Isopin & Electromagnetic Corrections to Weak Matrix Elements

or QED Corrections to Hadronic Processes in Lattice QCD

Benasque August 6th 2015

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PLAN OF THE TALK

1) Physics Motivations

2) Lattice Calculations of QED corrections to the hadron Spectrum

4) QED corrections to the hadronic amplitudes

5) $\pi+ \rightarrow \mu+ \nu_\mu \ (\gamma)$

6) Conclusion & Outlook

work done in collaboration with N.Carrasco, V.Lubicz, C.T.Sachrajda, F.Sanfillipo, N.Tantalo, C.Tarantino, M.Testa

G.M. de Divitiis, P. Dimopoulos, R. Frezzotti, R. Petronzio, G.C. Rossi and S. Simula

The accuracy of lattice calculations of the hadron spectrum (and hence of the quark masses) and of the decay constants and form factors is such that isospin breaking effects cannot be neglected anymore:

FLAG Collaboration, arXiv:1310.8555

$N_f = 2$ $m_{ud} = 3.6(2)$ MeV $m_s = 101(3)$ MeV

$m_s/m_{ud} = 28.1(1.2)$ $\varepsilon = 3\%$-6%

$N_f = 2+1$ $m_{ud} = 3.42(6)(7)$MeV $m_s = 93.8(1.5)(1.9)$ MeV

$m_s/m_{ud} = 27.45(15)(41)$

$f_\pi = 130.2(1.4)$ MeV $f_K = 156.3(0.8)$ MeV $\varepsilon = 0.5\%$-1.1%

$f_K/f_\pi = 1.194(5)$ $\varepsilon = 0.4\%$ $F^{K\pi}(0) = 0.967(4)$ $\varepsilon = 0.4\%$

(0.966(3))

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Phenomenological relevance of precision physics in the Standard Model and beyond

$|V_{us}| F^{K\pi}(0) = 0.2163(5)$ - exp ε =0.2%

$|V_{ud}| f_\pi / |V_{us}| f_\pi$ =0.2758(5) ε =0.2%
see discussion below

$|V_{ud}| = 0.97425(22)$ ε =0.02%

$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$ in the SM ($|V_{ub}|^2 \approx 1.6\ 10^{-5}$)

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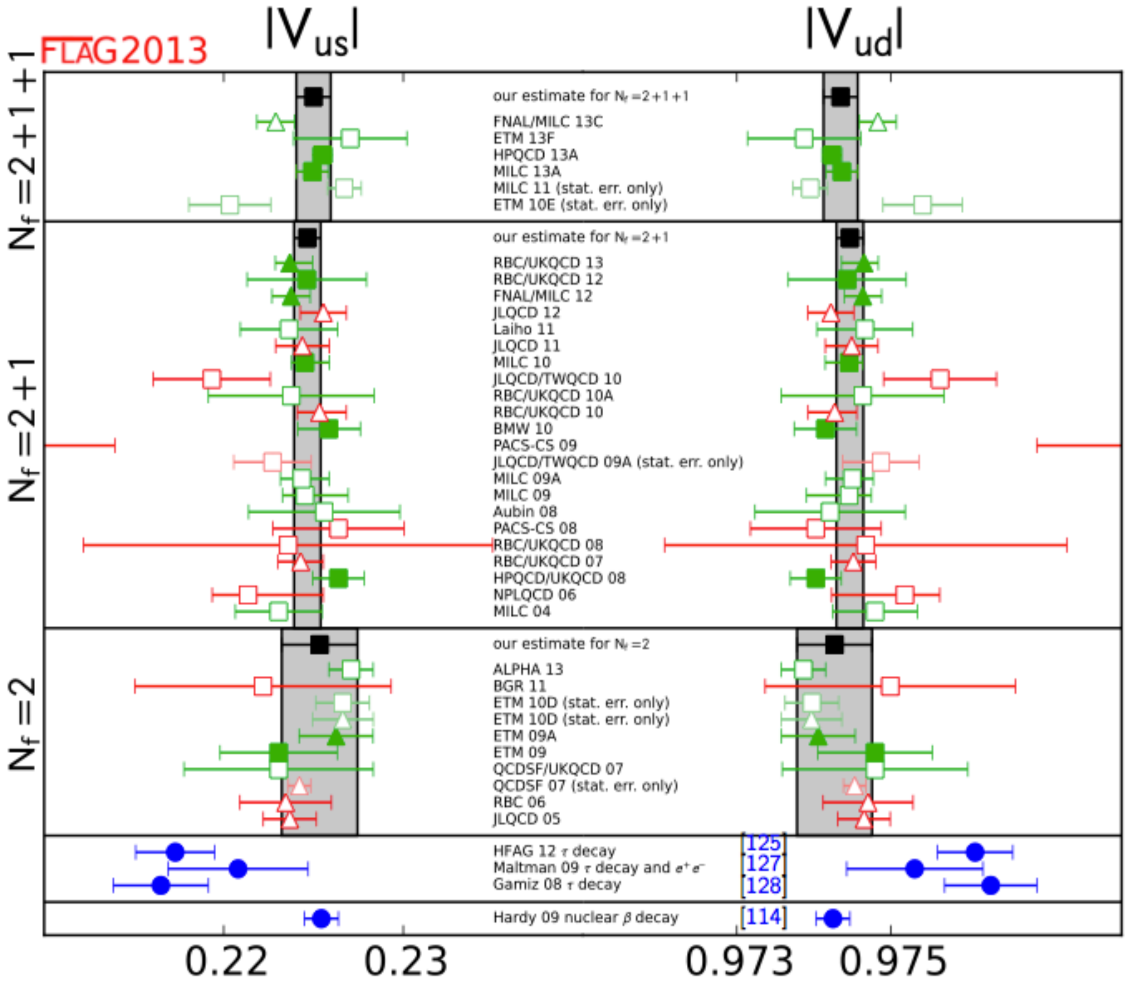
FLAG: lattice predictions within the SM



STANDARD MODEL UNITARITY TRIANGLE ANALYSIS (FLAG)

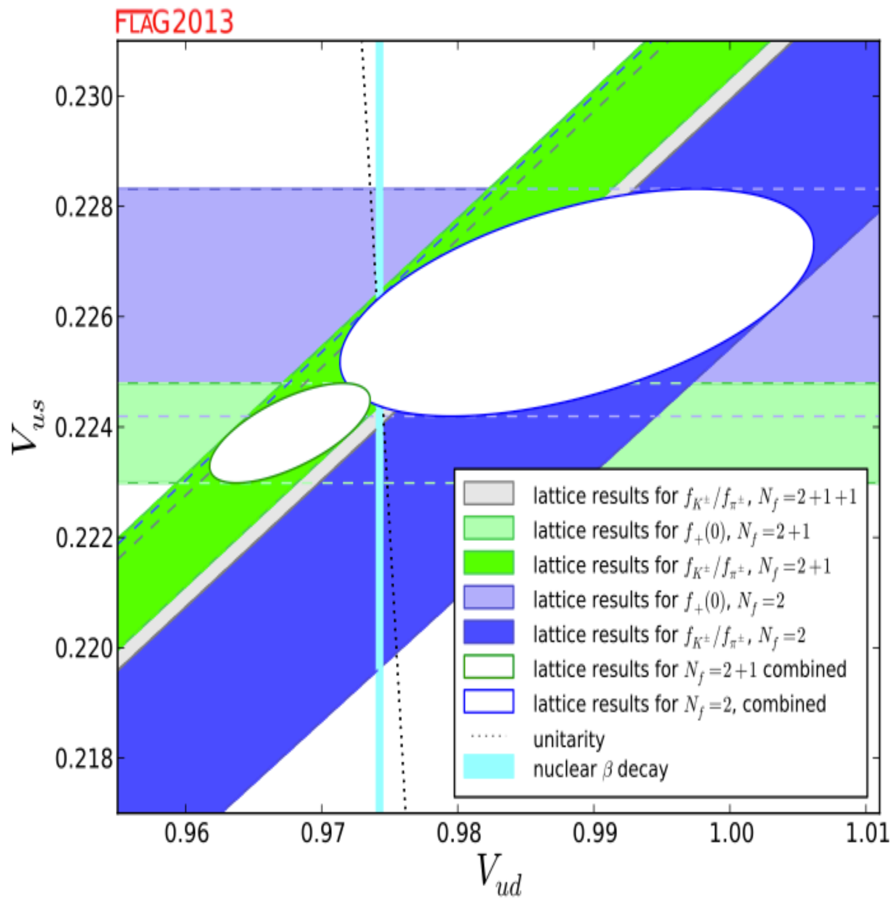
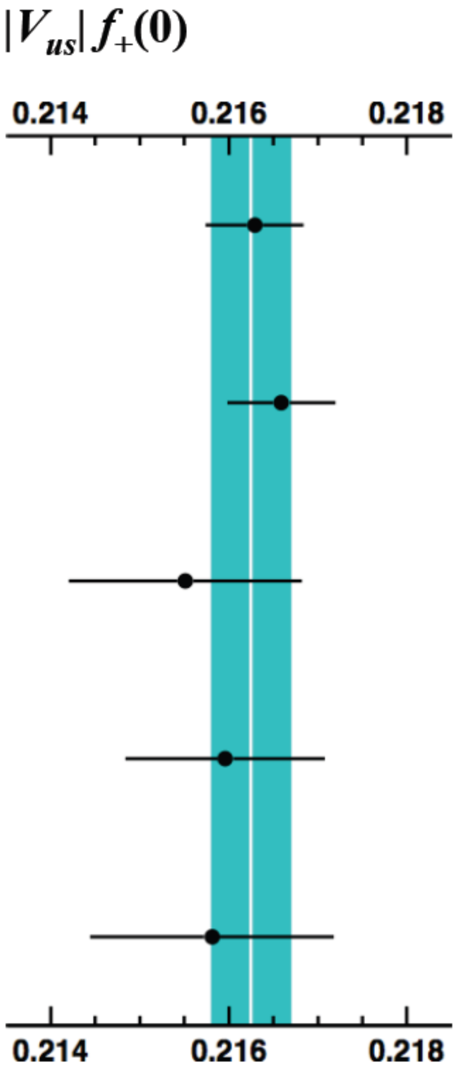


Figure 5: The plot compares the information for $|V_{ud}|$, $|V_{us}|$ obtained on the lattice wit the experimental result extracted from nuclear β transitions. The dotted arc indicates tl correlation between $|V_{ud}|$ and $|V_{us}|$ that follows if the three-flavour CKM-matrix is unitary.

