbL;had}} \times 10^{11}$ . The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

| Contribution                               | BPP            | HKS             | KN          | MV           | BP           | PdRV         | N/JN         |
|--|----------------|-----------------|-------------|--------------|--------------|--------------|--------------|
| $\pi^0, \eta, \eta'$                       | $85 \pm 13$    | $82.7 \pm 6.4$  | $83 \pm 12$ | $114 \pm 10$ | –            | $114 \pm 13$ | $99 \pm 16$  |
| $\pi, K$ loops                             | $-19 \pm 13$   | $-4.5 \pm 8.1$  | –           | –            | –            | $-19 \pm 19$ | $-19 \pm 13$ |
| $\pi, K$ loops + other subleading in $N_c$ | –              | –               | –           | $0 \pm 10$   | –            | –            | –            |
| axial vectors                              | $2.5 \pm 1.0$  | $1.7 \pm 1.7$   | –           | $22 \pm 5$   | –            | $15 \pm 10$  | $22 \pm 5$   |
| scalars                                    | $-6.8 \pm 2.0$ | –               | –           | –            | –            | $-7 \pm 7$   | $-7 \pm 2$   |
| quark loops                                | $21 \pm 3$     | $9.7 \pm 11.1$  | –           | –            | –            | 2.3          | $21 \pm 3$   |
| total                                      | $83 \pm 32$    | $89.6 \pm 15.4$ | $80 \pm 40$ | $136 \pm 25$ | $110 \pm 40$ | $105 \pm 26$ | $116 \pm 39$ |

→ Jegerlehner, Nyffeler 2009

- pseudoscalar pole contribution most important
- pion-loop second most important
- differences between models, large uncertainties

## How to improve HLbL calculation?



- make use of fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- relate HLbL to experimentally accessible quantities



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## The HLbL tensor: definitions

- hadronic four-point function:

$$\begin{aligned} & \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) \\ &= -i \int dx dy dz e^{-i(q_1 x + q_2 y + q_3 z)} \langle 0 | T j_{\text{em}}^\mu(x) j_{\text{em}}^\nu(y) j_{\text{em}}^\lambda(z) j_{\text{em}}^\sigma(0) | 0 \rangle \end{aligned}$$

- EM current:

$$j_{\text{em}}^\mu = \sum_{i=u,d,s} Q_i \bar{q}_i \gamma^\mu q_i$$

- Mandelstam variables:

$$s = (q_1 + q_2)^2, \quad t = (q_1 + q_3)^2, \quad u = (q_2 + q_3)^2$$

- for  $(g-2)_\mu$ , the external photon is on-shell:

$$q_4^2 = 0, \quad \text{where } q_4 = q_1 + q_2 + q_3$$

## The HLbL tensor

- a priori 138 ‘naive’ Lorentz structures:

$$\begin{aligned} \Pi^{\mu\nu\lambda\sigma} &= g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 \\ &+ \sum_{i,k,l,m} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi_{ijkl}^4 \\ &+ \sum_{i,j} g^{\lambda\sigma} q_i^\mu q_j^\nu \Pi_{ij}^5 + \dots \end{aligned}$$

- in 4 space-time dimensions: 2 linear relations among the 138 Lorentz structures → [Eichmann et al., 2014](#)
- six dynamical variables, e.g. two Mandelstam variables  $s, t$  and the photon virtualities  $q_1^2, q_2^2, q_3^2, q_4^2$

## HLbL tensor: gauge invariance

- Ward identities

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\} \Pi_{\mu\nu\lambda\sigma} = 0$$

imply 95 linear relations between scalar functions  $\Pi_i$

- off-shell basis:  $138 - 95 - 2 = 43 - 2$  structures
- corresponding to 41 helicity amplitudes
- relations between  $\Pi_i$  imply kinematic zeros

## HLbL tensor: Lorentz decomposition

Problem: find a decomposition

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

with the following properties:

- Lorentz structures  $T_i^{\mu\nu\lambda\sigma}$  manifestly gauge invariant:

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\} T_{\mu\nu\lambda\sigma}^i = 0$$