- $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2 = 0.9993(5)$ or $1.0000(6)$ from semileptonic and leptonic respectively

|Vus| f+(0) from world data: 2012

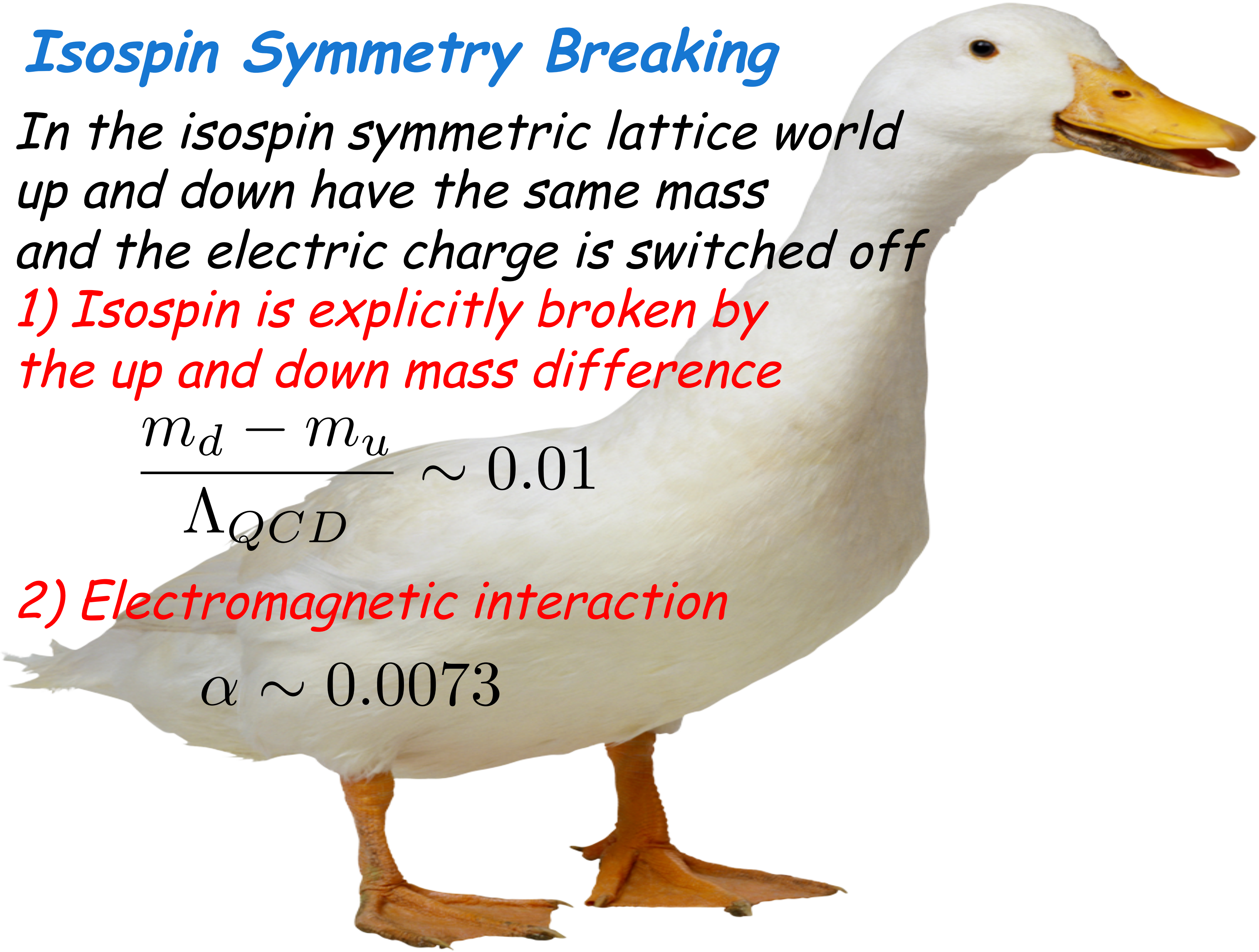


Approx. contrib. to % err from:

		% err	BR	τ	$\delta_{SU,EM}$	Int
$K_L e3$	**0.2163(5)**	**0.26**	0.09	0.20	0.11	0.05
$K_L \mu 3$	**0.2166(6)**	**0.28**	0.15	0.18	0.11	0.06
$K_S e3$	**0.2155(13)**	**0.61**	0.60	0.02	0.11	0.05
$K^\pm e3$	**0.2160(11)**	**0.52**	0.31	0.09	0.41	0.04
$K^\pm \mu 3$	**0.2158(13)**	**0.63**	0.47	0.08	0.41	0.06

Average: $|V_{us}|\,f_+(0)$ = 0.2163(5) **χ^2/ndf = 0.84/4 (93%)**

Isospin Symmetry Breaking



In the isospin symmetric lattice world up and down have the same mass and the electric charge is switched off

1) Isospin is explicitly broken by the up and down mass difference

$$\frac{m_d - m_u}{\Lambda_{QCD}} \sim 0.01$$

2) Electromagnetic interaction

$$\alpha \sim 0.0073$$

Non-compact lattice QED

- Naively discretised **Maxwell action:**

$$S[A_\mu] = \frac{1}{4}\sum_{\mu,\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$

- Pure gauge theory is **free**, it can be solved **exactly**
- Gauge invariance is preserved

QED Corrections to Hadron Masses, or $SU(3)_c \times U(1)$ on the Lattice

QED corrections to the hadron masses only require an ultraviolet cutoff

1) We need a physical condition for any renormalizable coupling to fix the scale i.e. to renormalize the strong (and the electromagnetic) coupling;
2) We must fix the masses of a certain number of hadrons, corresponding to the different flavors, to their physical value;
3) All the other hadron masses are finite and can be predicted
4) Quark masses are determined in your preferred renormalization scheme

QED$_{\mathrm{TL}}$ finite-volume effects

- Example — 1-loop QED$_{\mathrm{TL}}$ [BMWc, 2014]:

$$m(T,L) \underset{T,L\to+\infty}{\sim} m\left\{1-q^2\alpha\left[\frac{\kappa}{2mL}\left(1+\frac{2}{mL}\left[1-\frac{\pi}{2\kappa}\frac{T}{L}\right]\right)\right.\right.$$
$$\left.\left.-\frac{3\pi}{(mL)^3}\left[1-\frac{\coth(mT)}{2}\right]-\frac{3\pi}{2(mL)^4}\frac{L}{T}\right]\right\}$$

up to exponential corrections, with $\kappa = 2.83729\ldots$

Finite volume effects depend on the regulator of the zero mode, but this is not relevant to the following discussion.

<u>Hadron masses are infrared finite</u>

Full QCD + QED projects

	RBC-UKQCD	PACS-CS	QCDSF-UKQCD	BMWc
arXiv	1006.1311	1205.2961	1311.4554 and Lat. 2014	1406.4088
fermions	DWF	clover	clover	clover
N_f	2+1	1+1+1	1+1+1	1+1+1+1
method	reweighting	reweighting	RHMC	RHMC
$\min(M_\pi)$ (MeV)	420	135	250	195
a (fm)	0.11	0.09	0.08	0.06 — 0.10
$\#a$	1	1	1	4
L (fm)	1.8	2.9	1.9 — 2.6	2.1 — 8.3
$\#L$	1	1	2	11

Portelli @ Lattice 2014 - Calculation at several values of α, then extrapolation/interpolation. not really ``full" : linear extrapolation to 1/137 without the renormalization of α

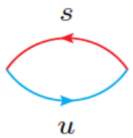
QED & Isospin Corrections to Hadronic Masses: The RM123 approach

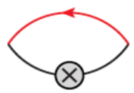
- Identify the isospin breaking term in the action and expand in $\Delta m = (m_d - m_u)/2$

$$S_m = \sum_x \left[m_u \bar{u}u + m_d \bar{d}d \right] = \sum_x \left[\frac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d) - \frac{1}{2}(m_d - m_u)\underbrace{(\bar{u}u - \bar{d}d)}_{} \right] = S_0 - \Delta m \hat{S}$$

$$\langle O \rangle = \frac{\int D\phi\, O\, e^{-S_0 + \Delta m\, \hat{S}}}{\int D\phi\, e^{-S_0 + \Delta m\, \hat{S}}} \overset{1st}{\simeq} \frac{\int D\phi\, O\, e^{-S_0} \left(1 + \Delta m\, \hat{S}\right)}{\int D\phi\, e^{-S_0} \left(1 + \Delta m\, \hat{S}\right)} \simeq \frac{\langle O \rangle_0 + \Delta m \langle O\, \hat{S} \rangle_0}{1 + \Delta m \cancel{\langle \hat{S} \rangle_0}} = \langle O \rangle_0 + \Delta m \langle O\, \hat{S} \rangle_0$$

- For the kaon decay constant:

$C_{K^+K^-}(t) = -$ 

$= -$  $-$  $+ \mathcal{O}(\Delta m_{ud})^2$

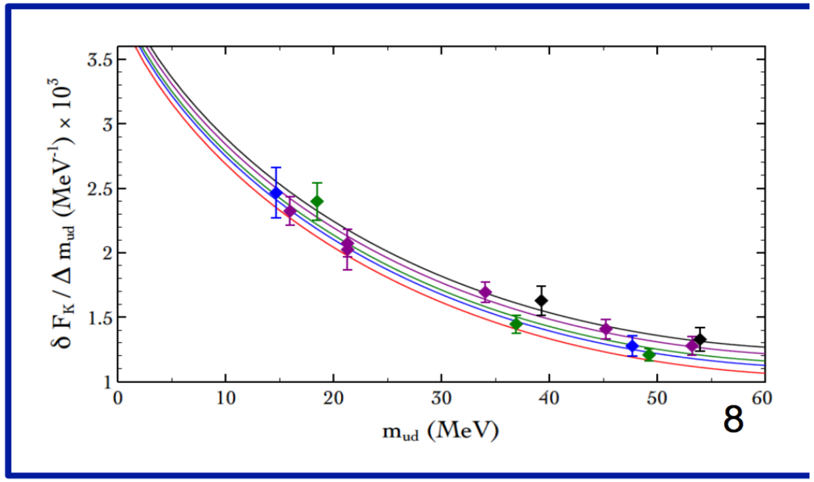
$$\boxed{\delta_{SU(2)} = -0.0080\,(7)}$$

Lattice - Nf=2
RM123 collab. (2012)

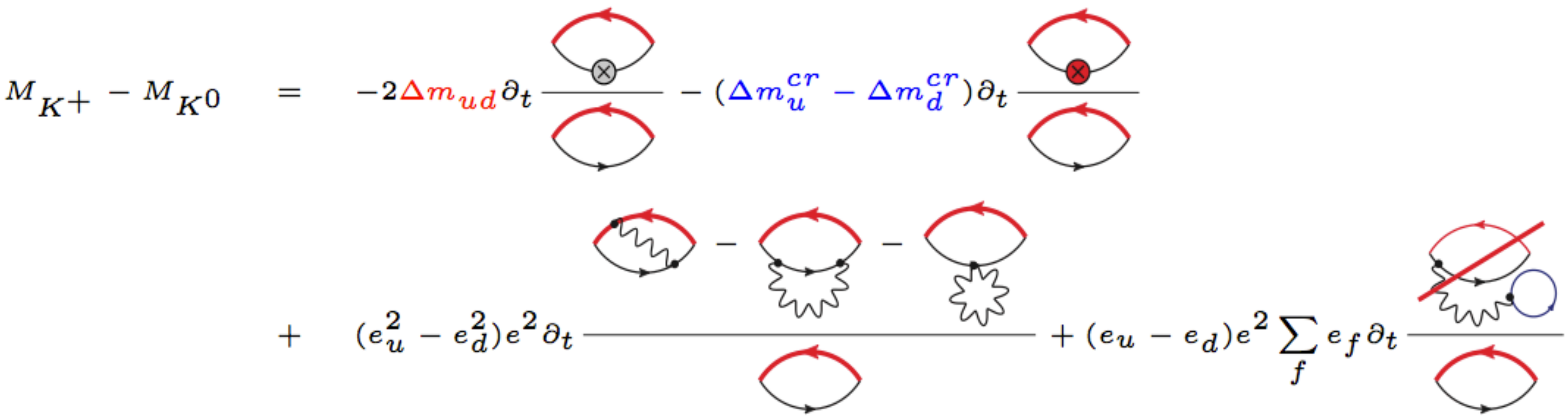
which is ~2.6 σ larger than

$$\boxed{\delta_{SU(2)} = -0.0044\,(12)}$$

ChPT
Cirigliano, Neufeld (2011)



QED & Isospin Corrections to Hadronic Masses: The RM123 approach



Expand the action in the ``small terms" namely in
α and $(m_u=m_d)/\Lambda_{QCD.}$

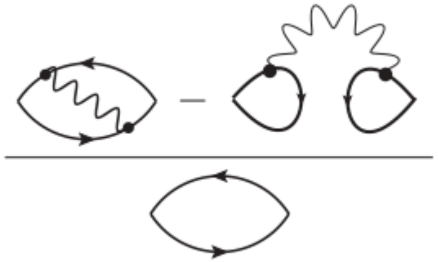
Advantge: We compute the insertion of operators of O(1) and no extrapolation $\alpha \rightarrow 1/137$ is needed;

Disadvantage: Complicated ``disconnected diagrams" must be computed;

Unavoidable: in electromagnetic corrections to hadronic amplitudes

LIB corrections to pion masses (ii)

$$M_{\pi^+} - M_{\pi^0} \quad = \quad \frac{(e_u - e_d)^2}{2} e^2 \partial_t$$



- there are no contributions proportional to $\hat{m}_d - \hat{m}_u$: the pion mass difference at this order is a pure QED effect
- note: sea quark effects are not neglected, they cancel in the difference!
- the electric charge does not renormalize at this order (a problem that *must* instead be faced at higher orders) and the previous expression is finite,

Some remark on QED Corrections to Hadron Masses

FLAG:

We distinguish the physical mass M_P, $P \in \{\pi^+, \pi^0, K^+, K^0\}$, from the mass $\hat{M}_P$ within QCD alone. The e.m. self-energy is the difference between the two, $M_P^\gamma \equiv M_P - \hat{M}_P$.

however, a world without electromagnetism where we can measure the masses of the mesons and fix the scale and the quark masses does not exist thus

$M_P{}^\gamma$ cannot be a physical quantity and
indeed it depends on the convention

It is not clear to me that when comparing the different results these do correspond to the same convention

although useful for a comparison with χpth, $M_P{}^\gamma$ should be abandoned:
without QED you only know that the error is of $O(\alpha)$, but you cannot compute it,
with QED the precise determination of error that you would have made depends on the convention, thus who cares?

People who live in glass houses should'nt throw stones

Chi è senza peccato scagli la prima pietra

Even RM123, following the common lore …..

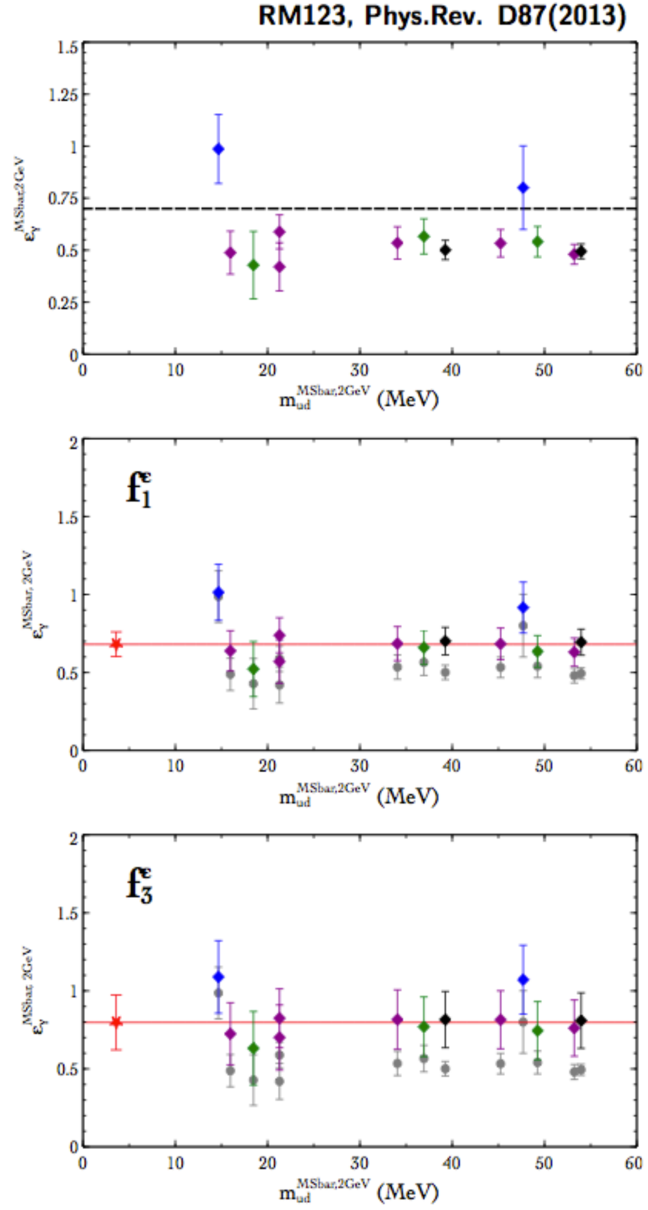
- the value of ε_γ depends upon the renormalization prescription used to separate QED from QCD IB effects

$$\varepsilon_\gamma = \frac{\left[M^2_{K^+} - M^2_{K^0}\right]^{QED} - \left[M^2_{\pi^+} - M^2_{\pi^0}\right]^{QED}}{M^2_{\pi^+} - M^2_{\pi^0}}$$

- it is needed to calculate the light quark masses by starting from QCD ($\hat{m}_u \neq \hat{m}_d$) lattice simulations and using the QCD contribution to the kaon mass splitting as "experimental" input

$$\begin{aligned} \varepsilon_\gamma &= 0.79(18)(18) \\ \hat{m}_u/\hat{m}_d &= 0.50(2)(3) \end{aligned}$$

- note: these results are scale and scheme dependent, $\overline{MS}$ 2 GeV, and *depend* upon the matching prescription used to separate QED from QCD contributions



N. Tantalo @ CERN 2014

QED (Isospin) Corrections in Hadronic Processes

After the renormalization of the $SU(3)_c \times U(1)$ Lagrangian you still need

1) The renormalization of the operators mediating the physical process of interest (e.g. the Weak effective Hamiltonian). But this is not a novelty;
2) A complex procedure to remove the infrared cutoff because in general the amplitudes, contrary to the masses, are infrared divergent.

A method to solve this problem is presented . This will be done by discussing an explicit example and will allow the discussion of some important theoretical subtelties

How to solve the problem of the infrared divergences discussed through an explicit example

$$\pi \to \ell + \nu_\ell + (\gamma)$$

N.Carrasco, V.Lubicz, G.M., C.T.Sachrajda, F.Sanfillipo, N.Tantalo, C.Tarantino, M.Testa

in preparation

NOTE: Chiral Perturbation Theory is NOT Used

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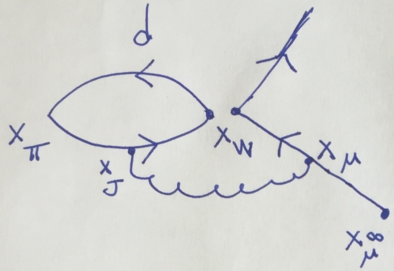

$\pi^+ \to \mu^+ \nu_\mu$

$$- \sum_{\vec{x}_\pi, x_J, x_\mu, \vec{x}_\mu^\infty, \vec{x}_\nu} \mathrm{tr}\left(\Gamma_\pi S_d(x_\pi, x_W) \gamma_L^\rho S_u(x_W, x_J) \gamma^\sigma S_u(x_J, x_\pi) \right)$$

$$\cdot\ \phi_\sigma(x_J) \qquad e^{-i\vec{p}_\nu \cdot \vec{x}_\nu} S(x_\nu, x_W) \gamma_\rho^L S_\mu(x_W, x_\mu) \gamma^\lambda S(x_\mu - x_\mu^\infty)$$

$$\phi_\lambda(x_\mu) \qquad e^{i\vec{p}_\mu \cdot \vec{x}_\mu^\infty}$$

$$\sum_{\vec{x}_\mu^\infty} S(x_\mu - x_\mu^\infty) e^{i\vec{p}_\mu \cdot \vec{x}_\mu^\infty} = \int \frac{dq_0}{(2\pi)} \int \frac{d^3\vec{q}}{(2\pi)^3} e^{iq_0(t_\mu - t_\mu^\infty)}$$

$$e^{i\vec{q}\cdot\vec{x}_\mu} \quad \frac{e^{i(\vec{p}_\mu - \vec{q}_\mu)\vec{x}_\mu^\infty}}{q_0^2 + \vec{q}^{\,2} + m_\mu^2}$$

$$= \int \frac{dq_0}{2\pi} e^{i\vec{p}_\mu \cdot \vec{x}_\mu} \frac{1}{\left(q_0^2 + \vec{p}_\mu^{\,2} + m_\mu^2\right)} e^{iq_0(t_\mu - t_\mu^\infty)} \to e^{E_\mu t_\mu}$$

$$\times e^{i\vec{p}_\mu \cdot \vec{x}_\mu} \times \left(\frac{e^{-E_\mu t_\mu^\infty}}{2E_\mu}\right)$$

(2)

www.ictp.it

$$- \sum_{\vec{x}_\pi, x_J, x_\mu} e^{-i\vec{p}_\mu \cdot \vec{x}_w + E_\nu t_w} \, t_2 \, \Gamma_\pi S_d(x_\pi, x_w) \gamma_L^\rho S_u(x_w, x_J) \gamma^\sigma S_u(x_J, x_\pi)$$

$$\phi_\sigma(x_J) \quad e^{i\vec{p}_\mu \cdot \vec{x}_\mu + E_\mu t_\mu}$$

$$\bar{u}(p_\nu = -p_\mu) \gamma_\rho^L S_\mu(x_w - x_\mu) \gamma^\lambda u(p_\mu) \phi_\lambda(x_\mu)$$

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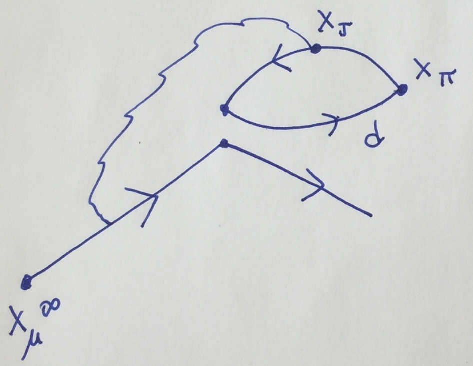
(3)



$$\mu^- \to \pi^- + \nu_\mu$$

$$- t_2 \, \Gamma_\pi S_d(x_\pi, x_w) \gamma_L^\rho S_u(x_w, x_J) \gamma^\sigma S_u(x_J, x_\pi) \, e^{i\vec{p}_\nu \cdot (\vec{x}_\pi - \vec{x}_w)} \phi_\sigma(x_J)$$

$$\bar{u}(p_\nu) \gamma_\rho^L S_\mu(x_w - x_\mu) \gamma^\lambda u(p_\mu = p_\mu^0 = m_\mu) \, \phi_\lambda(x_\mu) \, e^{-m_\mu t_\mu}$$

Leptonic decays at tree level

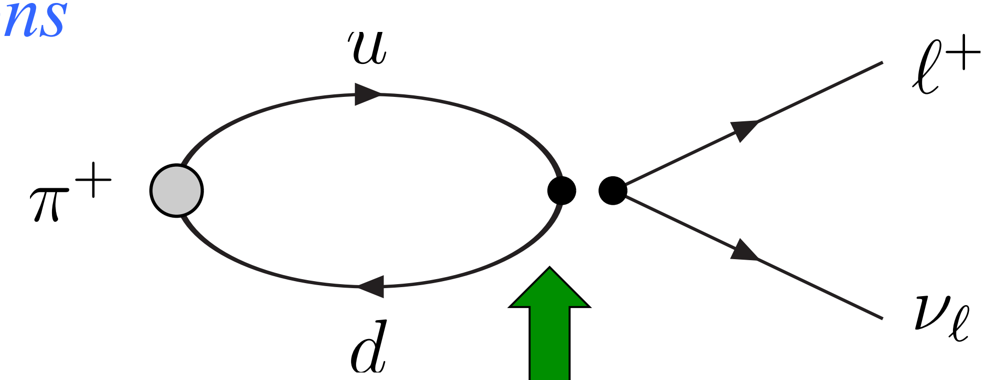
Since the mass of the pion is much lower than M_W we use the effective Hamiltonian

$$\boxed{H_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{ud}^{*} (\bar{d}\gamma^{\mu}(1-\gamma^5)u)\,(\bar{\nu}_{\ell}\gamma_{\mu}(1-\gamma^5)\ell)}$$

from which we compute

$$\boxed{\Gamma_0^{\text{tree}}(\pi^+ \to \ell^+\nu_{\ell}) = \frac{G_F^2\,|V_{ud}|^2 f_{\pi}^2}{8\pi}\, m_{\pi}\, m_{\ell}^2 \left(1-\frac{m_{\ell}^2}{m_{\pi}^2}\right)^2}$$

- *0 in Γ_0 means zero photons*
- *G_F is the Fermi constant defined from μ decay*
- *f_{π} is computed in lattice QCD*



Leptonic decays at O(α) – The ultraviolet matching in the ``W Regularization”

If G_F is the Fermi constant defined at O(α) from μ decay in the standard (convention dependent) way

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1 - \frac{8m_e^2}{m_\mu^2}\right]\left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]$$

S.M.Berman, PR 112 (1958) 267; T.Kinoshita and A.Sirlin, PR 113 (1959) 1652

then the effective Hamiltonian in the W-regularization is given by (Sirlin PRD 22 (80) 971)

$$H_{\rm eff} = -\frac{G_F}{\sqrt{2}} V_{ud}^* \left(1 + \frac{\alpha}{\pi}\log\frac{M_Z}{M_W}\right)(\bar{d}\gamma^\mu(1-\gamma^5)u)\,(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell)$$

matching the (Wilson) lattice to the W-regularization.

$$O_1^{\rm W-reg} = \left(1 + \frac{\alpha}{4\pi}\left(2\log a^2 M_W^2 - 15.539\right)\right) O_1^{\rm bare} + \frac{\alpha}{4\pi}\left(0.536\, O_2^{\rm bare} + 1.607\, O_3^{\rm bare} - 3.214\, O_4^{\rm bare} - 0.804\, O_5^{\rm bare}\right)$$

Rate at O(α)

$$\Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E)$$

$|V_{ud}|$

where $$\Gamma(\Delta E) = \int_0^{\Delta E} dE_\gamma \frac{d\Gamma}{dE_\gamma}$$

contrary to the hadron masses

at O(α) both Γ_0 and $\Gamma_1(\Delta E)$ are

INFRARED DIVERGENT

although the divergence cancel in the sum

F. Bloch, A. Nordsieck Phys.Rev. 52 (1937) T.D. Lee, M. Nauenberg Phys.Rev. 133 (1964)

and the infinite volume limit cannot be separately taken

At this stage we propose to compute $\Gamma_1(\Delta E)$ in perturbation theory @ values of ΔE corresponding to photons which are sufficiently soft for the point-like approximation of the pion to be valid
($\Delta E \ll \Lambda_{QCD} \approx 4\pi f_\pi$)
but hard enough with respect to the experimental resolution.
A value of O(10-20 MeV) seems to be appropriate both theoretically and experimentally.
F. Ambrosino et al., KLOE Collaboration,
PLB 632 (2006) 76; EPJC 64 (2009) 627; 65 (2010) 703(E);
J. Bijnens, G. Ecker, J. Gasser, NPB 396 (1993) 81; V.Cirigliano, I.Rosell, JHEP 0710 (2007) 005

In the future, as techniques and resources improve, it may be better to compute $\Gamma_1(\Delta E)$ nonperturbatively over a larger range of photon energies
(about the analytical continuation to the Euclidean see later)

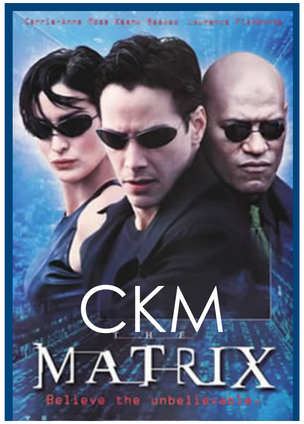
MASTER FORMULA for the rate at O(α)

$$\Gamma(\Delta E) = \lim_{V\to\infty} (\Gamma_0 - \Gamma_0^{pt}) + \lim_{V\to\infty} (\Gamma_0^{pt} + \Gamma_1(\Delta E))$$

pt = point-like & perturbative

- the infrared divergences in Γ_0 and Γ_0^{pt} are exactly the same and cancel in the difference
- $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ is infrared finite since is a physical, well defined quantity *F. Bloch, A. Nordsieck Phys.Rev. 52 (1937) T.D. Lee, M. Nauenberg Phys.Rev. 133 (1964)*
- the infrared divergences in $\Delta\Gamma_0(L) = \Gamma_0 - \Gamma_0^{pt}$ and $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ cancel separately hence they can be regulated with different infrared cutoff
- Γ_0 and Γ_0^{pt} are also ultraviolet finite

We now discuss the two terms, $\Delta\Gamma_0(L)$ and $\Gamma(\Delta E)$



Leptonic decays at O(α) – Perturbative Calculation of $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$

U.V. & Infrared finite but contains $\log(M_W)$ & $\log(\Delta E)$

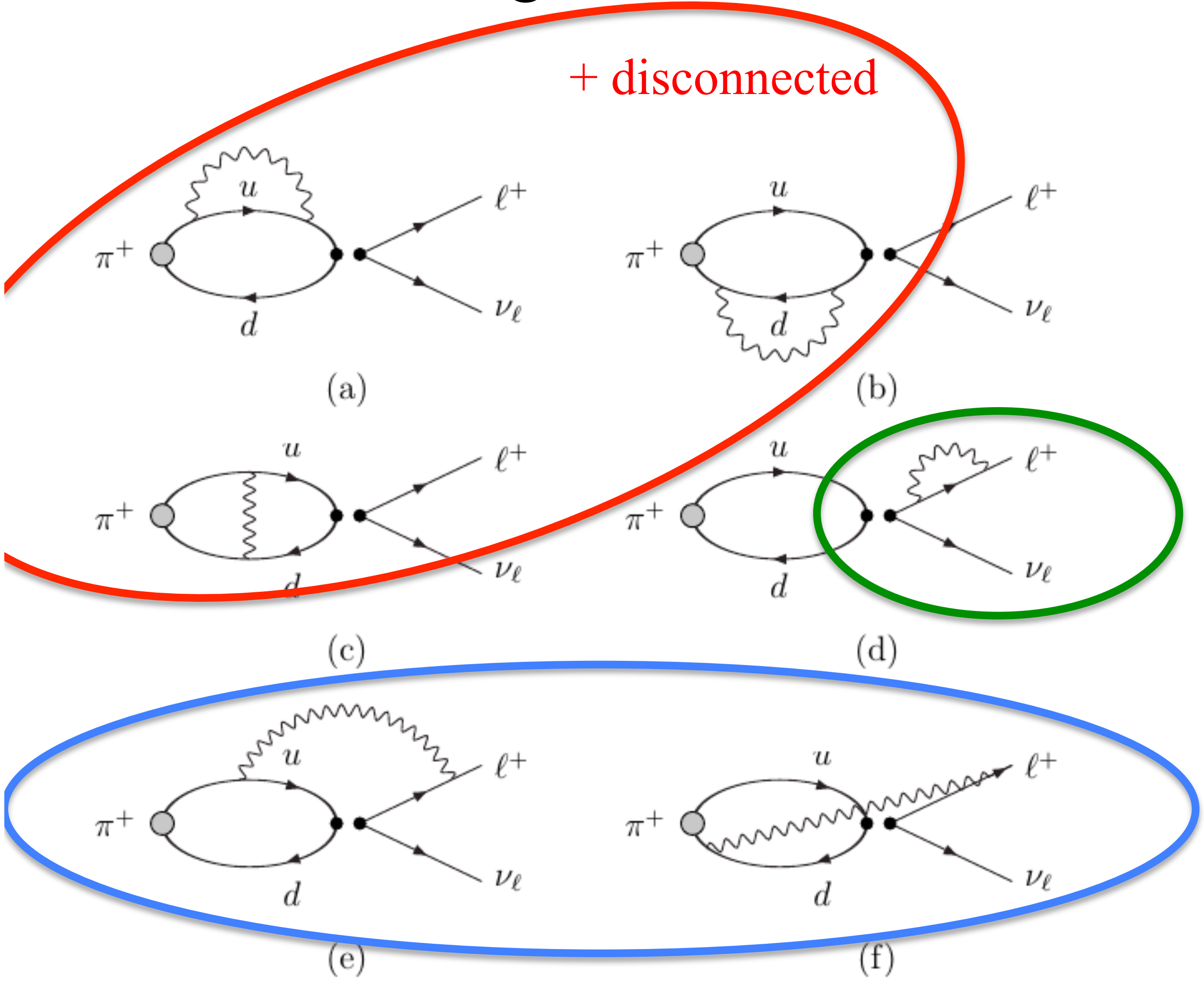
$$\Gamma(\Delta E) = \Gamma_0^{\text{tree}} \times \Bigg(1 + \frac{\alpha}{4\pi}\Bigg\{ 3\log\left(\frac{m_\pi^2}{M_W^2}\right) + \log\left(r_\ell^2\right) - 4\log(r_E^2) + \frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2)$$
$$-2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_E^2)\log(r_\ell^2) - 4\frac{1+r_\ell^2}{1-r_\ell^2}\text{Li}_2(1-r_\ell^2) - 3$$
$$+\left[\frac{3+r_E^2-6r_\ell^2+4r_E(-1+r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E) + \frac{r_E(4-r_E-4r_\ell^2)}{(1-r_\ell^2)^2}\log(r_\ell^2)\right.$$
$$\left. -\frac{r_E(-22+3r_E+28r_\ell^2)}{2(1-r_\ell^2)^2} - 4\frac{1+r_\ell^2}{1-r_\ell^2}\text{Li}_2(r_E)\right]\Bigg\}\Bigg)$$

$\Gamma(\Delta E_1)$ *T.Kinoshita, PRL 2 (1959) 477*

$$\boxed{r_E = \frac{2\Delta E}{m_\pi} \qquad r_\ell = \frac{m_\ell}{m_\pi}}$$

Leptonic decays at O(α) – The first term of the Master Formula $\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$

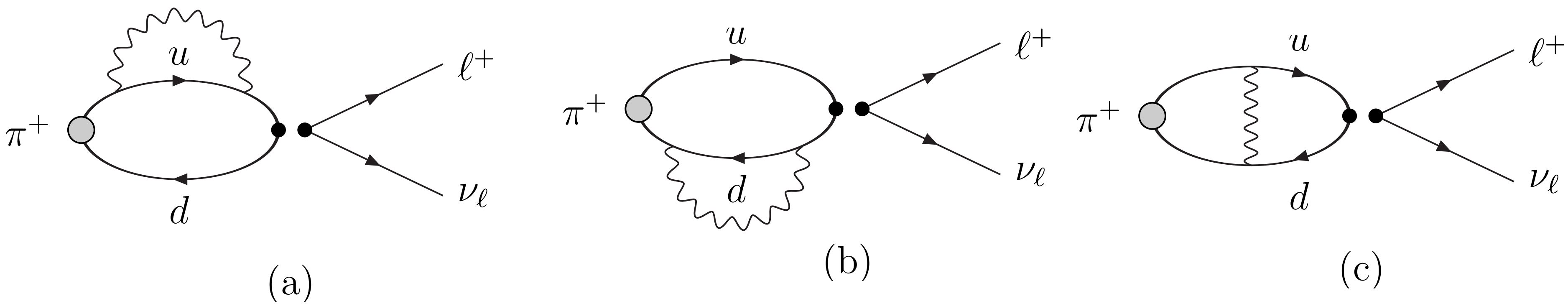
- Each of the two terms is U.V. finite but contains $\log(M_W)$
- Infrared divergences cancel in the difference



at this order we can take the difference of the amplitudes

Can be computed as discussed in arXiv: 1303.4896,Phys.Rev. D87(2013)

NOT by including the electromagnetic field in the action



The relevant correlation function is (the lepton leg is trivial)

$$C_1(t) = \frac{1}{2}\int d^3\mathbf{x}\, d^4x_1\, d^4x_2\ \langle 0|T\{J_W^{\nu}(0)\, j^{\mu}(x_1) j_{\mu}(x_2)\phi^{\dagger}(\mathbf{x},t)\}\,|0\rangle\ \Delta(x_1,x_2)$$

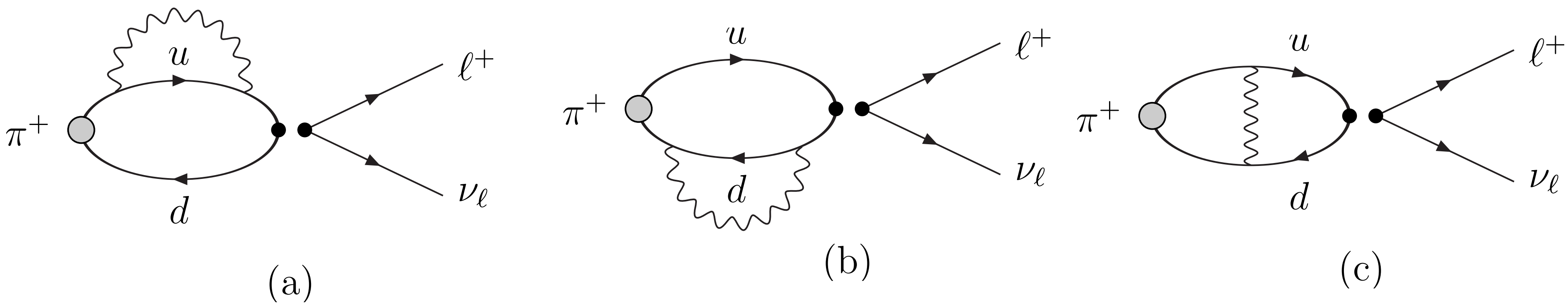
weak V-A current

electromagnetic current

$$j_\mu(x) = \sum_f Q_f\, \bar{f}(x)\gamma_\mu f(x)$$

this is the same set of diagrams used to compute the electromagnetic corrections to the pion (hadron) mass

(the lepton leg is completely irrelevant)



Combining $C_1(t)$ with the lowest order correlator

$$C_0(t) + C_1(t) \simeq \frac{e^{-m_\pi t}}{2m_\pi} Z^\phi \langle 0 \,|\, J_W^0(0) \,|\, \pi^+ \rangle$$

where the $O(\alpha)$ corrections are included; by writing

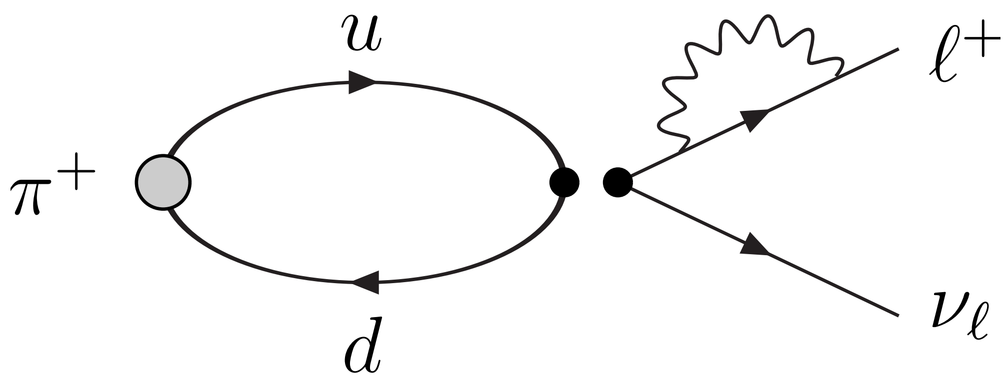
$$e^{-m_\pi t} \simeq e^{-m_\pi^0 t} (1 - \delta m_\pi \, t)$$

δm_π is infrared finite and gauge invariant

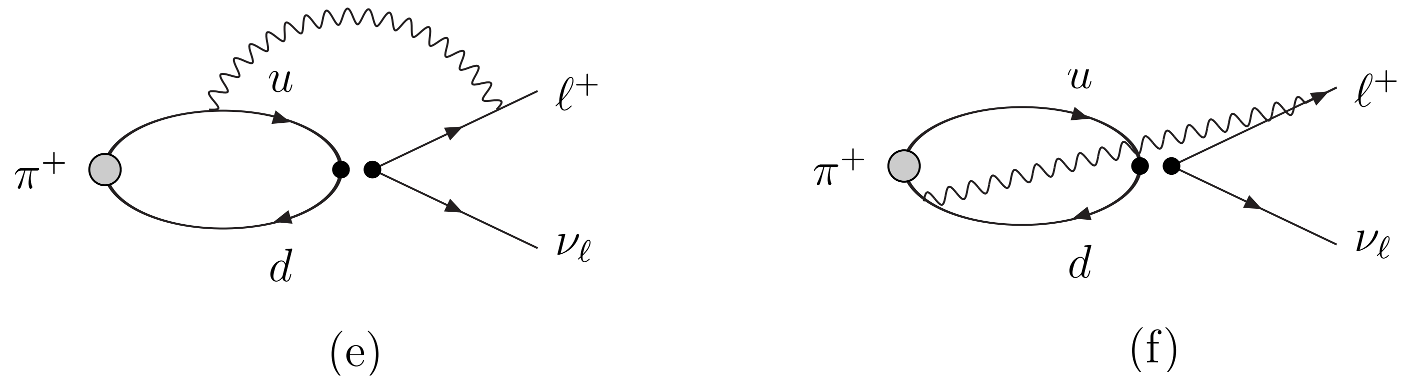
Z^ϕ and the matrix element of the axial current *however* are infrared divergent and cannot be interpreted as a correction to f_π

This diagram is an easy case: its contribution to $\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$ can be readily obtained in perturbation theory.

The recipe is simply to redefine the operator $O_1^{W\text{-}reg}$ and compute f_π in the numerical simulation

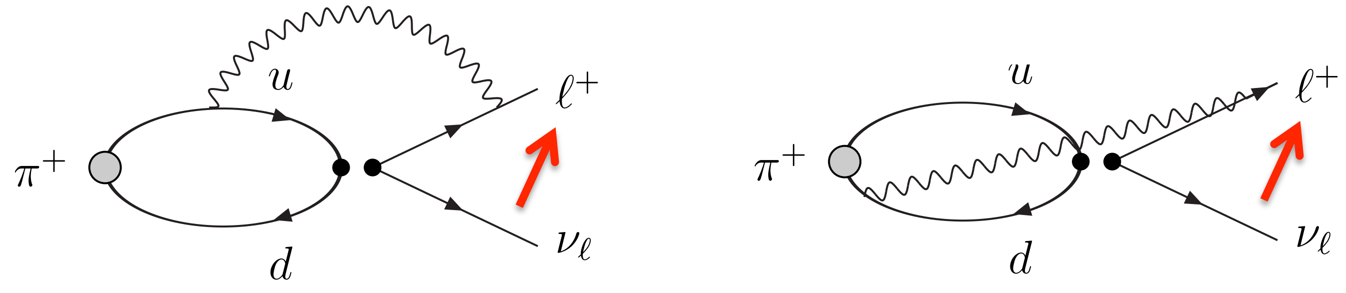


(d)



- *Certainly these diagrams are not simply a generalization of the evaluation of f_π ; they are also infrared divergent (there are also the disconnected diagrams)*
- *We have to isolate the finite volume ground state (necessity of a mass gap – Minkowski $\longleftrightarrow$ Euclidean continuation J. Gasser and G.R.S. Zarnauskas, Phys. Lett. B 693 (2010) 122)*
- *Finite volume effects, expected of the $O(1/L\,\Lambda_{QCD})$ after the cancellation of the infrared divergence, should be investigated in a numerical simulation.*

Calculation of the `nasty' diagrams in a lattice simulation



(e) (f)

The starting point is the Minkowski Green function

$$\int d^4x_1 d^4x_2 < 0|\, T(j_\mu(x_1) J_W^\nu(0))\, |\pi > \; iD_F(x_1 - x_2)\{\bar{u}(p_{\nu_\ell})\gamma^\nu(1-\gamma^5)(iS_F(x_2))\gamma^\mu v(p_\ell)\}e^{ip_\ell \cdot x_2}$$

from which we can compute the on-shell amplitude

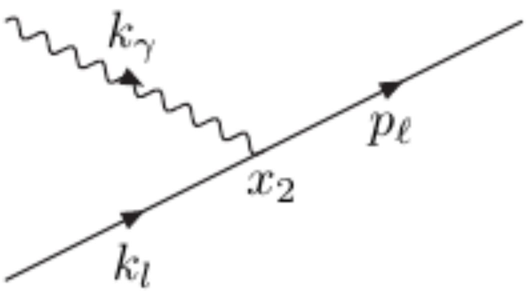
$$\bar{u}_\alpha(p_{\nu_\ell})(\bar{M}_1)_{\alpha\beta} v_\beta(p_\ell) = \; -i \lim_{k_0 \to m_\pi} ({k_0}^2 - m_\pi^2) \int d^4x_1 d^4x_2\, d^4x\, e^{-ik^0y^0} < 0|T(j_\mu(x_1) J_W^\nu(0)\pi(x))|0 >$$
$$\times iD_F(x_1 - x_2)\{\bar{u}(p_{\nu_\ell})\gamma_\nu(1-\gamma^5)(iS_F(x_2))\gamma^\mu v(p_\ell)\}e^{ip_\ell \cdot x_2}$$

which in the Euclidean simulation becomes

$$\bar{C}_1(t)_{\alpha\beta} = \; \int d^3\mathbf{x}\, d^4x_1\, d^4x_2\, \langle 0|T\{J_W^\nu(0)\, j_\mu(x_1)\phi^\dagger(\mathbf{x},t)\}|\,0\rangle\, \Delta(x_1 - x_2)$$
$$\times \left(\gamma_\nu(1-\gamma^5)S(x_2)\gamma^\mu\right)_{\alpha\beta} e^{E_\ell\, t_2} e^{-i\mathbf{p}_\ell \cdot \mathbf{x}_2}$$

A few technical but non trivial IMPORTANT slides:
the continuation from Minkowski to Euclidean

we need to ensure that the t_2 integration up to ∞ converges in spite of the factor $e^{E_l\, t_2}$ where $E_l = \sqrt{m_l^2 + p_l^2}$ is the energy of the outgoing charged lepton



1) Momentum conservation: since we integrate over x_2
$p_l = k_l + k_\gamma$

2) The integrations over the energies k_{4l} and $k_{4\gamma}$ lead to the exponential factor $e^{-(\omega_l + \omega_\gamma - E_l)\, t_2}$ where $\omega_l = \sqrt{m_l^2 + k_l^2}$, $\omega_\gamma = \sqrt{m_\gamma^2 + k_\gamma^2}$, and m_γ is the mass of the photon introduced as an infra-red cut-off.

A few technical but non trivial IMPORTANT slides: *the continuation from Minkowski to Euclidean*

3) … but $(\omega_l+\omega_\gamma) \geq \sqrt{(m_l+m_\gamma)^2 + p_l^2} > E_l = \sqrt{m_l^2+p_l^2}$

thus the argument of the exponent $e^{-(\omega_l + \omega_\gamma - E_l)\, t_2}$ is negative for every term appearing in the sum over the intermediate states and the integral over t_2 converges

4) note that the integration over t_2 is also convergent if we set $m_\gamma=0$ but remove photon zero mode in finite volume. In this case $(\omega_l+\omega_\gamma) > E_l+[1-(p_l/E_l)]\ (k_\gamma)_{min}$

- necessity of a mass gap
- absence of a lighter intermediate state

under these conditions

$$\bar{C}_1(t)_{\alpha\beta} \simeq Z_0^{\phi}\,\frac{e^{-m_\pi^0|t|}}{2m_\pi^0}\,(\bar{M}_1)_{\alpha\beta}$$

and the contribution to the amplitude from these diagrams is given by

$$\bar{u}_\alpha(p_{\nu_\ell})(\bar{M}_1)_{\alpha\beta}v_\beta(p_\ell)$$

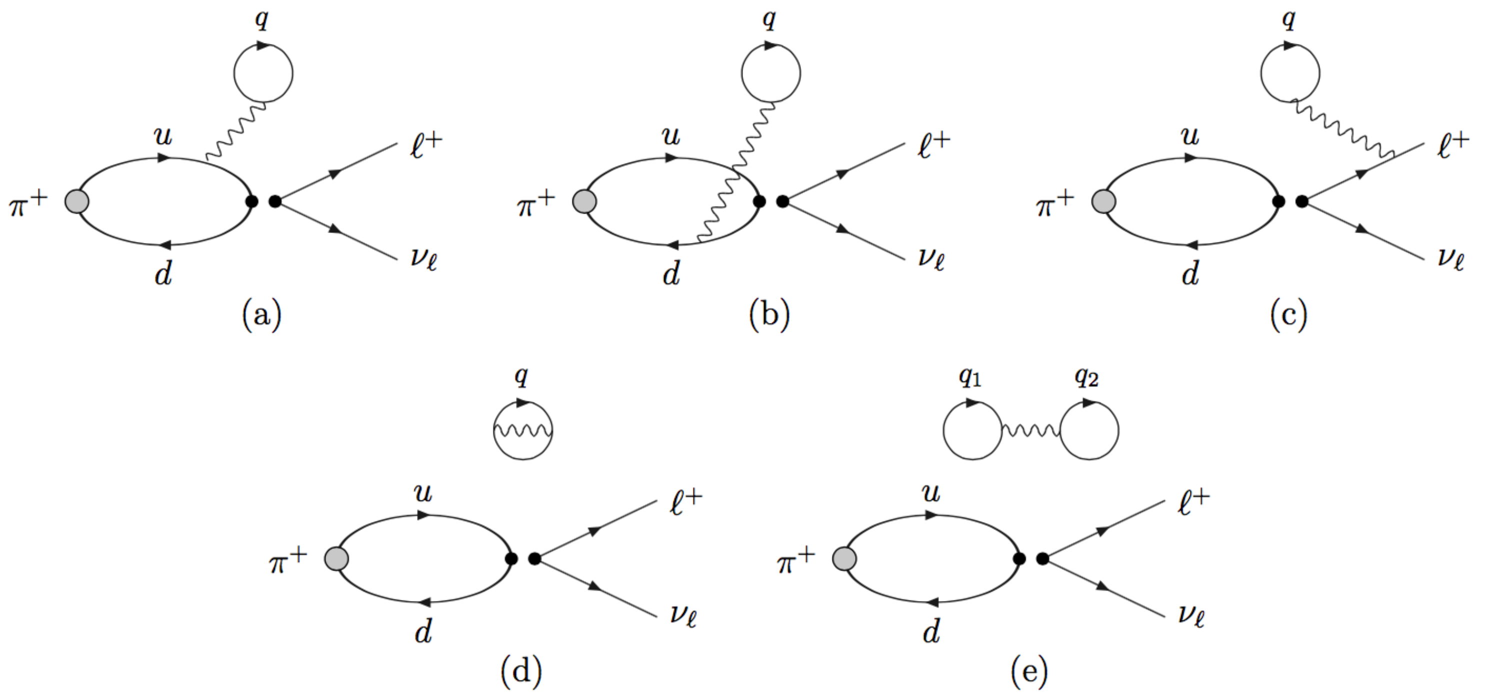


FIG. 6: Disconnected diagrams contributing at $O(\alpha)$ contribution to the amplitude for the decay $\pi^+ \to \ell^+\nu_l$. The curly line represents the photon and a sum over quark flavours q, q_1 and q_2 is to be performed.

$$\Gamma(\Delta E) = \boxed{\lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{\text{pt}}\right)} + \lim_{V\to\infty}\left(\Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E)\right)$$

- $\Gamma_0^{\text{pt}}(L)$ is calculated in perturbation theory with a pointlike pion

- UV divergences are regularized with the W-regularization
- IR divergences are regularized by the finite volume (same of $\Gamma_0(L)$)

- For the pion self energy, the result is:

Preliminary

$$\frac{1}{\pi}\left(\frac{2\pi}{L}\right)^3 \sum_{\vec q}\left\{\frac{1}{\left(M_W^4 - 4m_\pi^2 E_{W,\vec q}^2\right)^2}\left[16 m_\pi^4\left(\frac{\vec q^{\,2}}{E_{W,\vec q}} + \frac{M_W^2}{E_{W,\vec q}} + \frac{M_W^2}{E_{\pi,\vec q}}\right) + M_W^4\left(\frac{4\vec q^{\,2}}{E_{W,\vec q}} - \frac{4\vec q^{\,2}}{E_{\pi,\vec q}} + \frac{M_W^2}{E_{W,\vec q}} + \frac{M_W^2}{E_{\pi,\vec q}}\right)\right.\right.$$

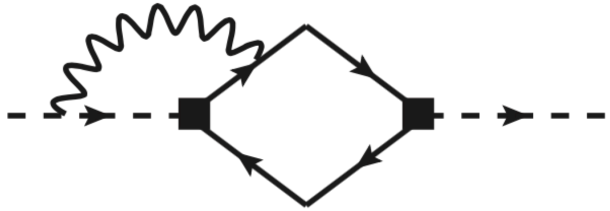
$$\left.\left. -4M_W^2 m_\pi^2\left(\frac{3\vec q^{\,2}}{E_{W,\vec q}} - \frac{3\vec q^{\,2}}{E_{\pi,\vec q}} + \frac{2M_W^2}{E_{W,\vec q}} + \frac{2M_W^2}{E_{\pi,\vec q}}\right)\right] - \left(M_W \to 0\right)\right\}$$

$$\boxed{\vec q = \frac{2\pi}{L}\left(n_x, n_y, n_z\right)} \qquad E_{X,\vec q} = \sqrt{M_X^2 + \vec q^{\,2}}$$

Courtesy by V. Lubicz

Γ_0^{pt} *The nasty diagram*

sum vs integral under study



$$\left(\frac{2\pi}{L}\right)^3 \sum_{\vec{q}} \frac{1}{\pi}$$

$$\Bigg(\frac{1}{m_W{}^2|\vec{q}\cdot\vec{q}|^{1/2}} - \frac{2m^2}{(m^2-m_\mu{}^2)m_W{}^2|\vec{q}\cdot\vec{q}|^{1/2}} + \frac{2m_\mu{}^2}{(m^2-m_\mu{}^2)m_W{}^2|\vec{q}\cdot\vec{q}|^{1/2}} + \frac{4|\vec{q}\cdot\vec{q}|^{1/2}}{m_W{}^4} - \frac{4m^2|\vec{q}\cdot\vec{q}|^{1/2}}{(m^2-m_\mu{}^2)m_W{}^4} +$$

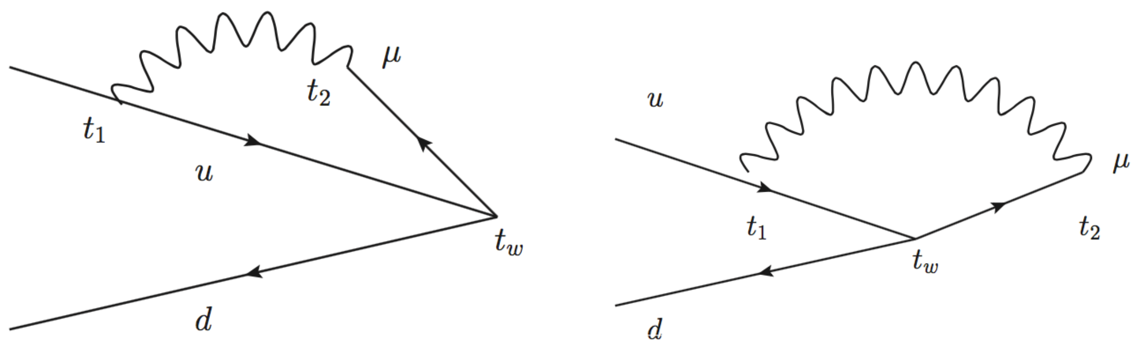
$$\frac{4m_\mu{}^2|\vec{q}\cdot\vec{q}|^{1/2}}{(m^2-m_\mu{}^2)m_W{}^4} + \frac{m^2\sqrt{\frac{1}{m^2+\vec{q}\cdot\vec{q}}}}{2(m^2-m_\mu{}^2)\vec{q}\cdot\vec{q}} + \frac{m_\mu{}^2\sqrt{\frac{1}{m^2+\vec{q}\cdot\vec{q}}}}{2(m^2-m_\mu{}^2)\vec{q}\cdot\vec{q}} - \frac{m_\mu{}^2\sqrt{\frac{1}{m_\mu{}^2+\vec{q}\cdot\vec{q}}}}{(m^2-m_\mu{}^2)\vec{q}\cdot\vec{q}} - \frac{3\sqrt{\frac{1}{m_W{}^2+\vec{q}\cdot\vec{q}}}}{m_W{}^2} -$$

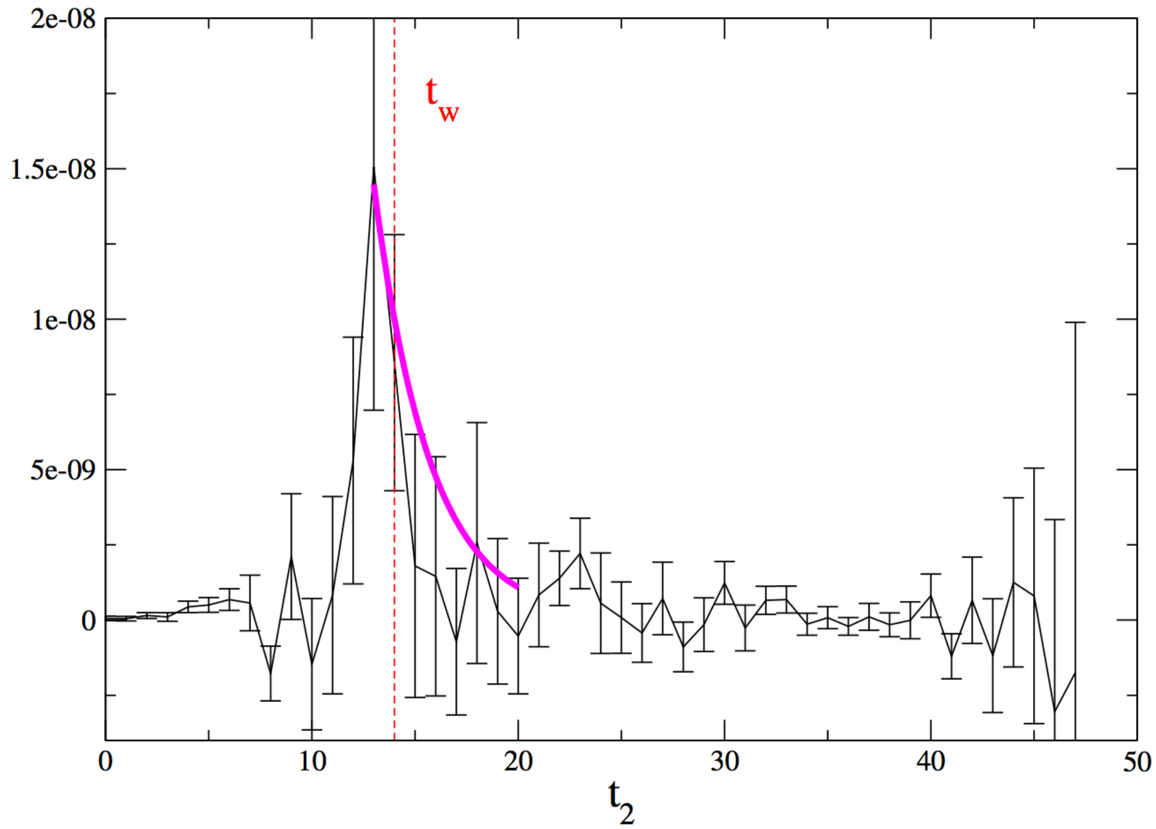
$$\frac{4\vec{q}\cdot\vec{q}\sqrt{\frac{1}{m_W{}^2+\vec{q}\cdot\vec{q}}}}{m_W{}^4} + \frac{4m^2\sqrt{m_W{}^2+\vec{q}\cdot\vec{q}}}{(m^2-m_\mu{}^2)m_W{}^4} - \frac{4m_\mu{}^2\sqrt{m_W{}^2+\vec{q}\cdot\vec{q}}}{(m^2-m_\mu{}^2)m_W{}^4} + \frac{m^2\left(\frac{1}{m_\gamma{}^2+\vec{q}\cdot\vec{q}}\right)^{3/2}\mathrm{Log}[r_\mu{}^2]}{2(m^2-m_\mu{}^2)} +$$

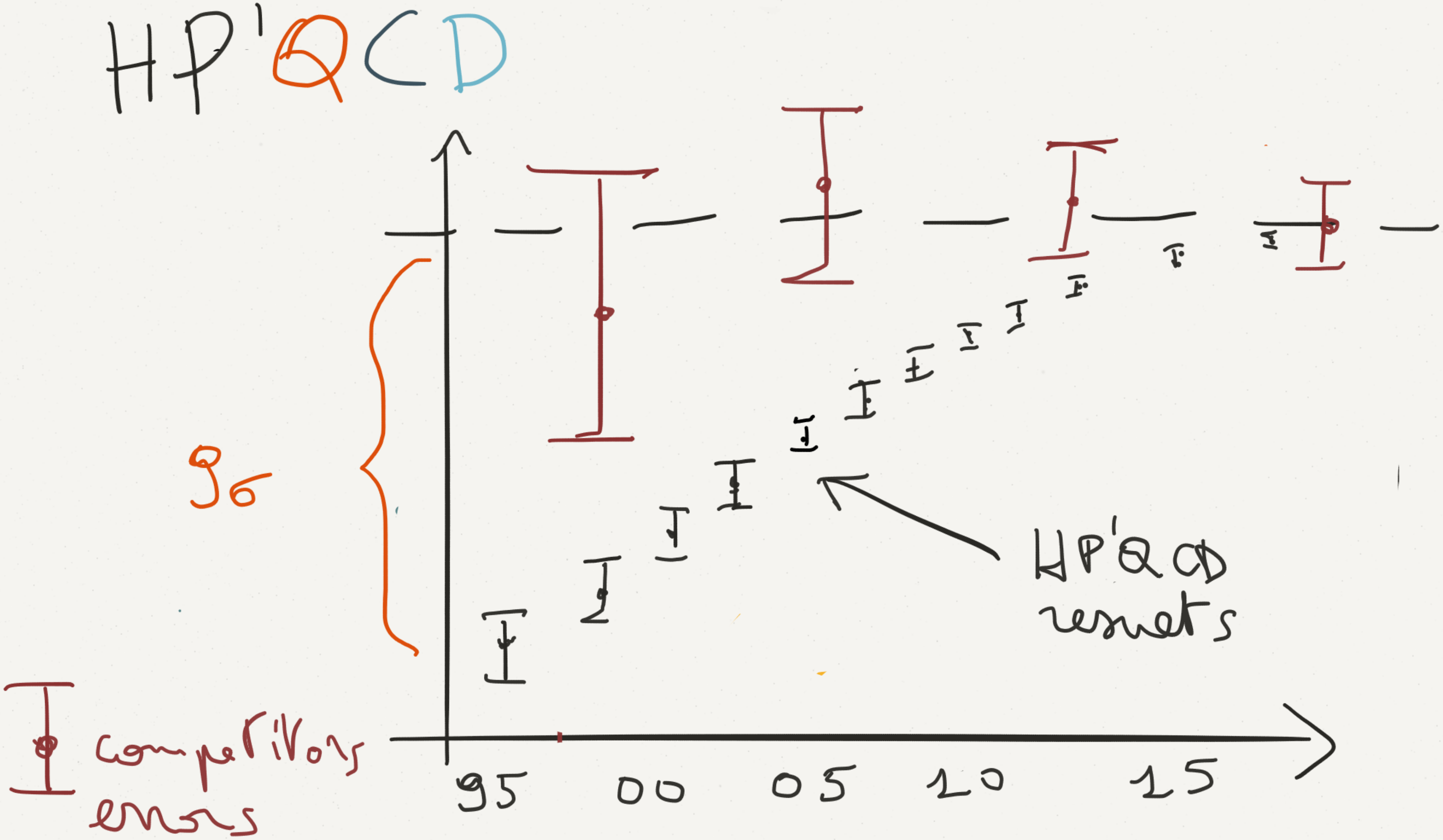
$$\frac{m_\mu{}^2\left(\frac{1}{m_\gamma{}^2+\vec{q}\cdot\vec{q}}\right)^{3/2}\mathrm{Log}[r_\mu{}^2]}{2(m^2-m_\mu{}^2)} - \frac{m^2\mathrm{Log}\left[\frac{1+|\vec{q}\cdot\vec{q}|^{1/2}\sqrt{\frac{1}{m^2+\vec{q}\cdot\vec{q}}}}{1-|\vec{q}\cdot\vec{q}|^{1/2}\sqrt{\frac{1}{m^2+\vec{q}\cdot\vec{q}}}}\right]}{2(m^2-m_\mu{}^2)\left(|\vec{q}\cdot\vec{q}|^{1/2}\right)^3} - \frac{m_\mu{}^2\mathrm{Log}\left[\frac{1+|\vec{q}\cdot\vec{q}|^{1/2}\sqrt{\frac{1}{m^2+\vec{q}\cdot\vec{q}}}}{1-|\vec{q}\cdot\vec{q}|^{1/2}\sqrt{\frac{1}{m^2+\vec{q}\cdot\vec{q}}}}\right]}{2(m^2-m_\mu{}^2)\left(|\vec{q}\cdot\vec{q}|^{1/2}\right)^3} +$$

$$\frac{m^2\mathrm{Log}\left[\frac{1+|\vec{q}\cdot\vec{q}|^{1/2}\sqrt{\frac{1}{m_\mu{}^2+\vec{q}\cdot\vec{q}}}}{1-|\vec{q}\cdot\vec{q}|^{1/2}\sqrt{\frac{1}{m_\mu{}^2+\vec{q}\cdot\vec{q}}}}\right]}{2(m^2-m_\mu{}^2)\left(|\vec{q}\cdot\vec{q}|^{1/2}\right)^3} + \frac{m_\mu{}^2\mathrm{Log}\left[\frac{1+|\vec{q}\cdot\vec{q}|^{1/2}\sqrt{\frac{1}{m_\mu{}^2+\vec{q}\cdot\vec{q}}}}{1-|\vec{q}\cdot\vec{q}|^{1/2}\sqrt{\frac{1}{m_\mu{}^2+\vec{q}\cdot\vec{q}}}}\right]}{2(m^2-m_\mu{}^2)\left(|\vec{q}\cdot\vec{q}|^{1/2}\right)^3}\Bigg)$$

2 Integrating $t_1 \leq t_w$, other reasonable time-orderings







HP'QCD = Hyperbolic Precision QCD

To conclude

- We have presented a method to compute QED corrections to hadronic processes;
- For these quantities the presence of infrared divergences in the intermediate stages of the calculation make the procedure much more complicated than in the case of the hadronic spectrum;
- In order to obtain the physical answer virtual corrections and real photon emissions must be combined together;
- It is not sufficient to add the electromagnetic interaction to the quark action, because separate explicit real and virtual emission diagrams must be evaluated for any given process;
- We have discussed a specific case, namely the radiative corrections to the leptonic decay of charged pseudoscalar mesons. The method can e however be extended to many other cases like for example to semileptonic decays.

To conclude

- The condition for the applicability of our strategy is that there is a mass gap between the decaying particle and the intermediate states generated by the emission of the photon, and that none of these states is lighter than the initial hadron.
- In the calculation of electromagnetic corrections a general issue is finite size effects. In this respect our method reduces to compute infrared finite, gauge invariant quantities for which we do expect finite size corrections which are comparable to those encountered for the spectrum. This expectation will be checked in forthcoming numerical studies, and eventually studied theoretically in chiral perturbation theory.
- The implementation of our method, although challenging, is within reach of the present lattice technology. The accuracy necessary to make the results phenomenologically interesting is not exceedingly high since the effect that we want to predict is, in general, of the order of a few percent.

THANKS FOR YOUR ATTENTION

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