- scalar functions  $\Pi_i$  free of kinematic singularities and zeros

## HLbL tensor: Lorentz decomposition

Recipe by Bardeen, Tung (1968) and Tarrach (1975):

- construct gauge projectors:

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4}$$

- gauge invariant themselves, e.g.

$$q_1^\mu I_{\mu\nu}^{12} = 0$$

- leave HLbL tensor invariant, e.g.

$$I_{12}^{\mu\mu'} \Pi_{\mu'\nu\lambda\sigma} = \Pi^\mu{}_{\nu\lambda\sigma}$$

## HLbL tensor: Lorentz decomposition

Following Bardeen, Tung (1968):

- apply gauge projectors to the 138 initial structures:  
95 immediately project to 0
- remove  $1/q_1 \cdot q_2$  and  $1/q_3 \cdot q_4$  poles by taking appropriate linear combinations
- BT basis: degenerate in the limits

$$q_1 \cdot q_2 \rightarrow 0, q_3 \cdot q_4 \rightarrow 0$$

## HLbL tensor: Lorentz decomposition

According to Tarrach (1975):

- no kinematic-free ‘basis’ of 43 elements exists
- degeneracies in the limits  $q_1 \cdot q_2 \rightarrow 0$ ,  $q_3 \cdot q_4 \rightarrow 0$ :

$$\sum_k c_k^i T_k^{\mu\nu\lambda\sigma} = q_1 \cdot q_2 X_i^{\mu\nu\lambda\sigma} + q_3 \cdot q_4 Y_i^{\mu\nu\lambda\sigma}$$

- extend basis by additional structures  $X_i^{\mu\nu\lambda\sigma}$ ,  $Y_i^{\mu\nu\lambda\sigma}$   
taking care of remaining kinematic singularities
- equivalent: implementing crossing symmetry



## HLbL tensor: Lorentz decomposition

Solution for the Lorentz decomposition:

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly gauge invariant
- crossing symmetry manifest: only 7 distinct structures, 47 follow from crossing
- scalar functions  $\Pi_i$  free of kinematics  
⇒ ideal quantities for a dispersive treatment

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## Master formula: contribution to $(g - 2)_\mu$

- from gauge invariance:

$$\Pi_{\mu\nu\lambda\rho} = -q_4^\sigma \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}$$

- for  $(g - 2)_\mu$ : afterwards take  $q_4 \rightarrow 0$
- no kinematic singularities in scalar functions: perform these steps with the derived Lorentz decomposition
- only 12 linear combinations of the scalar functions  $\Pi_i$  contribute to  $(g - 2)_\mu$

## Master formula: contribution to $(g - 2)_\mu$

$$a_\mu^{\text{HLbL}} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2] [(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$ : known integration kernel functions
- five loop integrals can be performed with Gegenbauer polynomial techniques
- Wick rotation possible even in the presence of anomalous thresholds

## Master formula: contribution to $(g - 2)_\mu$

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1 - \tau^2} Q_1^3 Q_2^3 \\ \times \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau),$$

- $T_i$ : known integration kernels
- $\bar{\Pi}_i$ : linear combinations of the scalar functions  $\Pi_i$
- Euclidean momenta:  $Q_i^2 = -q_i^2$
- $Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1Q_2\tau$

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## Analytic properties of scalar functions

- right- and left-hand cuts in each Mandelstam variable
- double-spectral regions (box topologies)
- anomalous thresholds for large photon virtualities

## Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

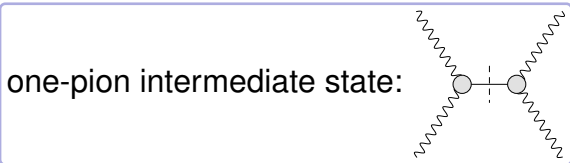
$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \dots$$



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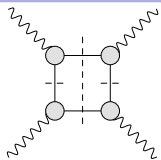


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two-pion intermediate state in both channels:

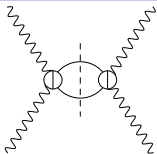


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two-pion intermediate state in first channel:



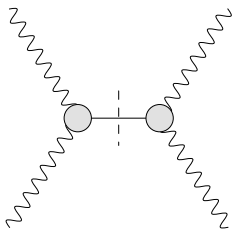
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neglected so far: higher intermediate states

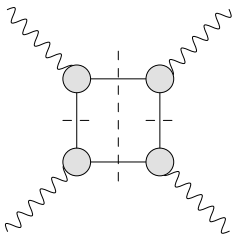
## Pion pole



- input: doubly-virtual and singly-virtual pion transition form factors  $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$  and  $\mathcal{F}_{\gamma^*\gamma\pi^0}$
- dispersive analysis of transition form factor:

→ Hoferichter et al., EPJC **74** (2014) 3180

## Box contributions



- simultaneous two-pion cuts in two channels
- Mandelstam representation explicitly constructed

$$\Pi_i = \frac{1}{\pi^2} \int ds' dt' \frac{\rho_i^{st}(s', t')}{(s' - s)(t' - t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

- $q^2$ -dependence: pion vector form factors  $F_\pi^V(q_i^2)$  for each off-shell photon factor out

## Box contributions

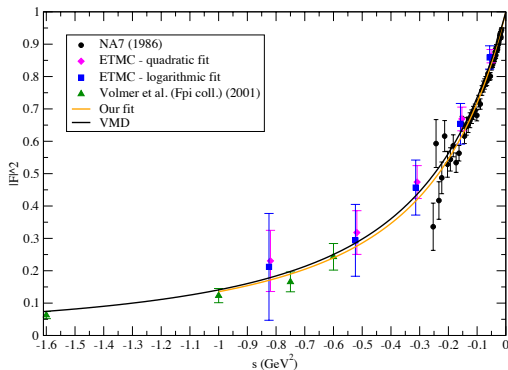
- sQED loop projected on BTT basis fulfils the same Mandelstam representation
- only difference are factors of  $F_\pi^V$
- $\Rightarrow$  box topologies are identical to FsQED:

$$\begin{aligned}
 & \text{Box diagram with 4 external wavy lines and a vertical dashed line} \\
 & \equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2) \\
 & \times \left[ \text{Bubble diagram} + \text{Triangle diagram} + \text{Box diagram} \right]
 \end{aligned}$$

- model-independent definition of pion loop

## Box contributions

Pion vector form factor in the space-like region:

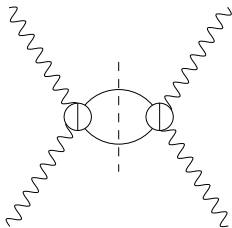


Preliminary results:

$$a_{\mu}^{\pi\text{-box}} = -15.9 \cdot 10^{-11}, \quad a_{\mu}^{\pi\text{-box, VMD}} = -16.4 \cdot 10^{-11}$$



## Rescattering contribution



- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel
- expansion into partial waves
- unitarity relates it to the helicity amplitudes of the subprocess

$$\gamma^* \gamma^{(*)} \rightarrow \pi \pi$$

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## Summary

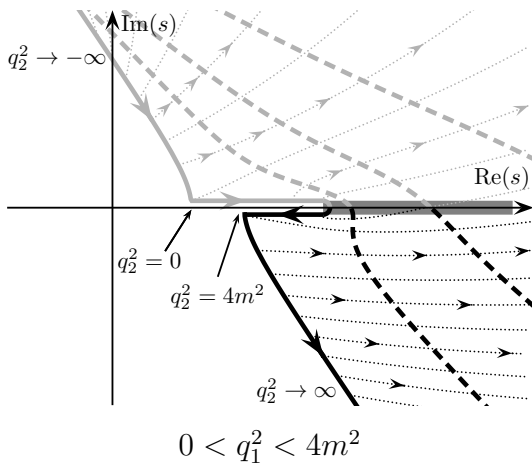
- our dispersive approach to HLbL scattering is based on fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- we take into account the lowest intermediate states:  
 $\pi^0$ -pole and  $\pi\pi$ -cuts
- relation to experimentally accessible (or again with data dispersively reconstructed) quantities
- a step towards a model-independent calculation of  $a_\mu$



Backup

## Wick rotation

Trajectory of triangle anomalous threshold:



## Wick rotation

Trajectory of triangle anomalous threshold:

