Multiparticle processes from the lattice

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Three particle scattering amplitudes from finite-volume simulations

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M.T. Hansen & S.R. Sharpe, arXiv:1408.5933 (PRD 2014) + arXiv:1504:04248 + In Progress

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Outline-v2

Introduction

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Summary of Results

Essential Issues in Derivation

Outlook

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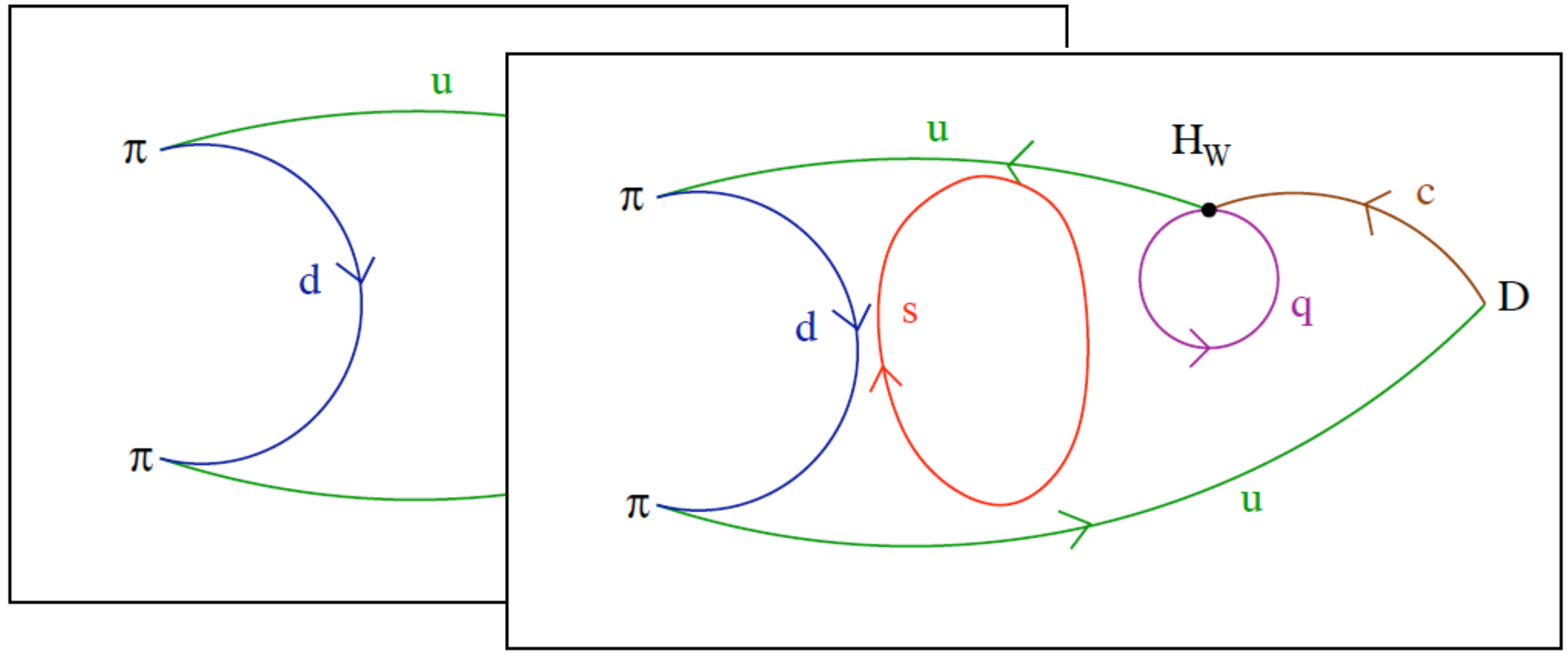
Why should you care?

- How related to "High Precision QCD at low Energy"?
- Example: CP violation in $D \to \pi\pi, KK$
  - Inspired by (now departed) hints of larger-than-expected CP violation from LHCb & others from a few years ago


- Can we use lattice QCD to calculate SM prediction?

Can LQCD predict D→ππ, KK?



- Not yet
  - Methods do exist for K→2π [Christof Lehner's talk: Lüscher, Lellouch-Lüscher]
  - Key new issue: multiparticle channels are significant, e.g. D→4π, 6π, …
  - Finite volume "ππ" state will be a mixture of ππ, KK, ηη, 4π, 6π,…
  - Need to disentangle these contributions, which requires understanding finite volume (FV) effects for multiparticle states
  - Understood for multiple two-particle channels [Hansen & SS, 2012]
  - Open problem for 4π, 6π, etc.

2, 3, 4, …

- 4 particle final states are too challenging for now
- Begin with 3 particles in a box!

✓

?

✗

Applications of 3-particle formalism

- Studying resonances with three particle decay channels

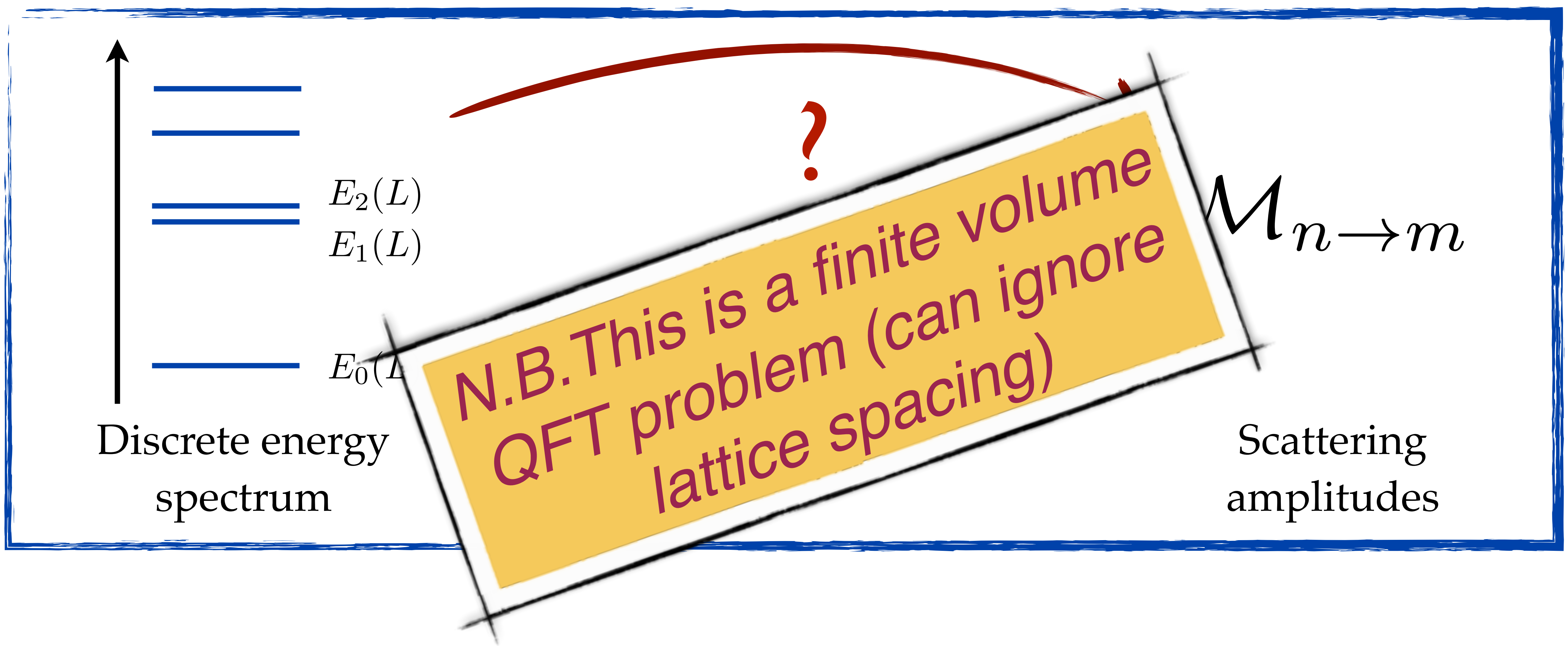
$$\omega(782) \to \pi\pi\pi \qquad K^* \longrightarrow K\pi\pi \quad N(1440) \to N\pi\pi$$

- Calculating weak decay amplitudes/form factors involving 3 particles, e.g. $K \to \pi\pi\pi$

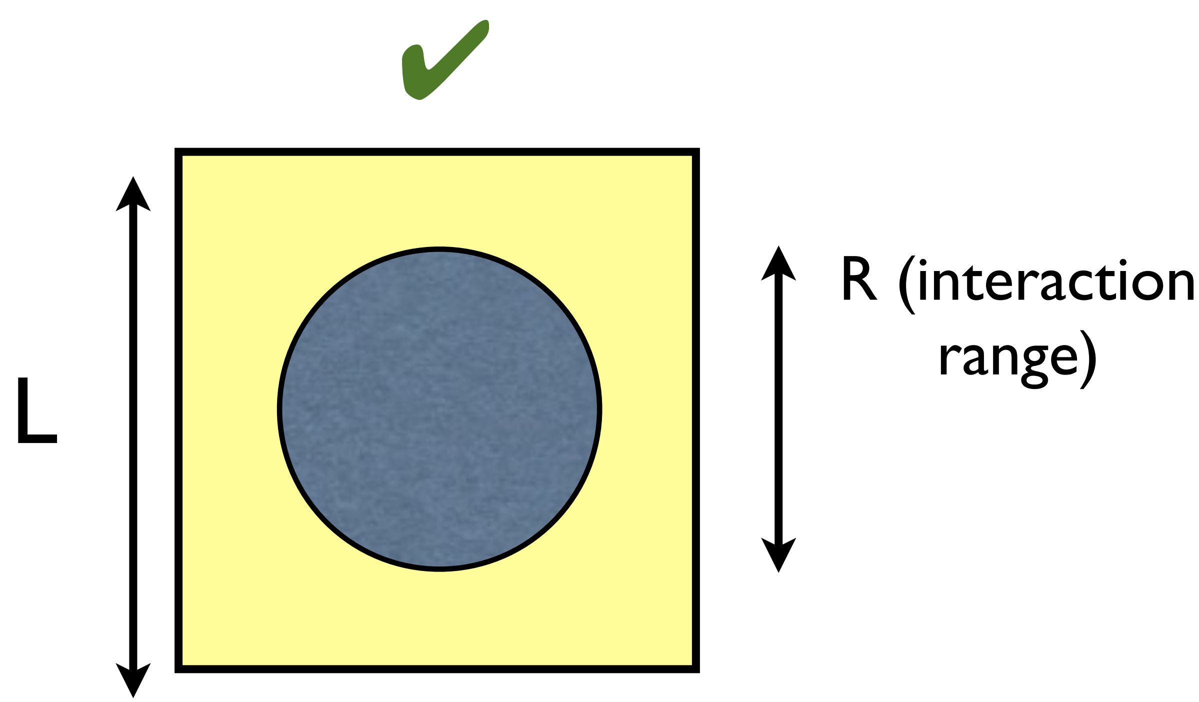
- Determining NNN interactions
  - Input for effective field theory treatments of larger nuclei & nuclear matter
  - Similarly, $\pi\pi\pi$, $\pi K\overline{K}$, … interactions needed for study of pion/kaon condensation

The fundamental issue

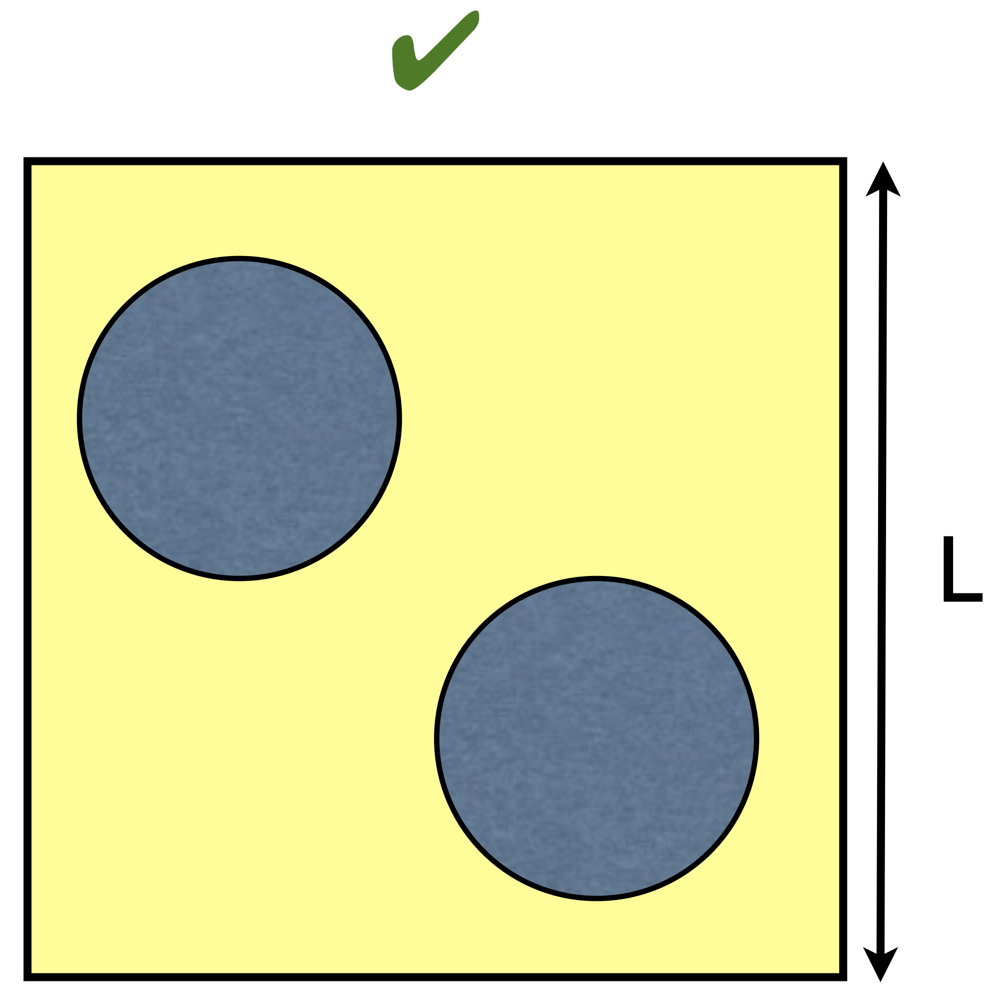
- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite volume scattering amplitudes (which determine resonance properties)?



When is spectrum related to scattering amplitudes?

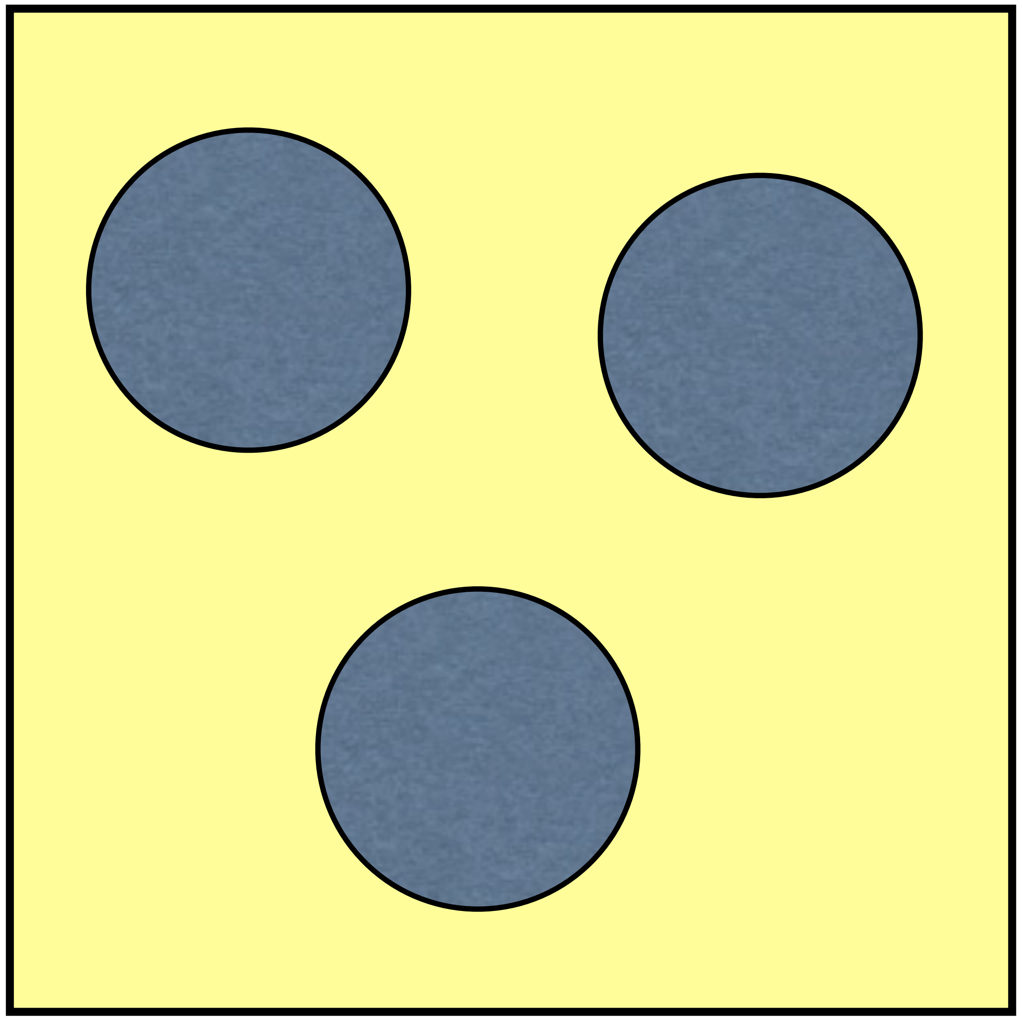


Single (stable) particle with L>R
Particle not "squeezed"
Spectrum same as in infinite volume up to corrections proportional to $e^{-M_\pi L}$
[Lüscher]



L>2R
There is an "outside" region.
Spectrum IS related to scatt. amps. up to corrections proportional to $e^{-M_\pi L}$
[Lüscher]
Theoretically understood;
numerical implementations mature.

Problem considered today



L>3R (?)

Spectrum IS related to 2→2, 2→3 & 3→3 scattering amplitudes up to corrections proportional to $e^{-M_\pi L}$ [Polejaeva & Rusetsky]

General relativistic formalism developed in simplest case [Hansen & SS]

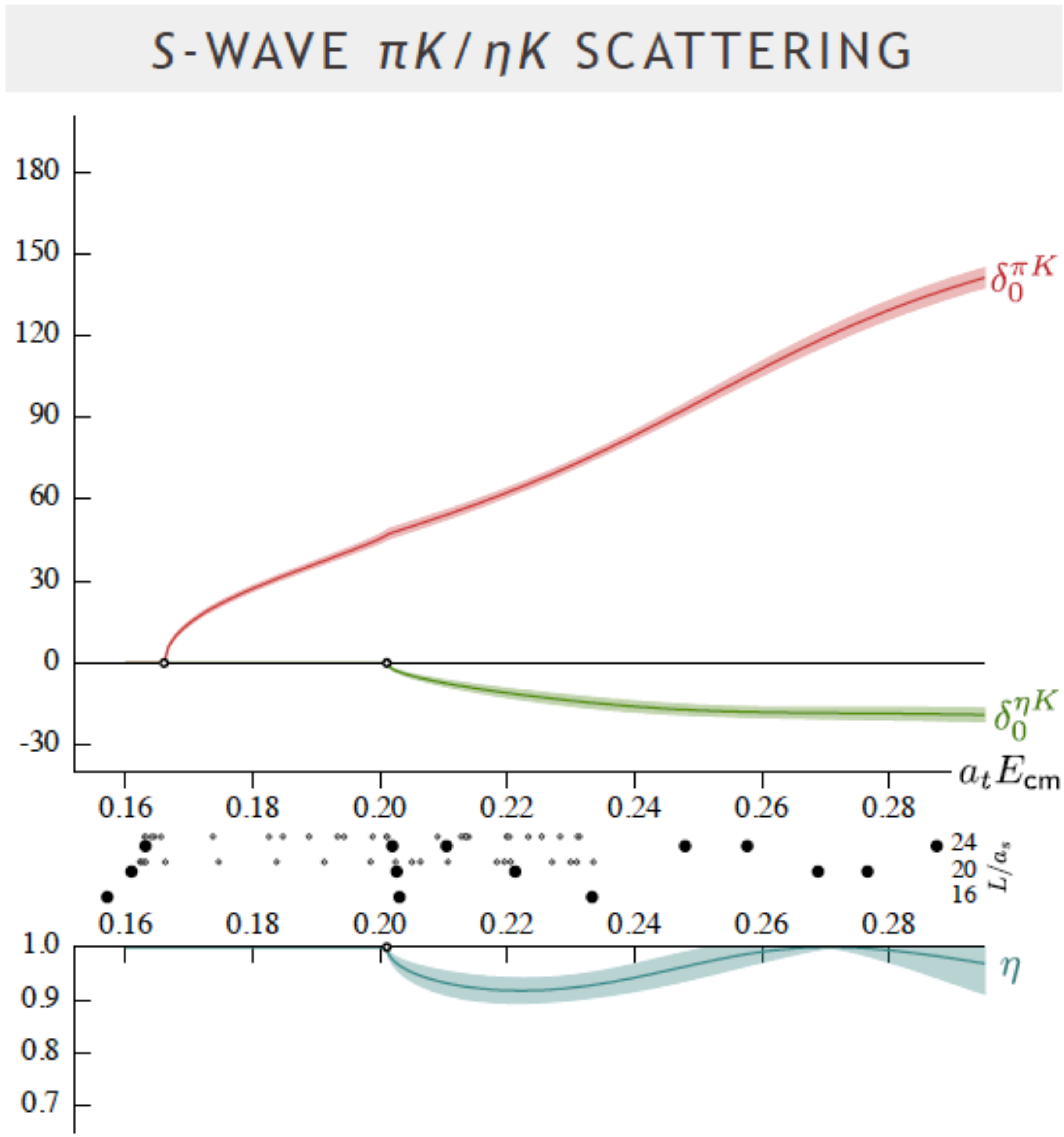
Practical applicability under investigation

Status for 2 particles

- Long understood in NRQM [Huang & Yang 57, ...]
- Quantization formula in QFT for energies below inelastic threshold converted into NRQM problem and solved by [Lüscher 86 & 91]
- Solution generalized to arbitrary total momentum **P**, multiple (2 body) channels, general BCs and arbitrary spins [Rummukainen & Gottlieb 85; Kim, Sachrajda & SS 05; Bernard, Lage, Meißner & Rusetsky 08; Hansen & SS 12; Briceño & Davoudi 12; …]
- Relation between finite volume $1\to2$ weak amplitude (e.g. $K\to\pi\pi$) and infinite volume decay amplitude determined [Lellouch & Lüscher 00]
- LL formula generalized to general **P**, to multiple (2 body) channels, to arbitrary currents, general BCs & arbitrary spin (e.g. $\gamma^*\pi\to\rho\to\pi\pi$, $\gamma^*N\to\Delta\to\pi N$, $\gamma D\to NN$) [Kim, Sachrajda & SS 05; Christ, Kim & Yamazaki 05; Meyer 12; Hansen & SS 12; Briceño & Davoudi 12; Agadjanov, Bernard, Meißner & Rusetsky 14; Briceño, Hansen & Walker-Loud 14; Briceño & Hansen 15;…]
- Leading order QED effects on quantization condition determined; do NOT fit into general formalism [Beane & Savage 14]

State of the art

[Dudek, Edwards, Thomas & Wilson 14]



Coupled two-body channels

$m_\pi \sim 391\,\mathrm{MeV}$

Status for 3 particles

- [Beane, Detmold & Savage 07 and Tan 08] derived threshold expansion for n particles in NRQM, and argued it applied also in QFT
- [Polejaeva & Rusetsky 12] showed in NREFT that 3 body spectrum determined by infinite-volume scattering amplitudes, using integral equation
- [Briceño & Davoudi 12] used a dimer approach in NREFT, with s-wave interactions only, to determine relation between spectrum and a finite volume quantity, itself related to infinite-volume amplitudes by an integral equation
- [Hansen & SS 14, 15] derived quantization condition in (fairly) general, relativistic QFT relating spectrum and $\mathcal{M}_2$ and 3-body scattering *quantity* $K_{df,3}$; relation between $K_{df,3}$ & $\mathcal{M}_3$ via integral equations now known
- [Meißner, Rios & Rusetsky 14] determined volume dependence of 3-body bound state in unitary limit

HALQCD method

- There is an alternative approach, followed by the HALQCD collaboration [Aoki et al.], using the Bethe-Salpeter wave-function calculated with lattice QCD to determine scattering amplitudes
- Extended from 2 particle to 3 (and higher) particle case in non-relativistic domain
- Potentially more powerful than the Lüscher-like methods I discuss today, but based on certain assumptions

Summary of Results

Single-channel 2-particle quantization condition

- At fixed L & P, the finite-volume spectrum E_1, E_2, ... is given by solutions to

$$\boxed{\Delta_{L,\vec{P}}(E) = \det\left[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2\right] = 0}$$

- $\mathcal{K}_2 \sim \tan\delta/q$ is the K-matrix, which is diagonal in *l,m* space

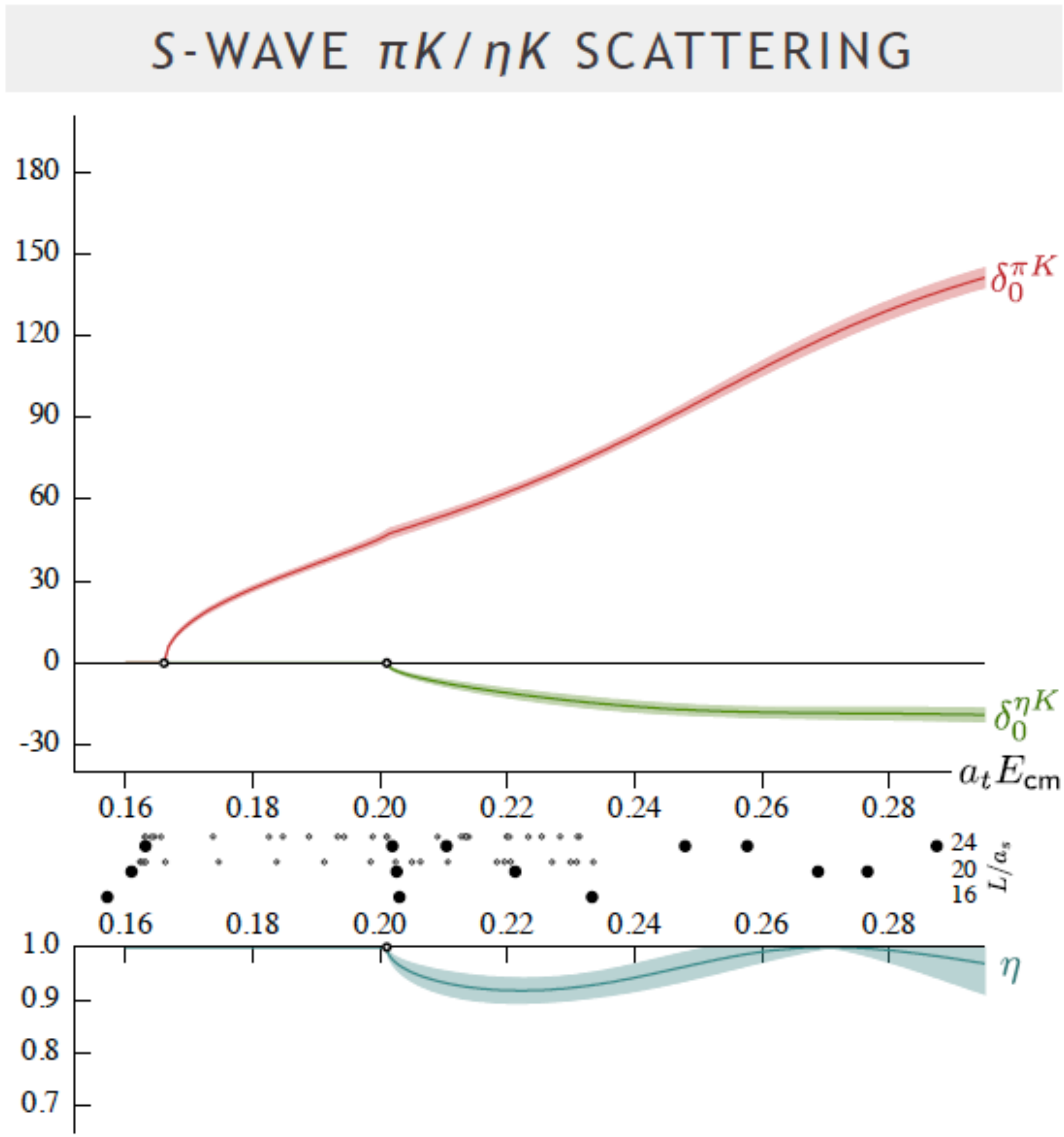
- F_{PV} is a known kinematical zeta-function, depending on the box shape & E; It is an off-diagonal matrix in *l,m*, since the box violates rotation symmetry

- Infinite dimensional determinant must be truncated to be practical; truncate by assuming that $\mathcal{K}_2$ vanishes above l_{max}. If $l_{max}=0$, then obtain:

$$\boxed{i\mathcal{K}_{2;00;00}(E_n^*) = \left[iF_{\widetilde{PV};00;00}(E_n,\vec{P},L)\right]^{-1}}$$

State of the art

[Dudek, Edwards, Thomas & Wilson 14]

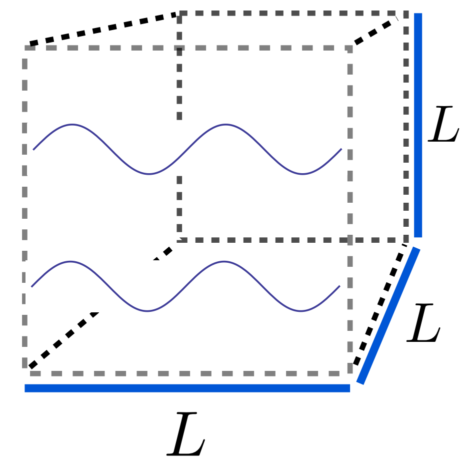


Coupled two-body channels

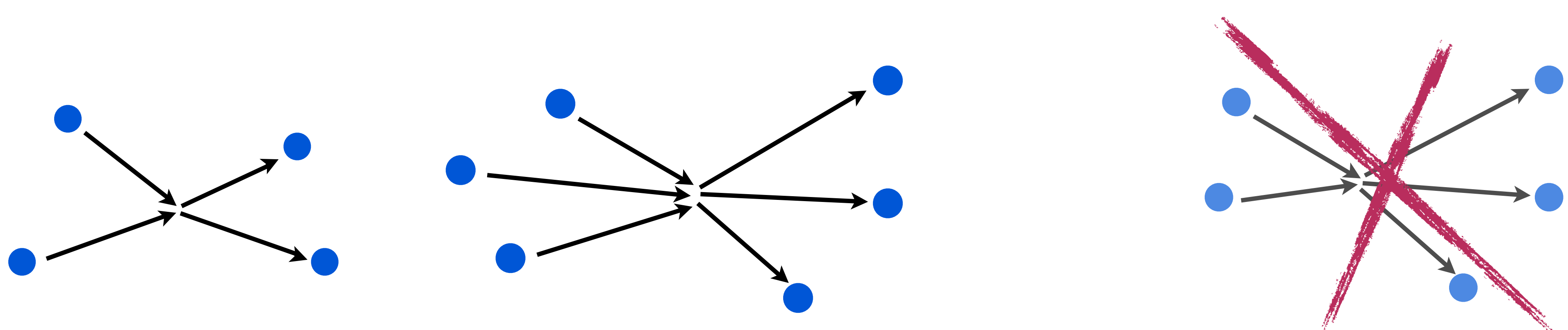
$m_\pi \sim 391\,\mathrm{MeV}$

Theory considered for 3 particles

- Work in continuum (assume that LQCD can control discretization errors)



- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\qquad \vec{k} = \frac{2\pi}{L}\vec{n}$
- Consider identical particles with physical mass m, interacting *arbitrarily* except for a Z_2 (G-parity-like) symmetry
  - Only vertices are 2→2, 2→4, 3→3, 3→1, 3→5, 5→7, etc.
  - Even & odd particle-number sectors decouple



Final result for 3 particles [Hansen & SS]

- Spectrum is determined (for given L, **P**) by solutions of

Infinite volume real 3-particle scattering quantity

$$\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

Matrices in the space describing 3-particle on-shell kinematics

$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + (1+\mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

Known kinematical quantity: essentially the same as F_{PV} in 2-particle analysis

$$G_{p,\ell',m';k,\ell,m} \equiv \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell',m'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell,m}(\hat{p}^*)}{2\omega_{kp}(E-\omega_k-\omega_p-\omega_{kp})}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

G is known kinematical quantity containing cut-off function H

- Superficially similar to 2-particle form …

$$\Delta_{L,\vec{P}}(E) = \det\left[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2\right] = 0$$

- … but F_3 contains both kinematical, finite-volume quantities (F_{PV} & G) and the dynamical, infinite-volume quantity $\mathcal{K}_2$

Final result for 3 particles

$$\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + (1+\mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

- All quantities are (infinite-dimensional) matrices, e.g. $(F_3)_{klm;pl'm'}$, with indices

[finite volume "spectator" momentum: **k**=2π**n**/L] x [2-particle CM angular momentum: *l,m*]

Three on-shell particles with total energy-momentum (E, **P**)

- For large **k** other two particles are below threshold; must include such configurations by analytic continuation up to a cut-off at k~m [provided by H(**k**)]

Final result for 3 particles

$$\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

- Important limitation: our present derivation requires that all two-particle sub-channels lie below resonance poles at the spectral energy under consideration
  - Resonances imply that $\mathcal{K}_2$ has a pole, and this leads to additional finite volume dependence not accounted for in the derivation
  - We only have an ugly solution—searching for something better

Truncation in 2 particle case

$$\Delta_{L,\vec{P}}(E) = \det\left[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2\right] = 0$$

- If $\mathcal{K}_2$ (which is diagonal in l,m) vanishes for $l > l_{max}$ then can show that need only keep $l \leq l_{max}$ in F (which is not diagonal) and so have finite matrix condition which can be inverted to find $\mathcal{K}_2(E)$ from energy levels

Truncation in 3 particle case

$$\boxed{\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{df,3}\right] = 0}$$

$$F_3 = \frac{F_{\widetilde{\rm PV}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1+(1+\mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\rm PV}}}\right]$$

- For fixed E & **P**, as spectator momentum |**k**| increases, remaining two-particle system drops below threshold, so F_{PV} becomes exponentially suppressed
  - Smoothly interpolates to $F_{PV}=0$ due to H factors; same holds for G
- Thus **k** sum is naturally truncated (with, say, N terms required)
- *l* is truncated if both $\mathcal{K}_2$ and $\mathcal{K}_{df,3}$ vanish for $l > l_{max}$
- Yields determinant condition truncated to $[N(2l_{max}+1)]^2$ block

Truncation in 3 particle case

$$\boxed{\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{df,3}\right] = 0}$$

$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1+(1+\mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

- Given prior knowledge of $\mathcal{K}_2$ (e.g. from 2-particle quantization condition) each energy level E_i of the 3 particle system gives information on $\mathcal{K}_{df,3}$ at the corresponding 3-particle CM energy E_i^*
- Probably need to proceed by parameterizing $\mathcal{K}_{df,3\to3}$, in which case one would need at least as many levels as parameters at given energy
- Given $\mathcal{K}_2$ and $\mathcal{K}_{df,3}$ one can reconstruct $\mathcal{M}_3$
- The locality of $\mathcal{K}_{df,3}$ is crucial for this program
- Clearly very challenging in practice, but there is an existence proof....

Isotropic approximation

- Assume $\mathcal{K}_{df,3}$ depends only on E* (and thus is indep. of **k**, *l*, *m*)
- Also assume $\mathcal{K}_2$ only non-zero for s-wave ($\Rightarrow l_{max}=0$) and known
- Truncated [N x N] problem simplifies: $\mathcal{K}_{df,3}$ has only 1 non-zero eigenvalue, and problem collapses to a single equation:

$$\boxed{1 + F_3^{\rm iso} \mathcal{K}_{{\rm df},3}^{\rm iso}(E^*) = 0}$$

Known in terms of
two particle scattering amplitude

$$F_3^{\rm iso} \equiv \sum_{\vec k, \vec p} \frac{1}{2\omega_k L^3} \left[F^s_{\widetilde{\rm PV}} \left(-\frac{2}{3} + \frac{1}{1 + [1 + \mathcal{K}_2^s G^s]^{-1} \mathcal{K}_2^s F^s_{\widetilde{\rm PV}}} \right) \right]_{k,p}$$

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Relating $\mathcal{K}_{df,3}$ to $\mathcal{M}_3$

- Three particle quantization condition depends on $\mathcal{K}_{df,3}$ rather than the three particle scattering amplitude $\mathcal{M}_3$
- $\mathcal{K}_{df,3}$ is an infinite volume quantity (loops involve integrals) but is not physical
  - Has a very complicated, unwieldy definition
  - Depends on the cut-off function H
  - It was forced on us by the analysis, and is some sort of local vertex
- To complete the quantization condition we must relate $\mathcal{K}_{df,3}$ to $\mathcal{M}_3$

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

Involve only $\mathcal{M}_2$ and G
so "known"

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}}\mathcal{R}_L\right]$$

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} i\mathcal{M}_{L,3\to3}$$

Sums over k go over
to integrals with iε pole prescription

- Result is an integral equation giving $\mathcal{M}_3$ in terms of $\mathcal{K}_{\mathrm{df},3}$
- Requires knowing $\mathcal{M}_2$ (including continued below threshold)
- Completes formalism—shows that finite volume spectrum is given by infinite-volume scattering amplitudes

Conclusions & Outlook

Summary: successes

- Obtained a 3-particle quantization condition
- Confirmed that 3-particle spectrum determined by infinite-volume scattering amplitudes in a general relativistic QFT
- Truncation to obtain a finite problem occurs naturally
- Threshold expansion and other checks give us confidence in the expression

Summary: limitations

- Relation of $\mathcal{K}_{df,3}$ to $\mathcal{M}_3$ requires solving integral equations
- $\mathcal{K}_2$ is needed below (as well as above) 2-particle threshold
- Formalism fails when $\mathcal{K}_2$ is singular $\Rightarrow$ each two-particle channel must have no resonances within kinematic range
- Applies only to identical, spinless particles, with Z_2 symmetry

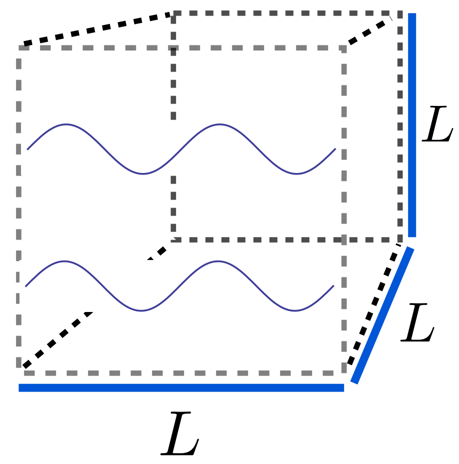
Many challenges remain!

- Fully develop 3 body formalism
  - Allow two particle sub-channels to be resonant
  - Extend to non-identical particles, particles with spin
  - Generalize LL factors to $1\rightarrow3$ decay amplitudes (e.g. for $K\rightarrow\pi\pi\pi$)
  - Include $1\rightarrow2$, $2\rightarrow3$, … vertices
- Develop models of amplitudes so that new results can be implemented in simulations
- Onwards to 4 or more particles?!?

Set-up & main ideas

Set-up

- Work in continuum (assume that LQCD can control discretization errors)



- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\vec{k} = \frac{2\pi}{L}\vec{n}$
- Consider identical particles with physical mass m, interacting *arbitrarily* except for a Z_2 (G-parity-like) symmetry
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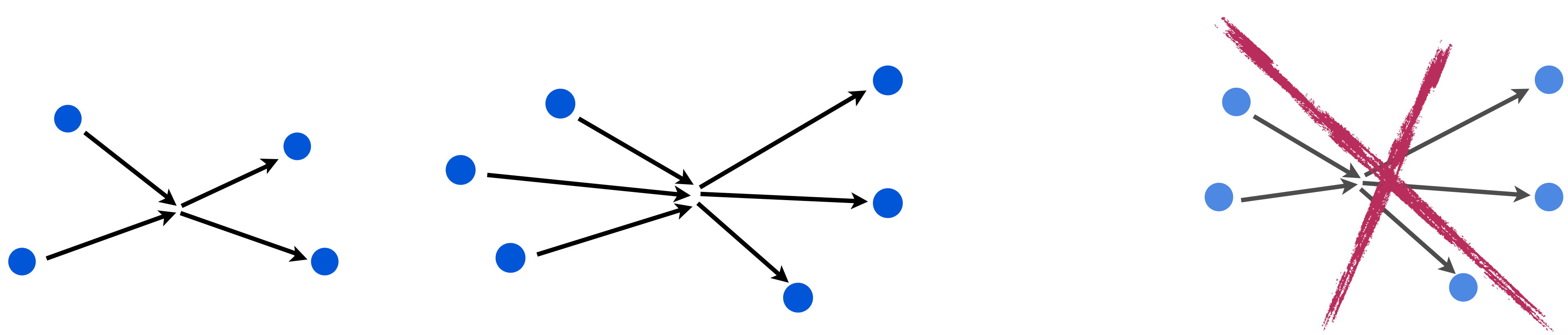
  - Even & odd particle-number sectors decouple



Methodology

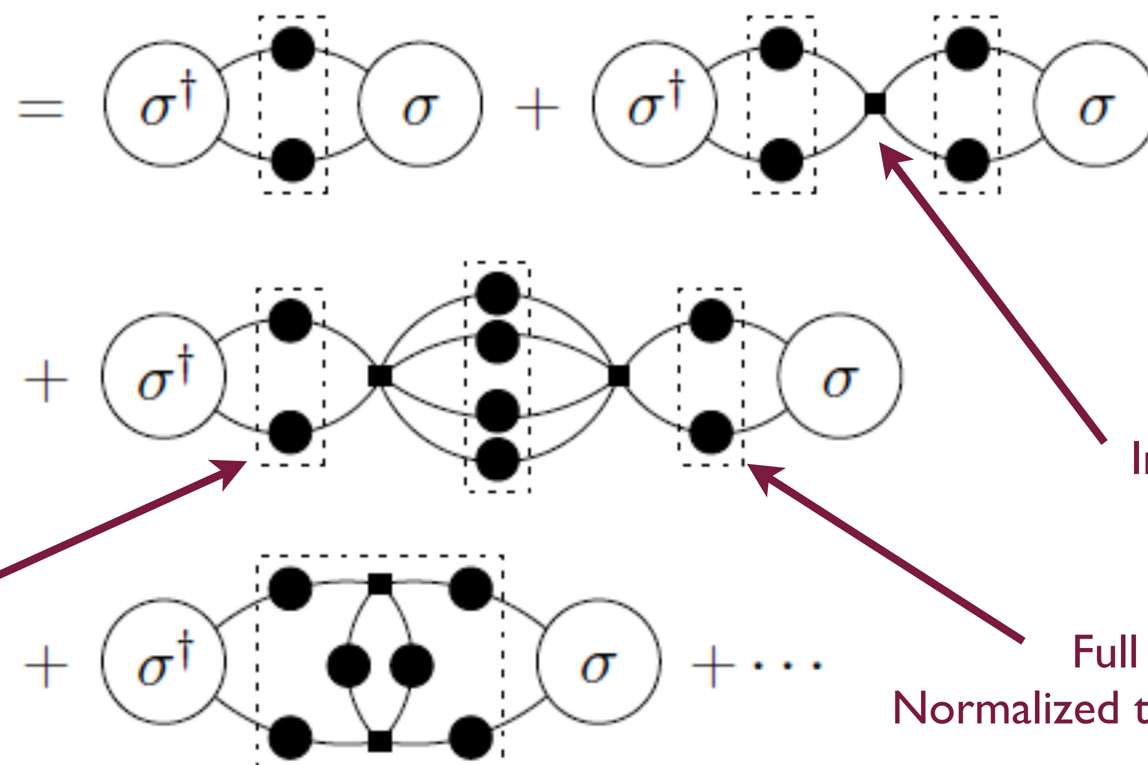
CM energy is
$E^* = \sqrt{(E^2 - P^2)}$

- Calculate (for some **$P = 2\pi n_P/L$**)

$$C_L(E,\vec{P}) \equiv \int_L d^4x \; e^{-i\vec{P}\cdot\vec{x}+iEt} \langle \Omega | T\sigma(x)\sigma^\dagger(0)|\Omega\rangle_L$$

- Poles in C_L occur at energies of finite-volume spectrum
- For 2 & 3 particle states, $\sigma \sim \pi^2$ & π^3, respectively
- E.g. for 2 particles:

$C_L(E,\vec{P}) =$



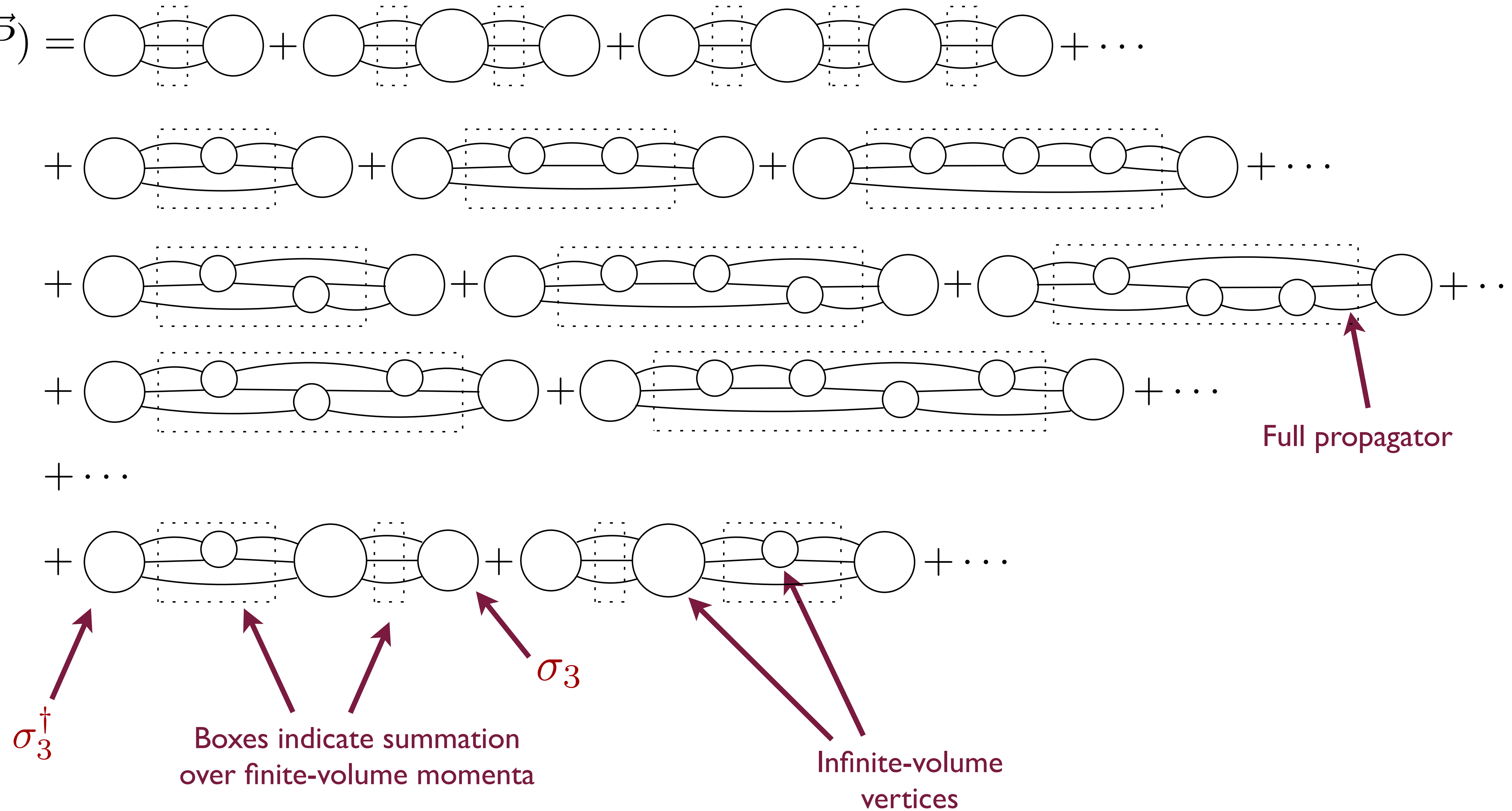
Infinite-volume vertices

Boxes indicated summation over finite-volume momenta

Full propagators
Normalized to unit residue at pole

3-particle correlator

$C_L(E, \vec{P}) =$



Key step 1

- Replace loop sums with integrals where possible
  - Drop exponentially suppressed terms ($\sim e^{-ML}$, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \cancel{\sum_{\vec{l}\neq\vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})}$$

Exp. suppressed if g(k) is smooth
and scale of derivatives of g is ~1/M

Key step 2

- Use "sum=integral + [sum-integral]" if integrand has pole, with [Kim,Sachrajda,SS 05]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int\frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\ \ (q^*,q^{*'})g^*(\hat{q}^{*'})\quad + \text{exp. suppressed}$$

q^* is relative momentum of pair on left in CM

Kinematic function

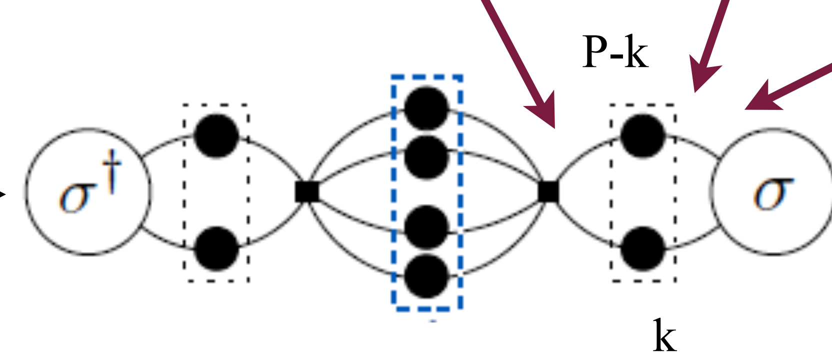
f & g evaluated for ON-SHELL momenta
Depend only on direction in CM

- Example

Focus on this loop

f is left-hand part of integrand

g is right-hand part of integrand

$P = (E, \vec{P}) \longrightarrow$ 

Key step 2

- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*}d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\;\;(q^*,q^{*'})g^*(\hat{q}^{*'})$$

- Decomposed into spherical harmonics, $\mathcal{F}$ becomes

$$F_{\;\ell_1,m_1;\ell_2,m_2} \equiv \;\; \eta\left[\frac{\mathrm{Re}q^*}{8\pi E^*}\delta_{\ell_1\ell_2}\delta_{m_1m_2} + \right.$$
$$\left. \frac{i}{2\pi EL}\sum_{\ell,m} x^{-\ell}\,\mathcal{Z}^P_{\ell m}[1;x^2]\int d\Omega\, Y^*_{\ell_1,m_1}Y^*_{\ell,m}Y_{\ell_2,m_2}\right]$$

$x \equiv q^*L/(2\pi)$ and $\mathcal{Z}^P_{\ell m}$ is a generalization of the zeta-function

Kinematic functions

$Z_{4,0}$ & $Z_{6,0}$ for **P=0** [Luu & Savage, `11]

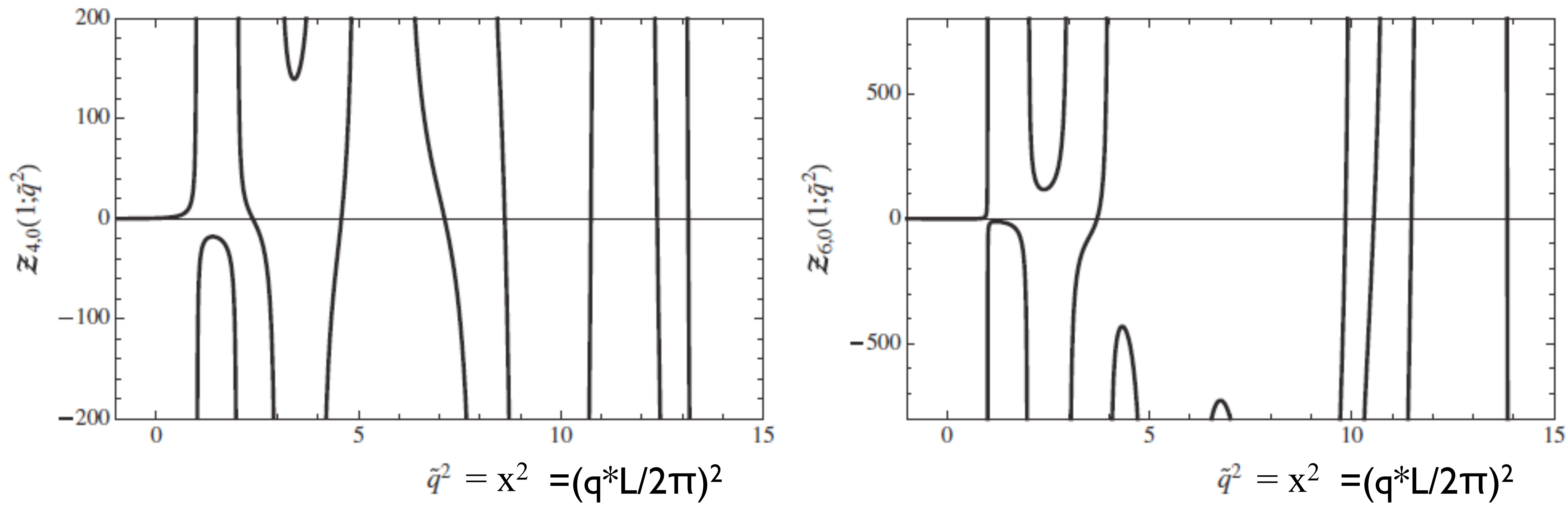


FIG. 29. The functions $Z_{4,0}(1;\tilde{q}^2)$ (left panel) and $Z_{6,0}(1;\tilde{q}^2)$ (right panel).

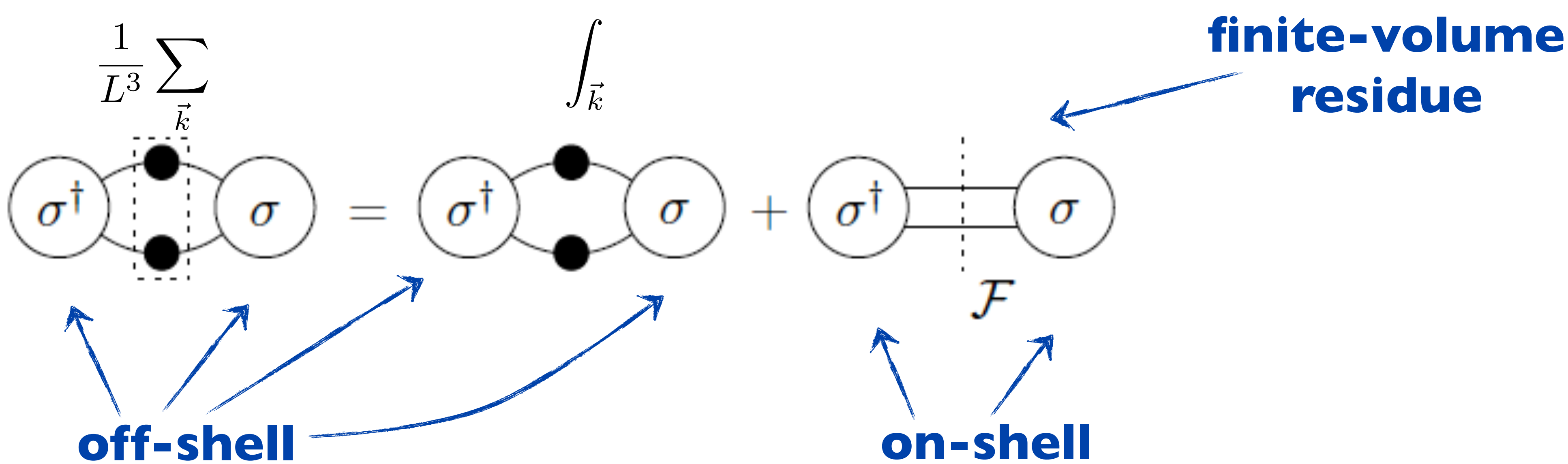
Key step 2

- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k) \frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon} g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\ \ (q^*, q^{*'}) g^*(\hat{q}^{*'})$$

- Diagrammatically



Variant of key step 2

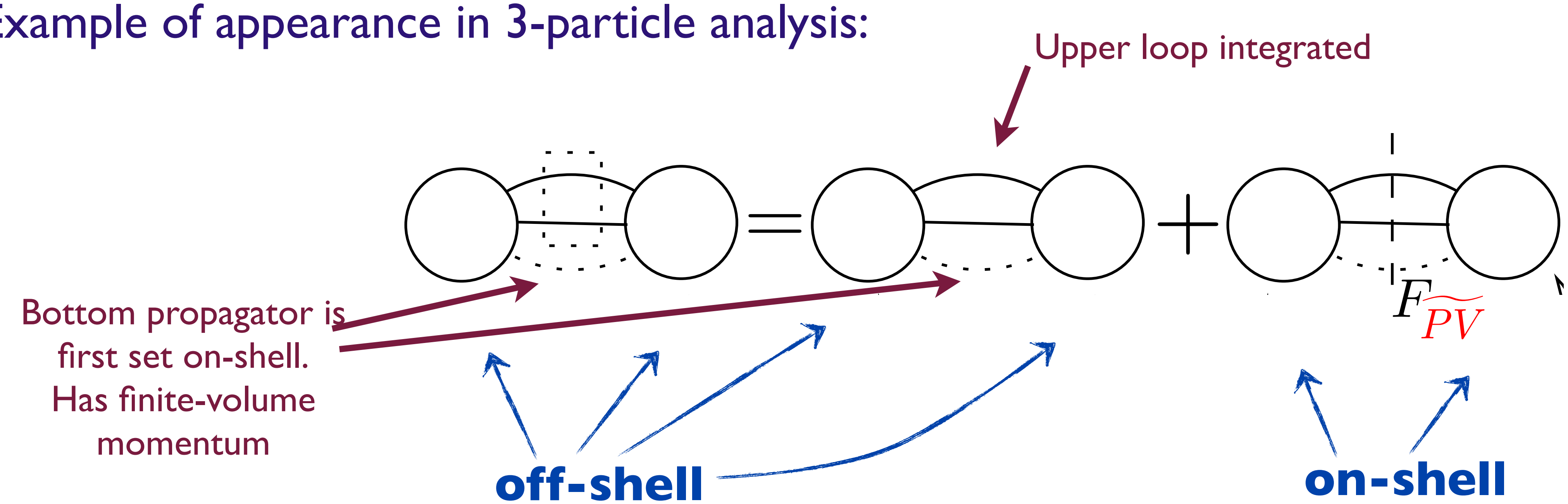
- For generalization to 3 particles use (modified) PV prescription instead of iε

$$\left(\int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \widetilde{PV}\!\!\int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + \cancel{i\epsilon}} \frac{1}{(P-k)^2 - m^2 + \cancel{i\epsilon}} g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*) \mathcal{F}_{\widetilde{PV}}(q^*, q^{*\prime}) g^*(\hat{q}^{*\prime})$$

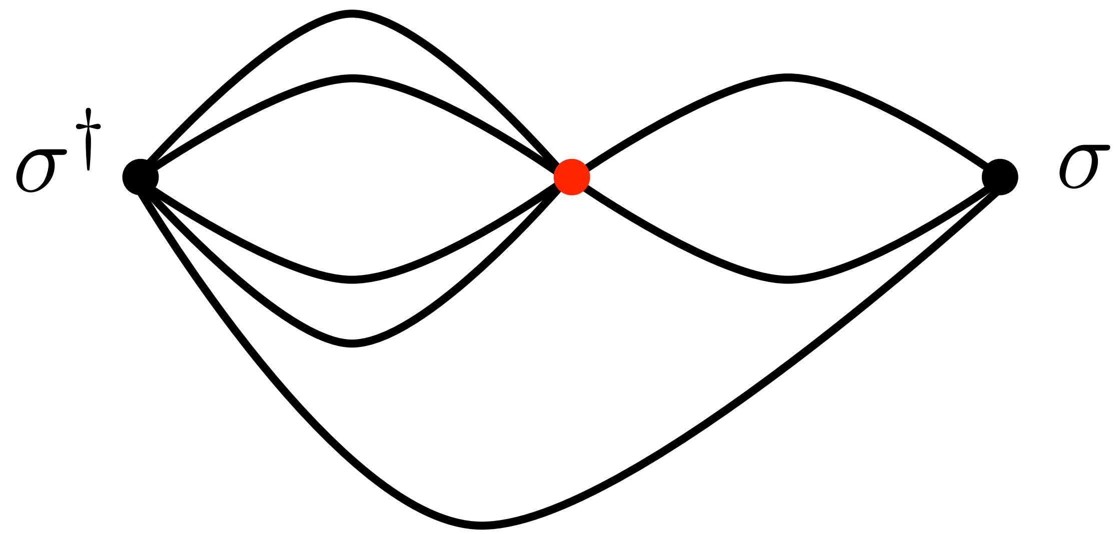
- Key properties of F_{PV} (discussed below): real and no unitary cusp at threshold

- Example of appearance in 3-particle analysis:



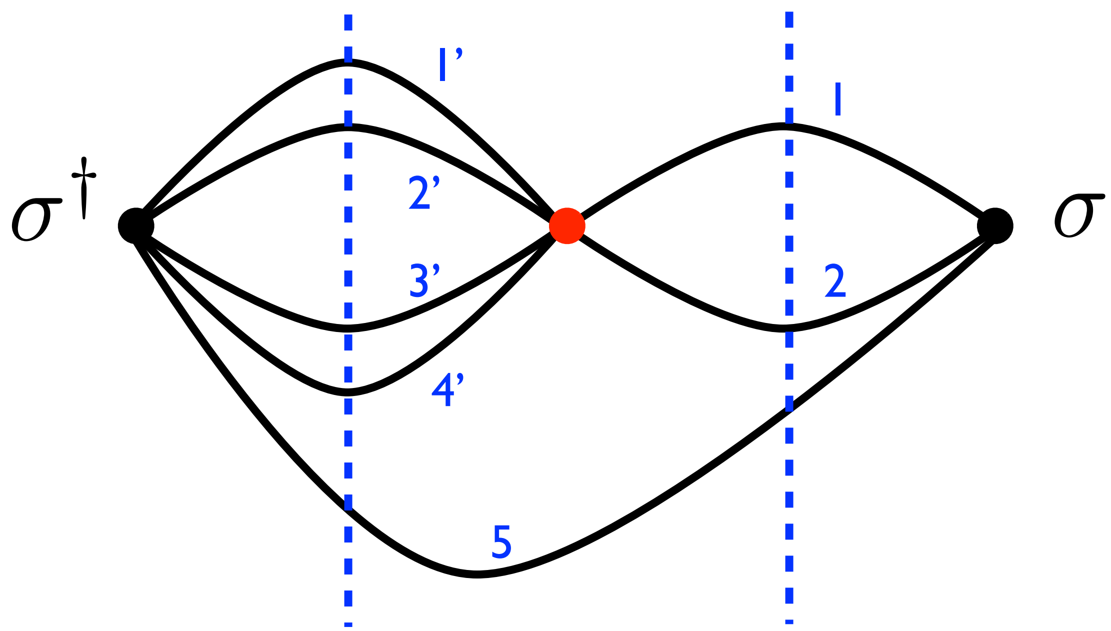
Key step 3

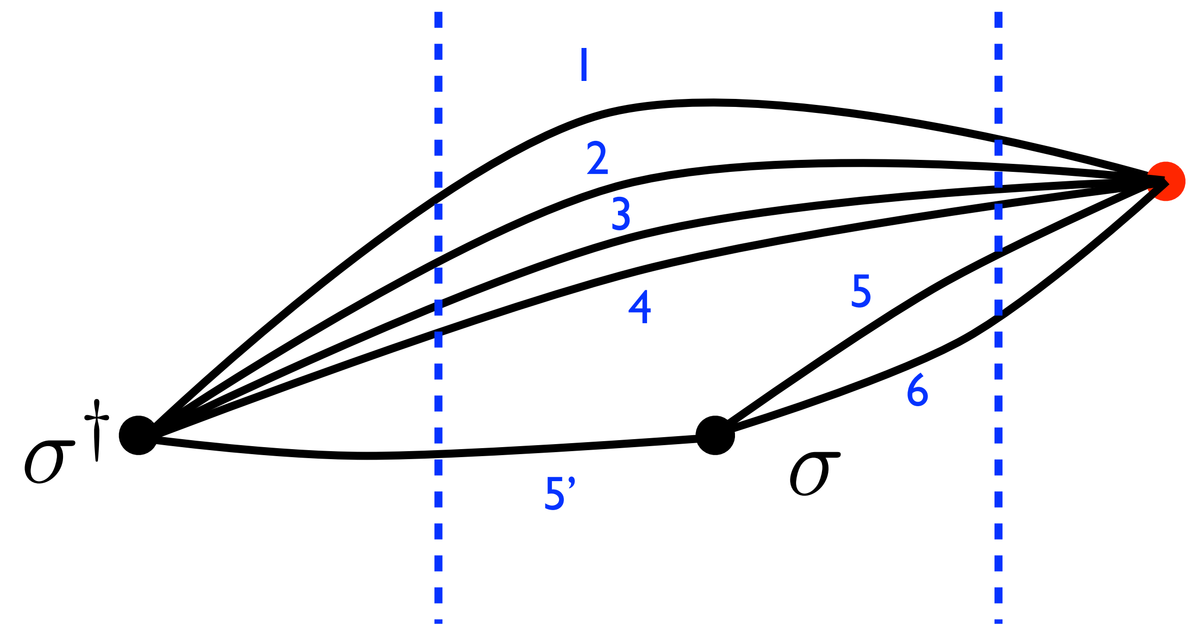
- Identify potential singularities: can use time-ordered PT (i.e. do k_0 integrals)
- Example



Key step 3

- 2 out of 6 time orderings:





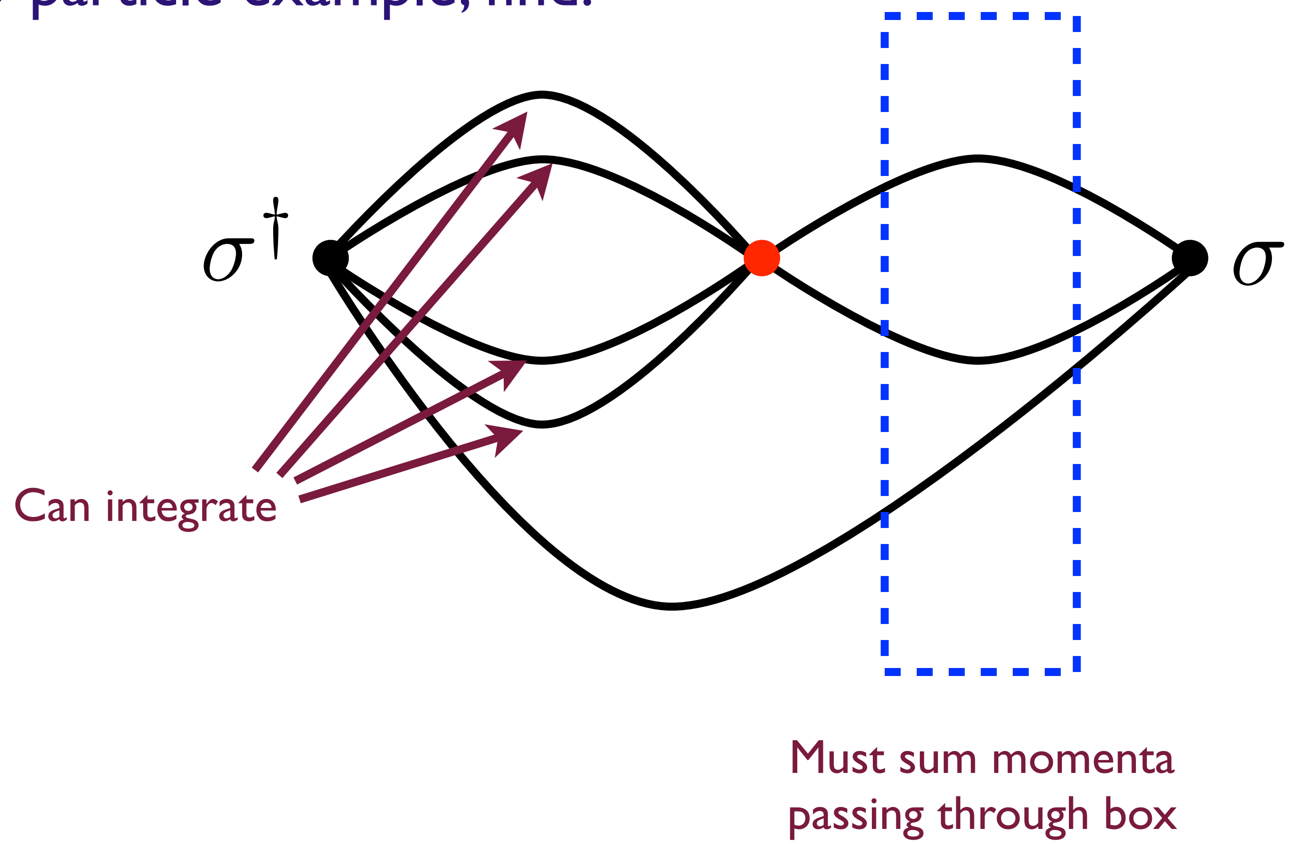
$$\frac{1}{E-\omega_1'-\omega_2'-\omega_3'-\omega_4'-\omega_5} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_5} \qquad \frac{1}{E-\omega_1-\omega_2-\omega_3-\omega_4-\omega_5'} \qquad \frac{1}{\sum_{j=1,6}\omega_j}$$

On-shell energy $\omega_j = \sqrt{\vec{k}_j^2 + M^2}$

- If restrict M < E*< 5M then only 3-particle "cuts" have singularities, and these occur only when all three particles to go on-shell

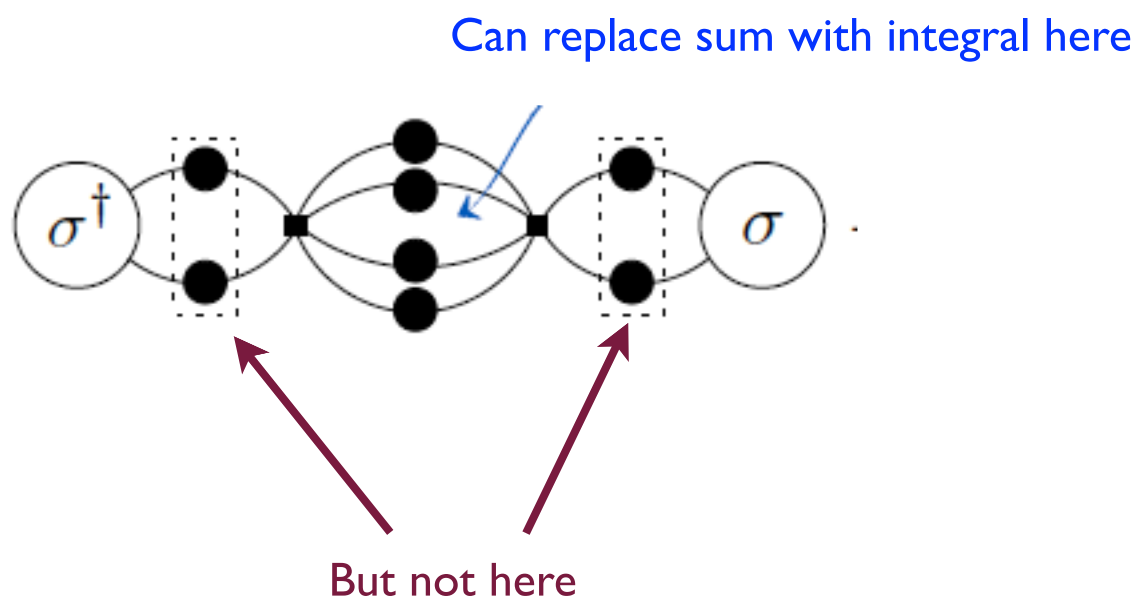
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our 3-particle example, find:



Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our 2-particle example, find:



- Then repeatedly use sum=integral + "sum-integral" to simplify

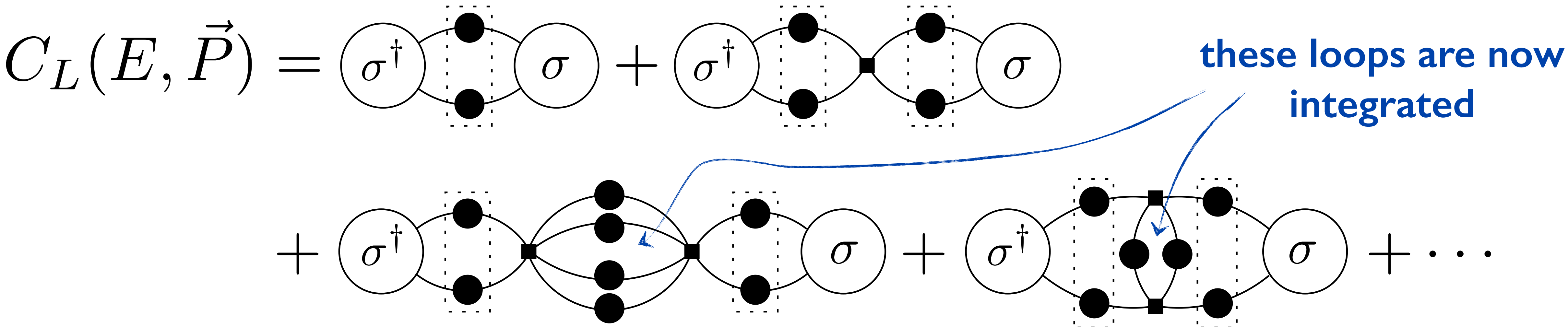
Key issues 4-6

- Dealing with cusps, avoiding divergences in 3-particle scattering amplitude, and dealing with breaking of particle interchange symmetry
- Discuss later!

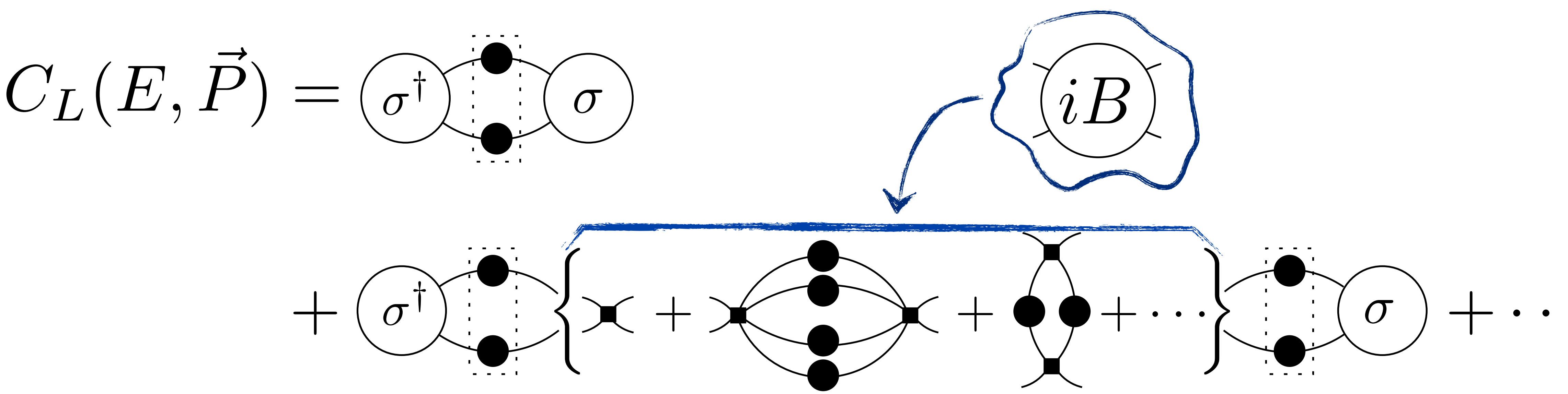
2-particle quantization condition

Following method of [Kim, Sachrajda & SS 05]

- Apply previous analysis to 2-particle correlator ($0 < E^* < 4M$)



- Collect terms into infinite-volume Bethe-Salpeter kernels



- Apply previous analysis to 2-particle correlator
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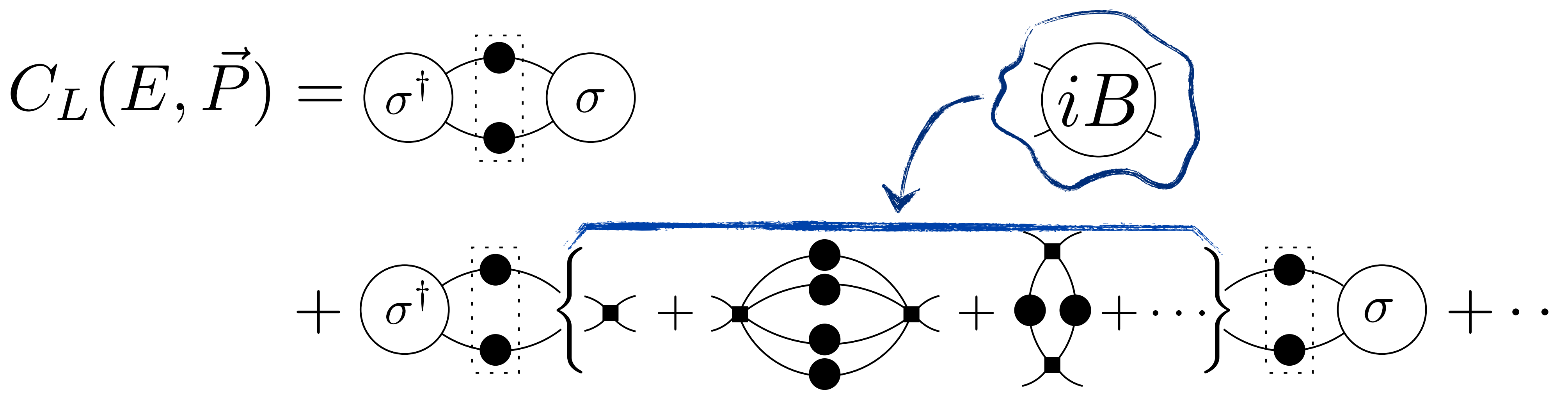
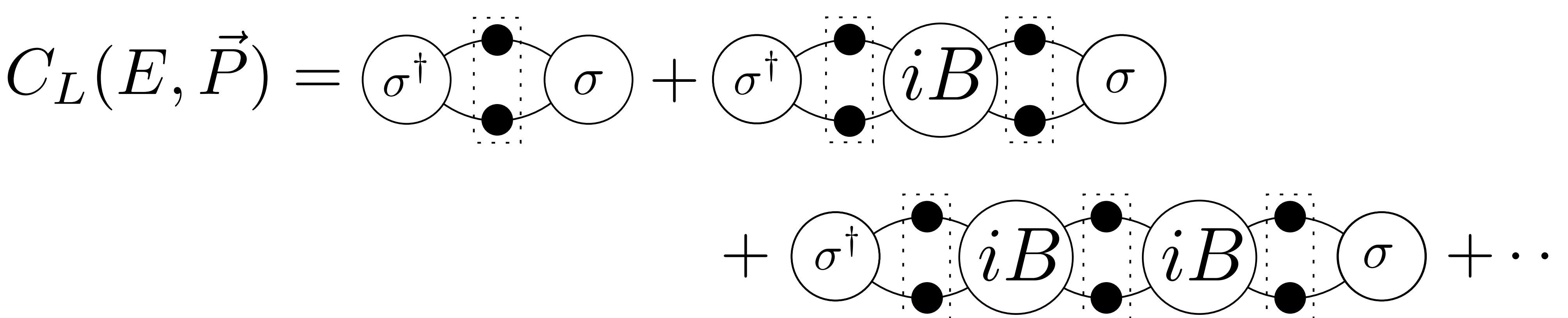
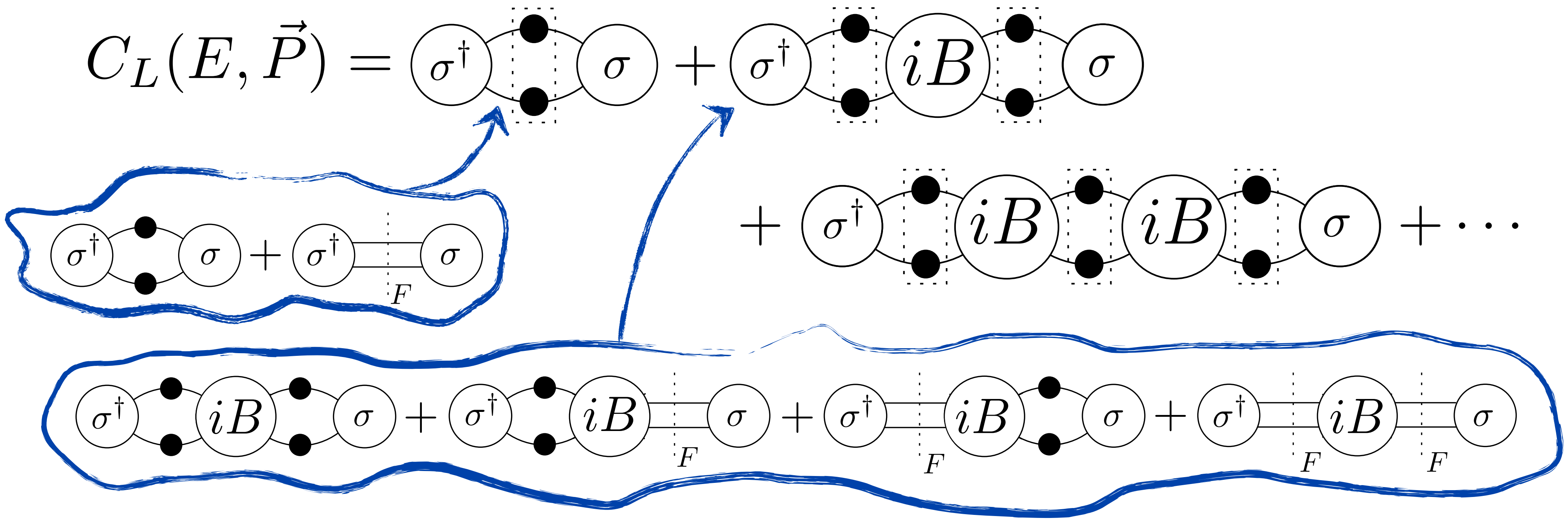


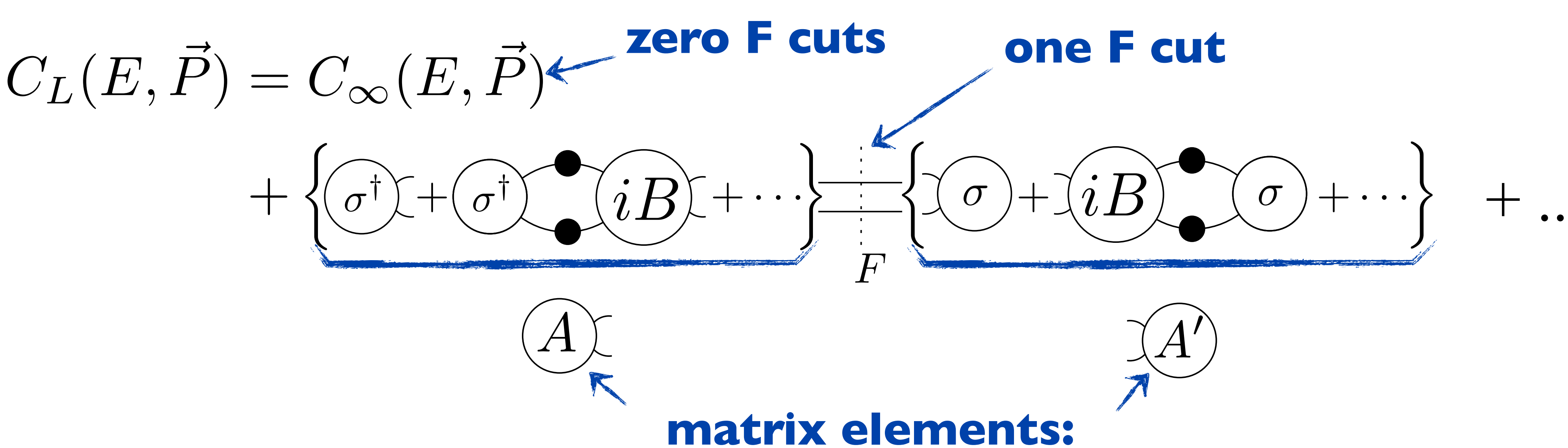
- Leading to



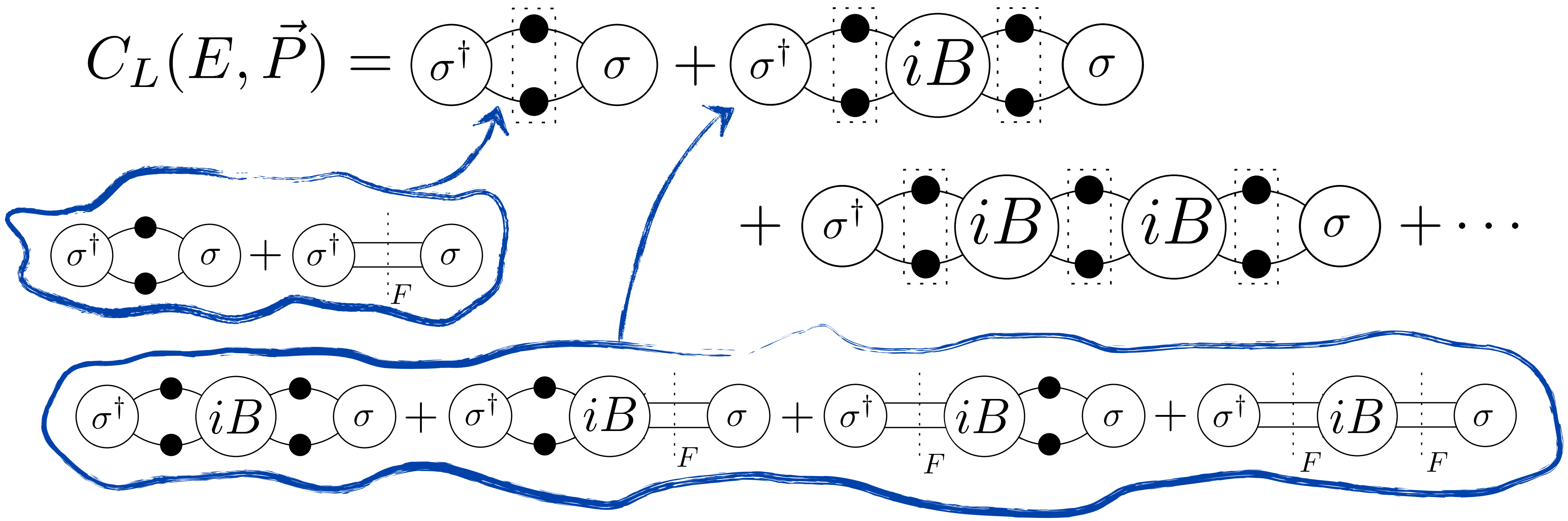
- Next use sum identity



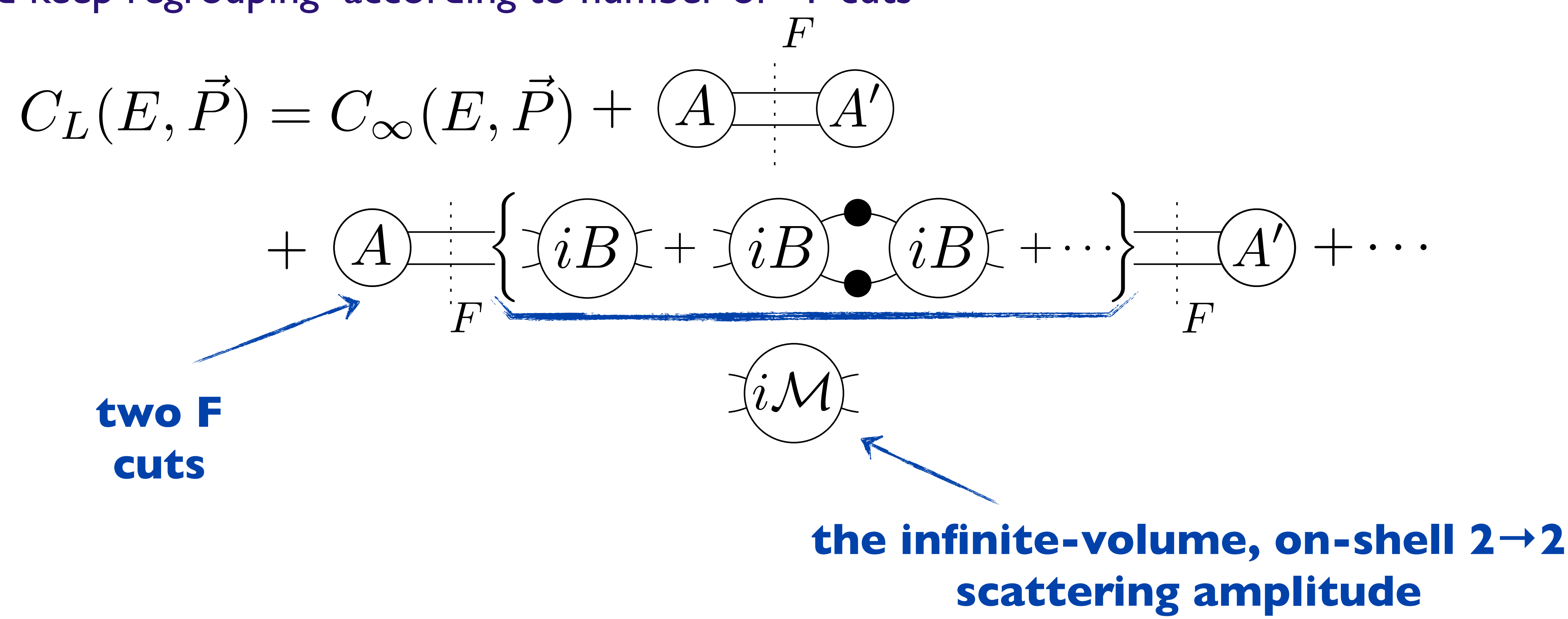
- And regroup according to number of "F cuts"



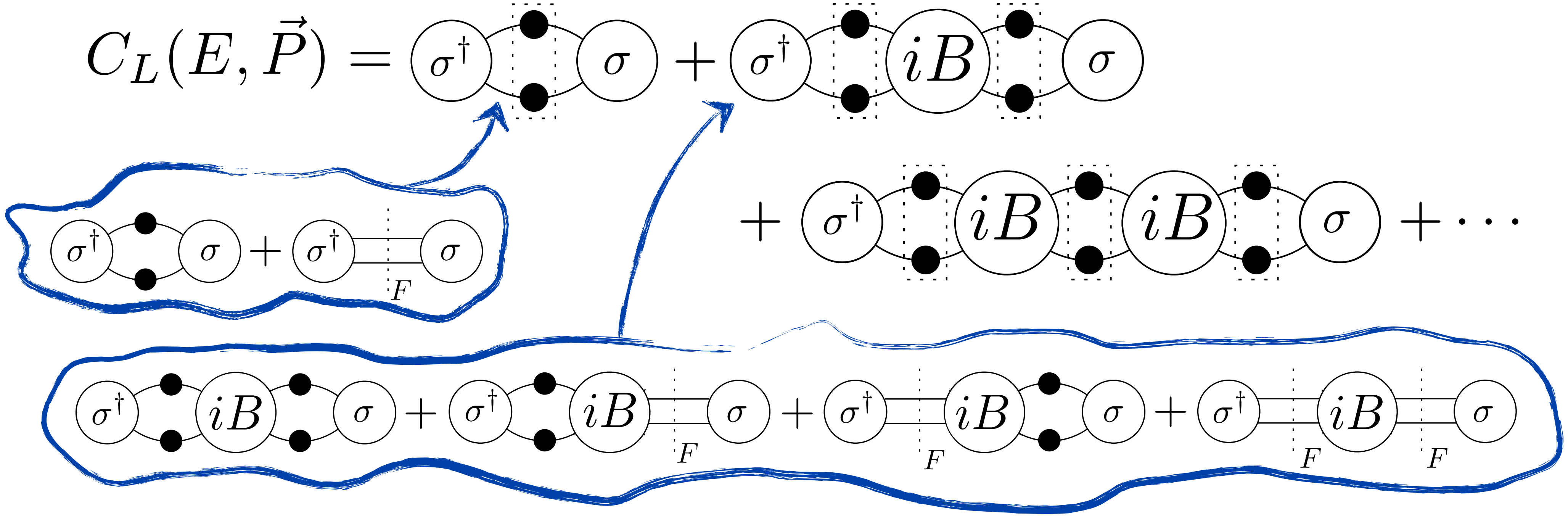
- Next use sum identity



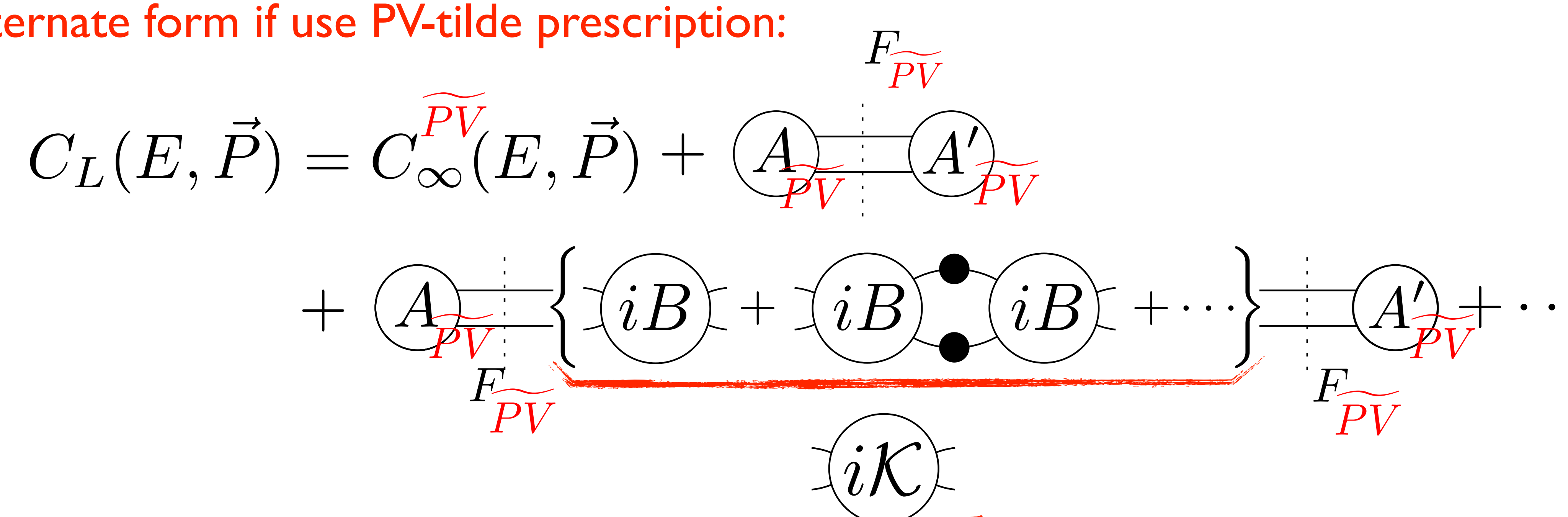
- And keep regrouping according to number of "F cuts"



- Next use sum identity



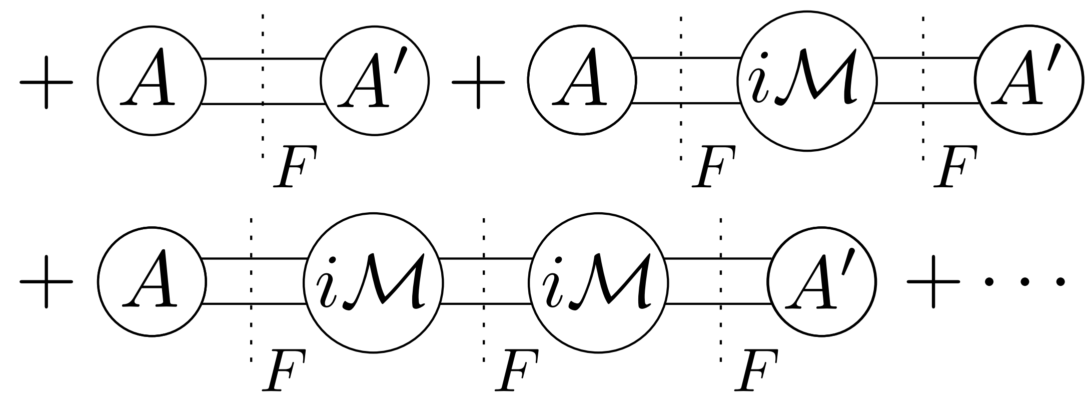
- Alternate form if use PV-tilde prescription:



the infinite-volume, on-shell 2→2 K-matrix

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$

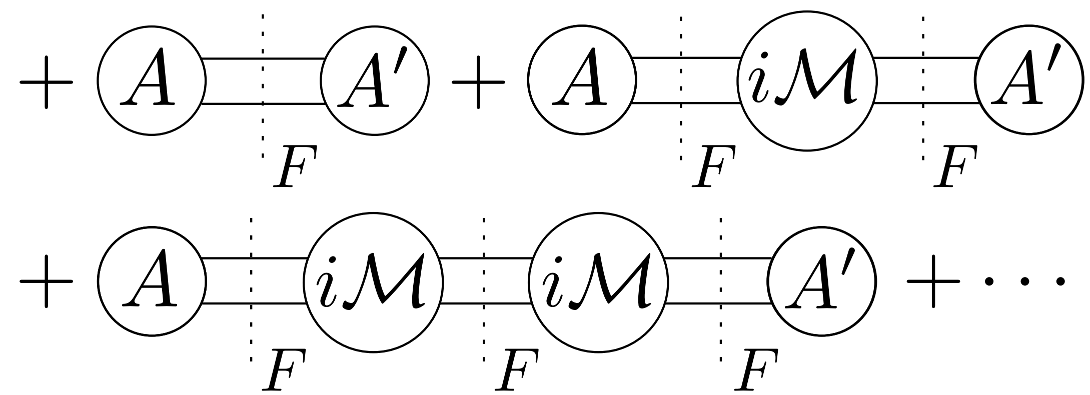


- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to 2} iF]^n A$$

- Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$



- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to2} iF]^n A$$

- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to2} iF} A$$

no poles, only cuts

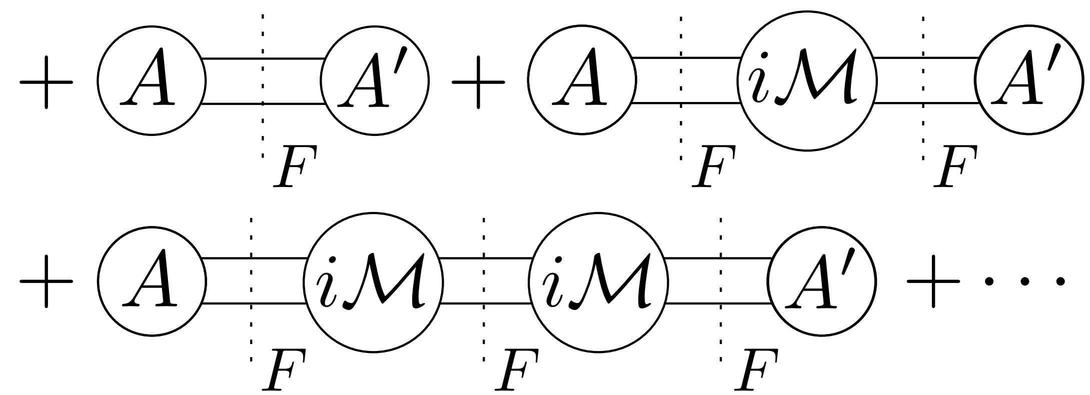
matrices in l,m space

no poles, only cuts

- $C_L(E,\vec{P})$ diverges whenever $iF \dfrac{1}{1 - i\mathcal{M}_{2\to2} iF}$ diverges

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$



- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF[i\mathcal{M}_{2\to2} iF]^n A$$

- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to2} iF} A$$

no poles, only cuts

no poles, only cuts

matrices in l,m space

$$\Rightarrow \quad \boxed{\Delta_{L,\vec{P}}(E) = \det\left[(iF)^{-1} - i\mathcal{M}_{2\to2}\right] = 0}$$

$$\Rightarrow \quad \Delta_{L,\vec{P}}(E) = \det\left[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2\right] = 0$$

Alternative form

2-particle quantization condition

- At fixed L & **P**, the finite-volume spectrum E_1, E_2, ... is given by solutions to

$$\boxed{\Delta_{L,\vec{P}}(E) = \det\left[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2\right] = 0}$$

- $\mathcal{K}_2$, F_{PV} are matrices in *l,m* space
- $\mathcal{K}_2$ is diagonal in *l,m*
- F_{PV} is off-diagonal, since the box violates rotation symmetry
- To make useful, truncate by assuming that $\mathcal{K}_2$ vanishes above l_{max}

$$\boxed{i\mathcal{K}_{2;00;00}(E_n^*) = \left[iF_{\widetilde{PV};00;00}(E_n, \vec{P}, L)\right]^{-1}}$$

Equivalent to generalization of s-wave Lüscher equation to moving frame [Rummukainen & Gottlieb]

3-particle quantization condition

Following [Hansen & SS 14]

Final result

- Spectrum is determined (for given L, **P**) by solutions of

$$\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

Infinite volume 3-particle scattering quantity

$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

Known kinematical quantity: essentially the same as F_{PV} in 2-particle analysis

$$G_{p,\ell',m';k,\ell,m} \equiv \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell',m'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell,m}(\hat{p}^*)}{2\omega_{kp}(E - \omega_k - \omega_p - \omega_{kp})}\left(\frac{p^*}{q_k^*}\right)^{\ell} \frac{1}{2\omega_k L^3}$$

G is known kinematical quantity containing cut-off function H

- Superficially similar to 2-particle form ...

$$\Delta_{L,\vec{P}}(E) = \det\left[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2\right] = 0$$

- ... but F_3 contains both kinematical, finite-volume quantities (F_{PV} & G) and the dynamical, infinite-volume quantity $\mathcal{K}_2$

Final result

$$\boxed{\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0}$$

$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + (1+\mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

- All quantities are (infinite-dimensional) matrices, e.g. $(F_3)_{klm;pl'm'}$, with indices

[finite volume "spectator" momentum: **k**=2π**n**/L] x [2-particle CM angular momentum: *l*,*m*]

Three on-shell particles with total energy-momentum (E, **P**)

- For large **k** other two particles are below threshold; must include such configurations by analytic continuation up to a cut-off at k~m [provided by H(**k**)]

Final result

$$\boxed{\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0}$$

$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

- Important limitation: our present derivation requires that all two-particle sub-channels are non-resonant at the spectral energy under consideration
  - Resonances imply that $\mathcal{K}_2$ has a pole, and this leads to additional finite volume dependence not accounted for in the derivation
  - We only have an ugly solution—searching for something better

Final result

$$\boxed{\Delta_{L,P}(E) = \det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0}$$

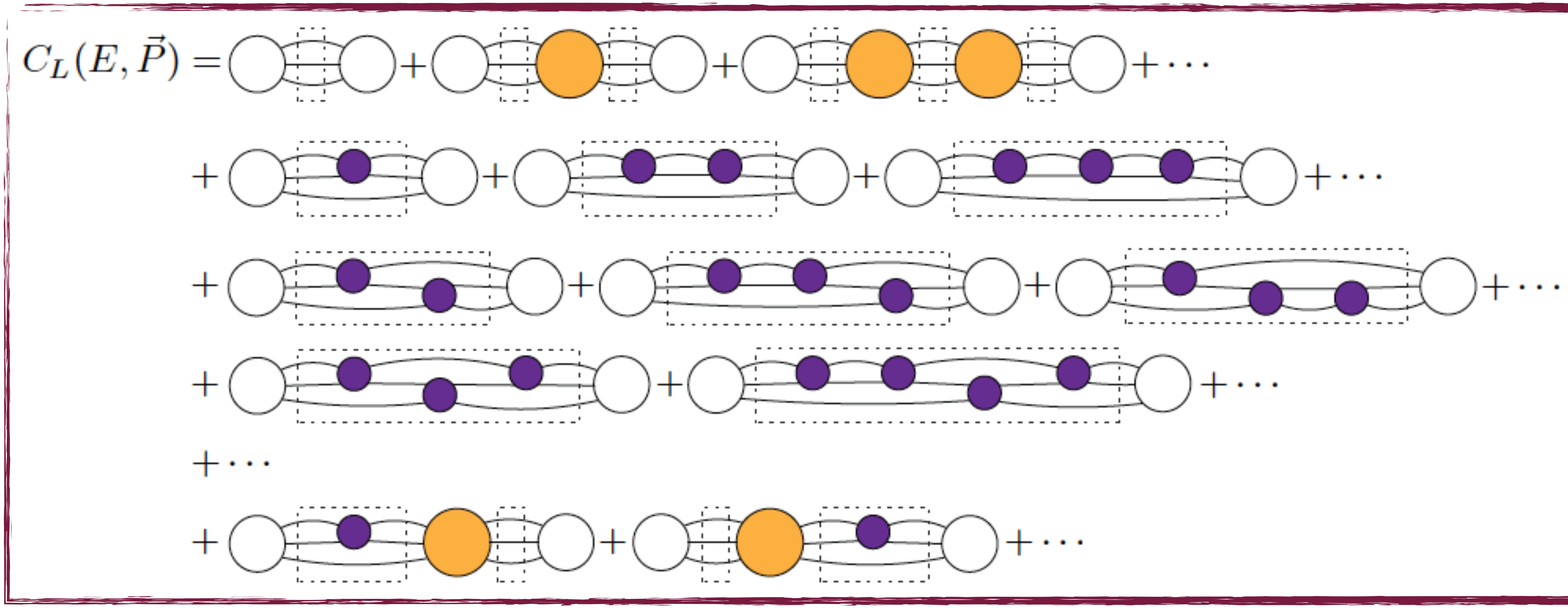
$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

- Successfully separated infinite volume quantities from finite volume kinematic factors, but….
  - What is $\mathcal{K}_{\mathrm{df},3}$?
  - How do we obtain this result?
  - How can it be made useful?

Key issue 4: dealing with cusps

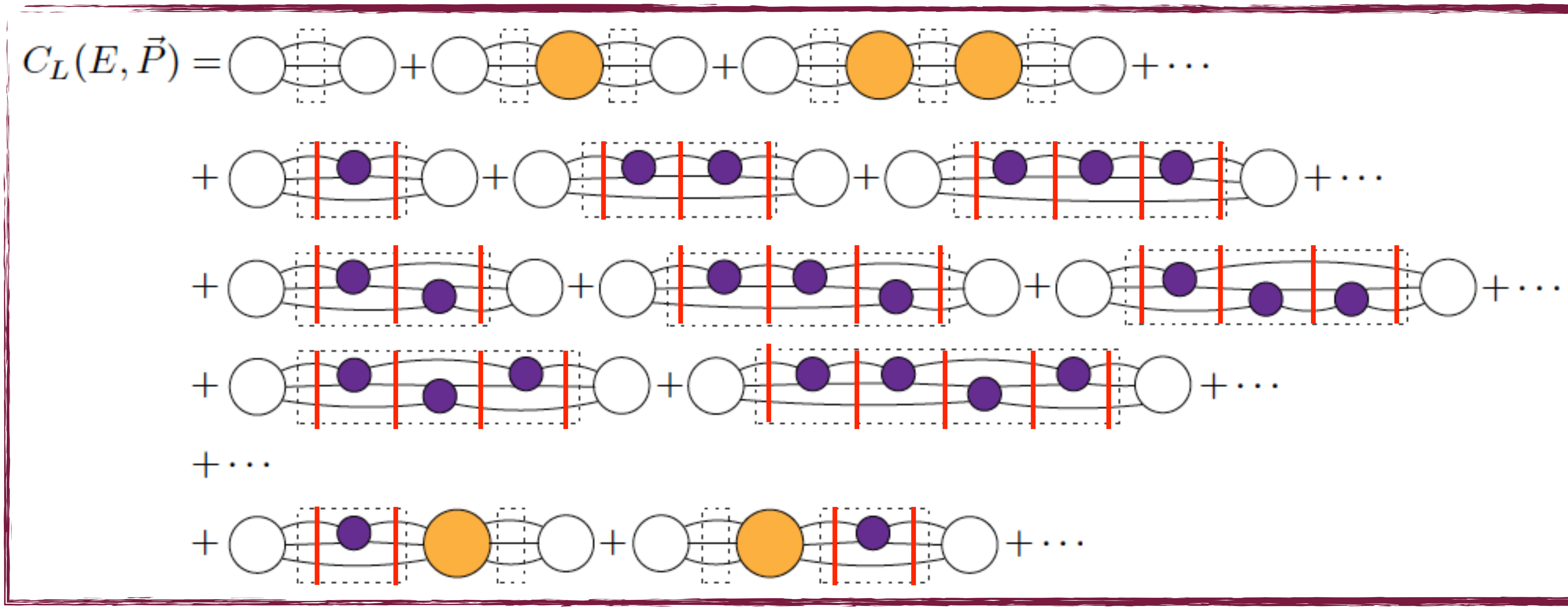
- Can sum subdiagrams without 3-particle cuts into Bethe-Salpeter kernels

⇒ Skeleton expansion in terms of Bethe-Salpeter kernels



Key issue 4: dealing with cusps

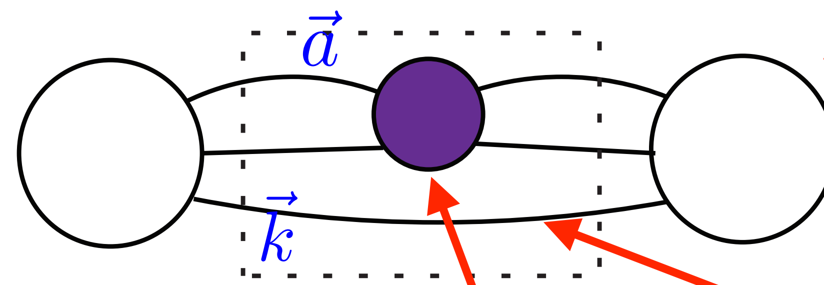
- Want to replace sums with integrals + F-cuts as in 2-particle analysis
- Straightforward implementation fails when have 3 particle intermediate states adjacent to 2→2 kernels



Cusp analysis (1)

- Aim: replace sums with integrals + finite-volume residue
- E.g.

$$(E, \vec{P}) \longrightarrow \quad \frac{1}{L^6} \sum_{\vec{k}} \sum_{\vec{a}}$$



- Can replace sums with integrals for smooth, non-singular parts of summand
- Singular part of left-hand 3-particle intermediate state

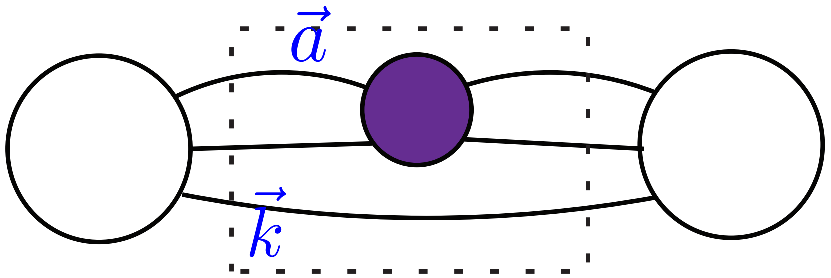
smooth functions

$$\frac{1}{L^6} \sum_{\vec{k}} \sum_{\vec{a}} \frac{A(\vec{k}, \vec{a}) B(\vec{k}, \vec{a})}{E - \omega_k - \omega_a - \omega_{ka}}$$

denominator vanishes on-shell

$\sqrt{\vec{k}^2 + m^2}$ $\qquad$ $\sqrt{\vec{a}^2 + m^2}$ $\qquad$ $\sqrt{(\vec{P} - \vec{k} - \vec{a})^2 + m^2}$

Cusp analysis (2)



$$\frac{1}{L^6}\sum_{\vec{k}}\sum_{\vec{a}} \frac{A(\vec{k},\vec{a})B(\vec{k},\vec{a})}{E-\omega_k-\omega_a-\omega_{ka}}$$

Step 1: treat sum over **a**

$$\frac{1}{L^3}\sum_{\vec{a}} \longrightarrow \int_{\vec{a}} + \left(\frac{1}{L^3}\sum_{\vec{a}} - \int_{\vec{a}}\right)$$

Step 2: treat sum over **k**

Difference gives zeta-function F with A & B projected on shell [Lüscher,...]

F has multiple singularities, so leave **k** summed for F-term

- Want to replace sum over **k** with integral for $\int_{\vec{a}}$ term
- Only possible if integral over **a** gives smooth function
- iε prescription and standard principal value (PV) lead to cusps at threshold ⇒ sum-integral ~1/L^4 [Polejaeva & Rusetsky]
- Requires use of modified $\widetilde{\mathrm{PV}}$ prescription

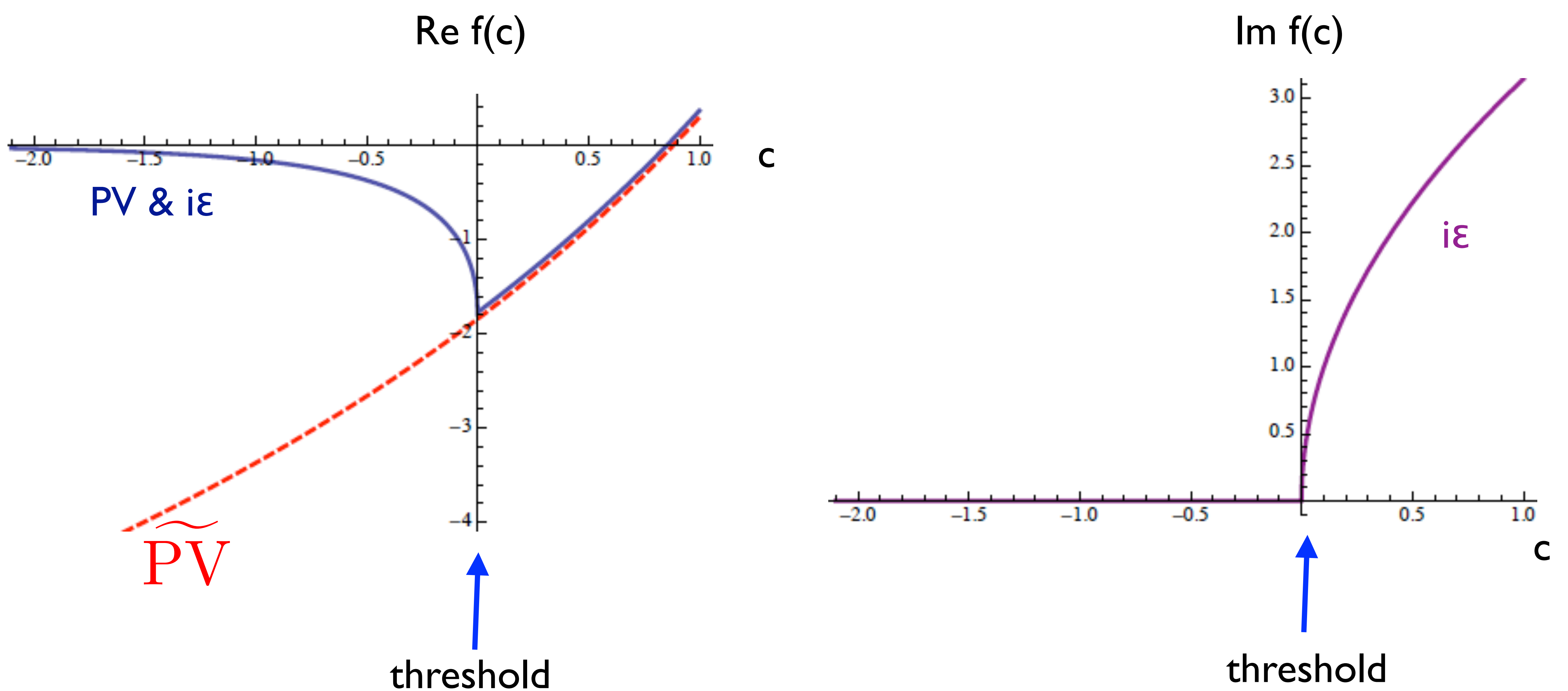
Result:

$$\frac{1}{L^6}\sum_{\vec{k}}\sum_{\vec{a}} = \int_{\vec{k}}\int_{\vec{a}} + \sum_{\vec{k}} \text{“F term”}$$

Cusp analysis (3)

- Simple example: $\int_{\vec{a}} \frac{A(\vec{k},\vec{a})B(\vec{k},\vec{a})}{E-\omega_k-\omega_a-\omega_{ka}} \longrightarrow f(c) = \int_0^\infty dx \frac{\sqrt{x}e^{-(x-c)}}{c-x}$

$x \sim (a^*)^2$

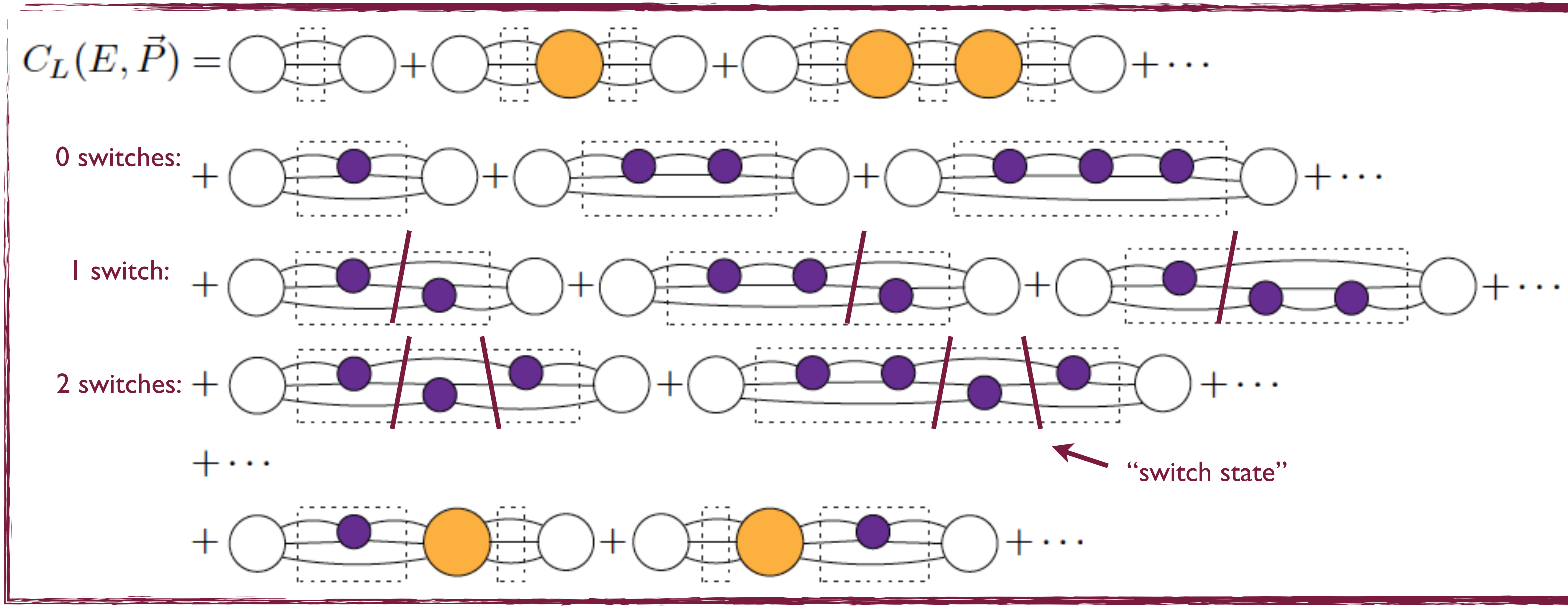


- Far below threshold, $\widetilde{\mathrm{PV}}$ smoothly turns back into PV

Cusp analysis (4)

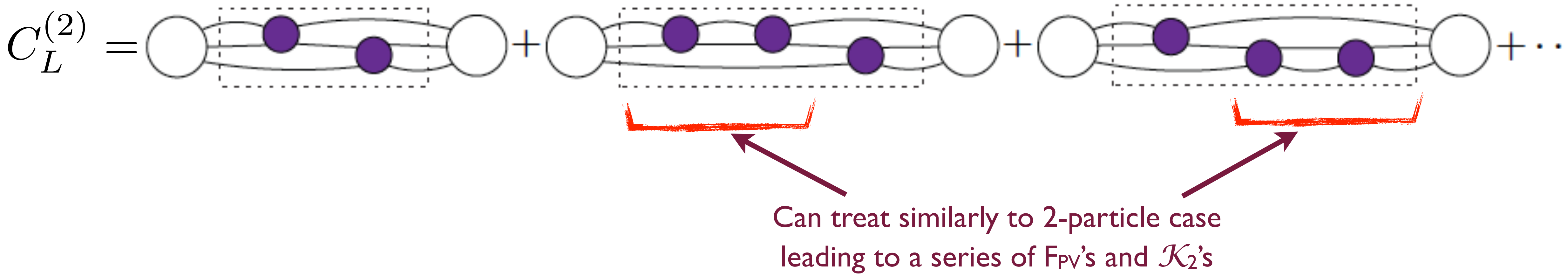
- Bottom line: must use $\widetilde{\mathrm{PV}}$ prescription for all loops
- This is why K-matrix $\mathcal{K}_2$ appears in 2-particle summations
- $\mathcal{K}_2$ is standard above threshold, and given below by analytic continuation (so there is no cusp)
- This prescription is that used previously when studying finite-volume effects on bound-state energies using two-particle quantization condition [Detmold, Savage,...]
- Far below threshold smoothly turns into $\mathcal{M}_2{}^{\ell}$

Key issue 5: dealing with "switches"

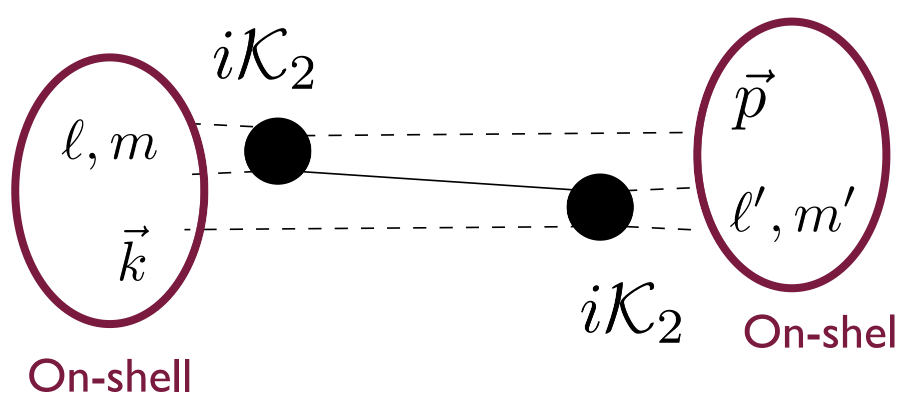


- With cusps removed, no-switch diagrams can be summed as for 2-particle case
- "Switches" present a new challenge

One-switch diagrams

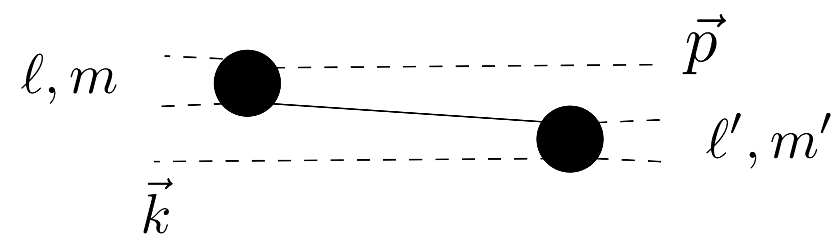


- End up with L-dependent part of C$^{(2)}$ having at its core:

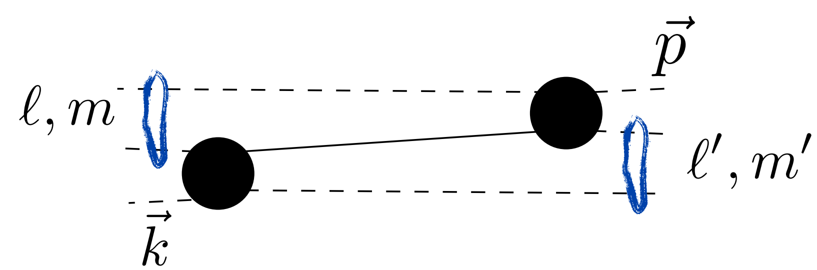


- This is our first contribution to the infinite-volume 3 particle scattering amplitude

One-switch problem



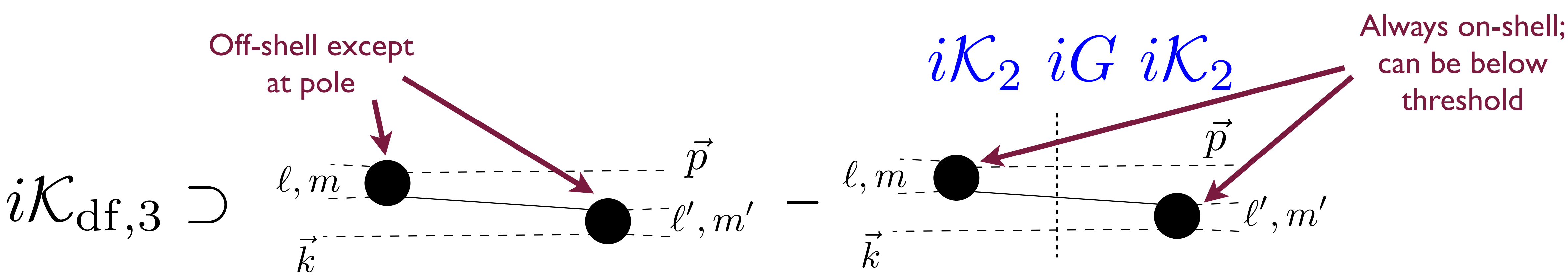
- Amplitude is **singular** for some choices of **k, p** in physical regime
  - Propagator goes on shell if top two (and thus bottom two) scatter elastically
- Not a problem per se, but leads to difficulties when amplitude is symmetrized
  - Occurs when include three-switch contributions



- Singularity implies that decomposition in $Y_{l,m}$ will not converge uniformly
  - Cannot usefully truncate angular momentum expansion

One-switch solution

- Define divergence-free amplitude by subtracting singular part
  - Utility of subtraction noted in [Rubin, Sugar & Tiktopoulos, '66]

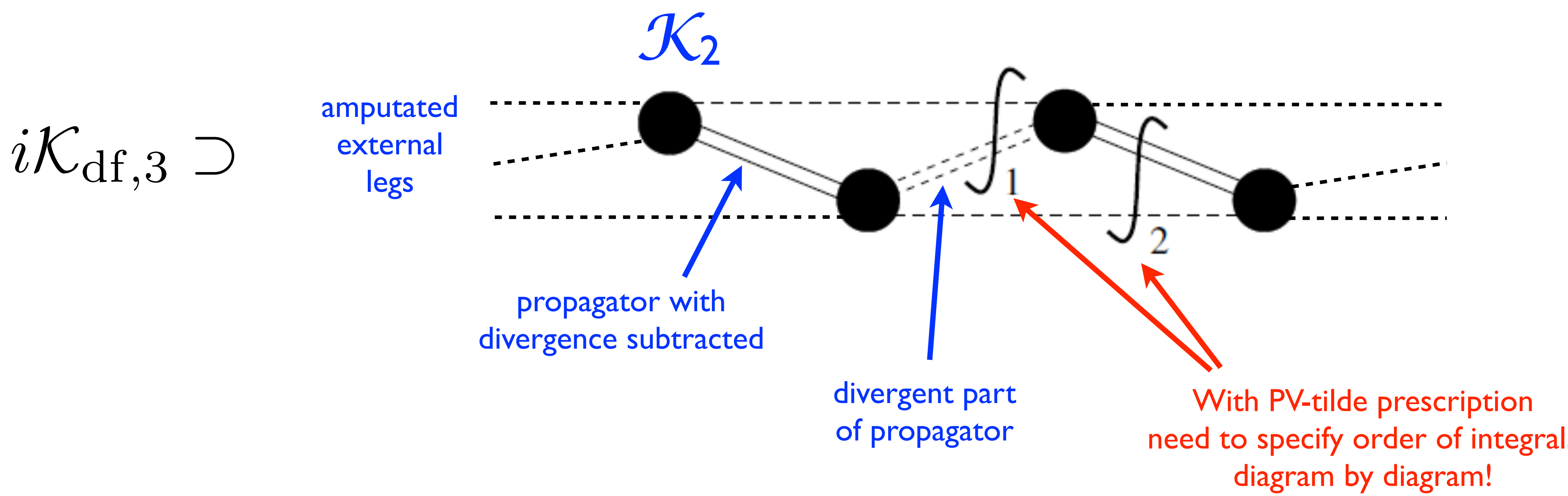


$$G_{p,\ell',m';k,\ell,m} \equiv \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell',m'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y^*_{\ell,m}(\hat{p}^*)}{2\omega_{kp}(E-\omega_k-\omega_p-\omega_{kp})} \left(\frac{p^*}{q_k^*}\right)^{\ell} \frac{1}{2\omega_k L^3}$$

- <u>Key point</u>: $\mathcal{K}_{df,3}$ is local and its expansion in harmonics can be truncated
- Subtracted term must be added back---leads to G contributions to F_3
- Can extend divergence-free definition to any number of switches

Key issue 6: symmetry breaking

- Using $\widetilde{\mathrm{PV}}$ prescription breaks particle interchange symmetry
  - Top two particles treated differently from spectator
  - Leads to very complicated definition for $\mathcal{K}_{\mathrm{df},3}$, e.g.



- Can extend definition of $\mathcal{K}_{\mathrm{df},3}$ to all orders, in such a way that it is symmetric under interchange of external particles

Key issue 6: symmetry breaking

- Final definition of $\mathcal{K}_{\mathrm{df},3}$ is, crudely speaking:
  - Sum all Feynman diagrams contributing to $\mathcal{M}_3$
  - Use $\widetilde{\mathrm{PV}}$ prescription, plus a (well-defined) set of rules for ordering integrals
  - Subtract leading divergent parts
  - Apply a set of (completely specified) extra factors ("decorations") to ensure external symmetrization
- $\mathcal{K}_{\mathrm{df},3}$ is an UGLY infinite-volume quantity related to scattering
- At the time of our initial paper, we did not know the relation between $\mathcal{K}_{\mathrm{df},3}$ and $\mathcal{M}_3$ & $\mathcal{M}_2$, although we had reasons to think that such a relationship exists
- Now we know the relationship

Infinite volume relation between $\mathcal{K}_{df,3}$ & $\mathcal{M}_3$

[Hansen & SS 15, in preparation]

The issue

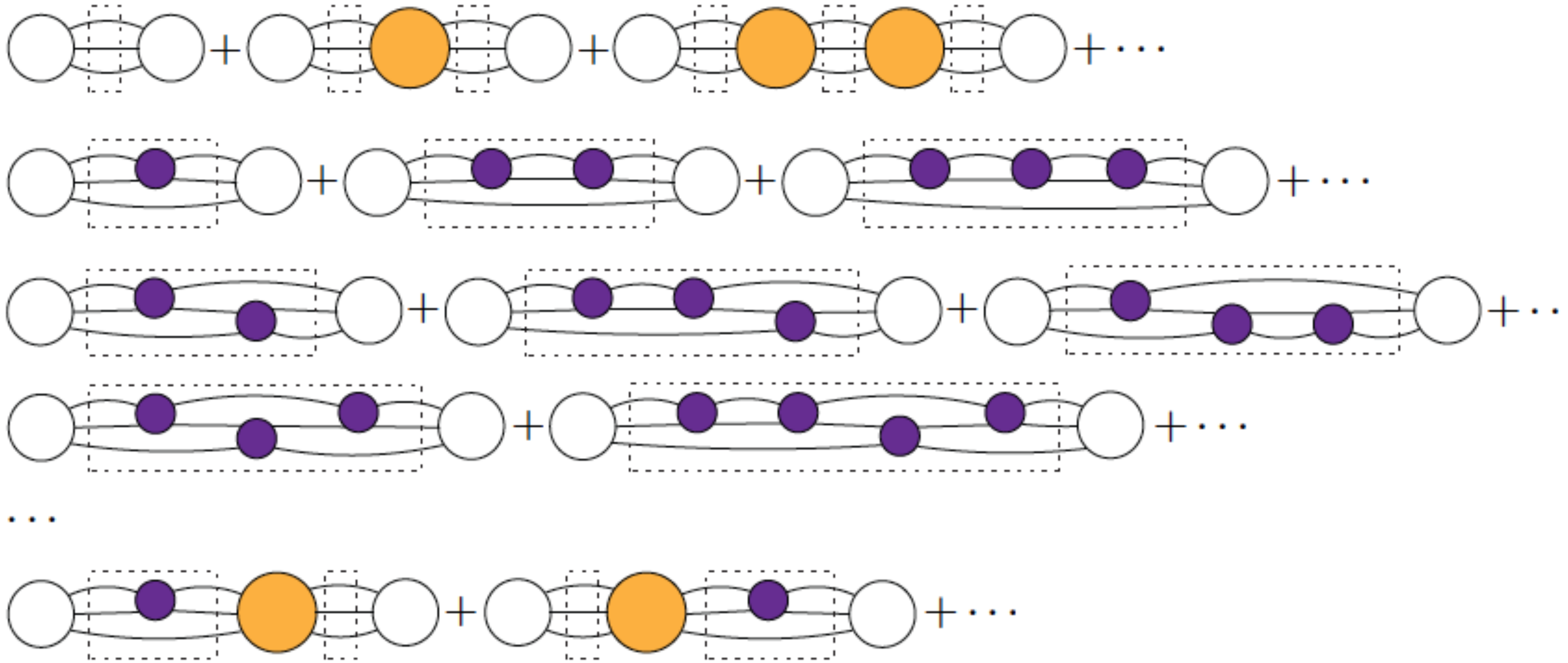
- Three particle quantization condition depends on $\mathcal{K}_{\mathrm{df},3}$ rather than the three particle scattering amplitude $\mathcal{M}_3$
- $\mathcal{K}_{\mathrm{df},3}$ is an infinite volume quantity (loops involve integrals) but is not physical
  - Has a very complicated, unwieldy definition
  - Depends on the cut-off function H
  - However, it was forced on us by the analysis, and is some sort of local vertex
- To complete the quantization condition we must relate $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

The method

- Define a "finite volume scattering amplitude" $\mathcal{M}_{L,3}$ which goes over to $\mathcal{M}_3$ in an (appropriately taken) $L \to \infty$ limit
- Relate $\mathcal{M}_{L,3}$ to $\mathcal{K}_{df,3}$ at finite volume—which turns out to require a small generalization of the methods used to derive the quantization condition
- Take $L \to \infty$, obtaining nested integral equations

Modifying C_L to obtain $\mathcal{M}_{L,3}$

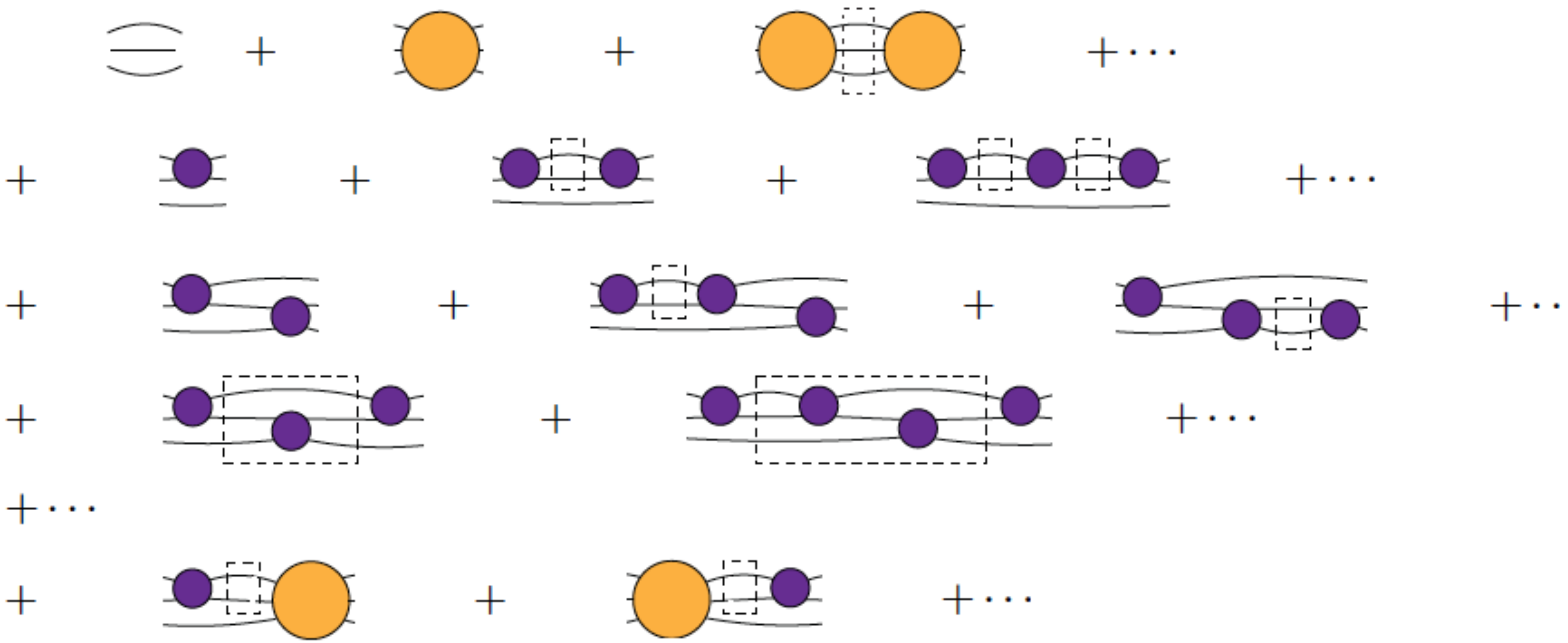
$C_L(E,\vec{P}) =$



$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A_3' iF_3 \frac{1}{1 - i\mathcal{K}_{\mathrm{df},3\to 3}\, iF_3} A_3$$

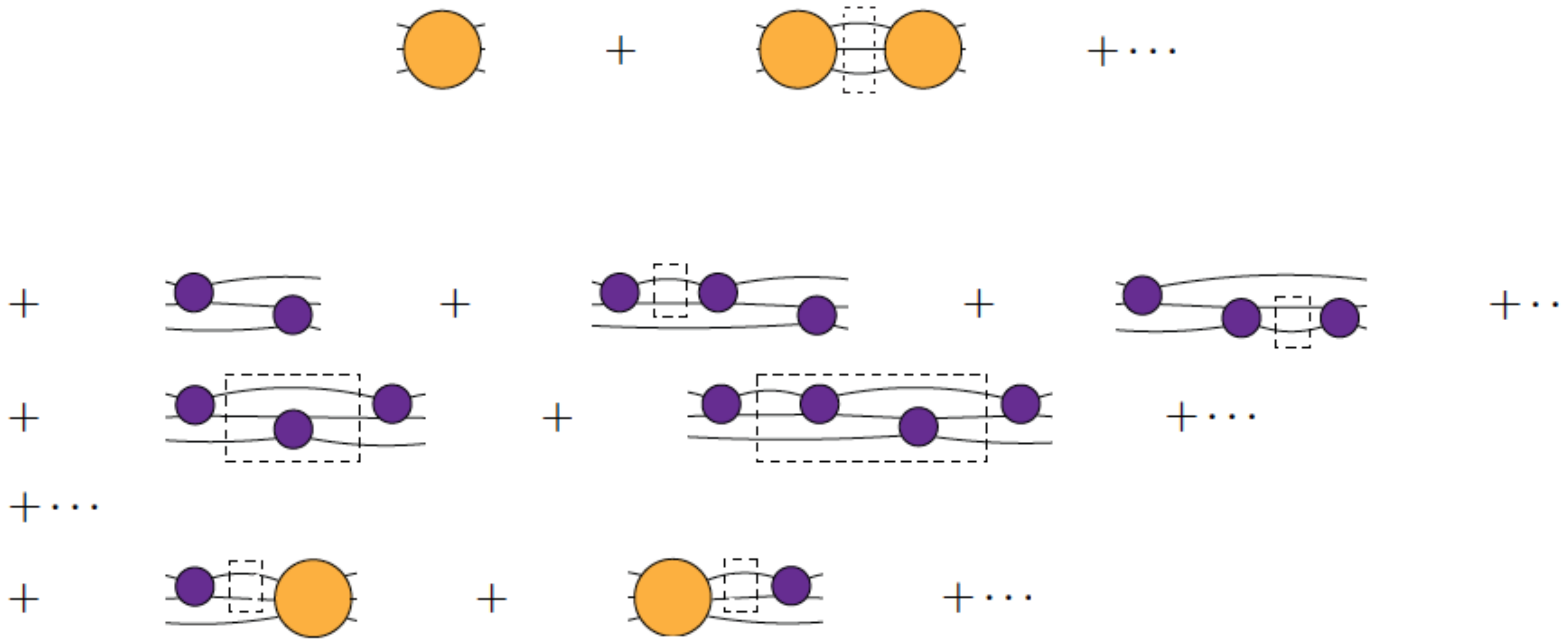
no poles **no poles** **no poles**

Modifying C_L to obtain $\mathcal{M}_{L,3}$



Step 1: "amputate"

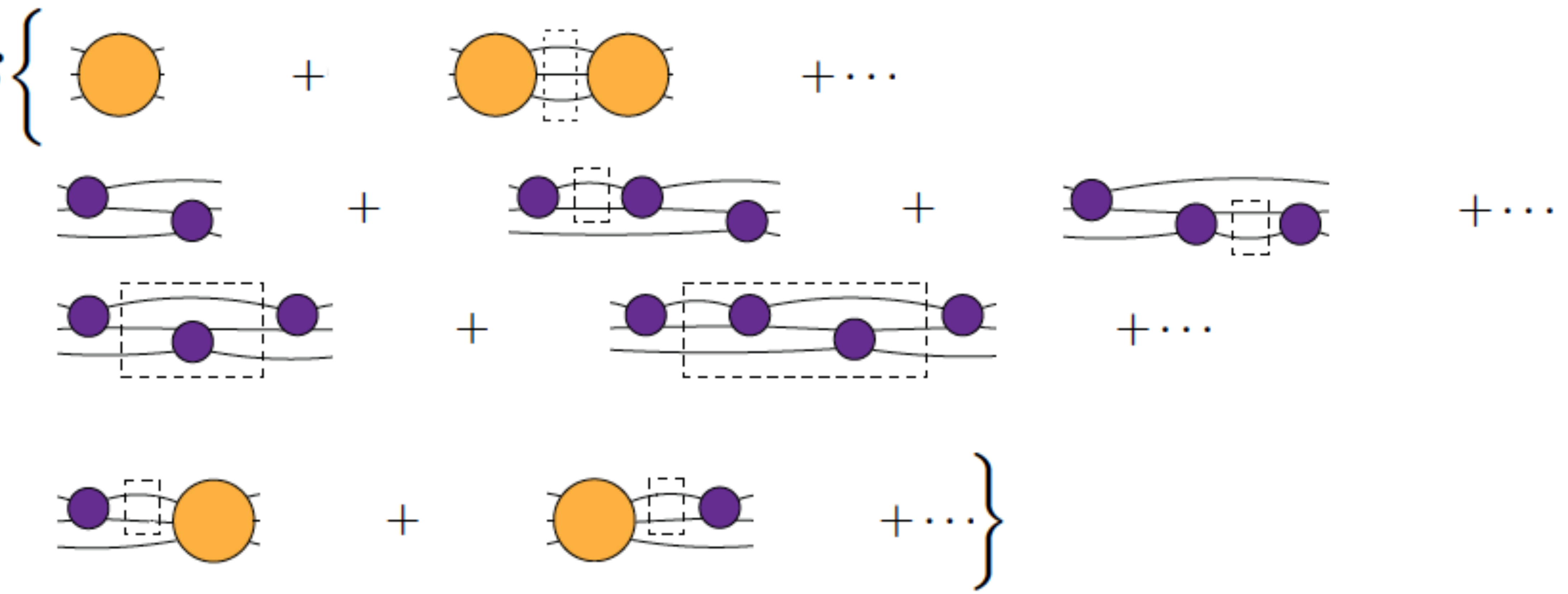
Modifying C_L to obtain $\mathcal{M}_{L,3}$



Step 2: Drop disconnected diagrams

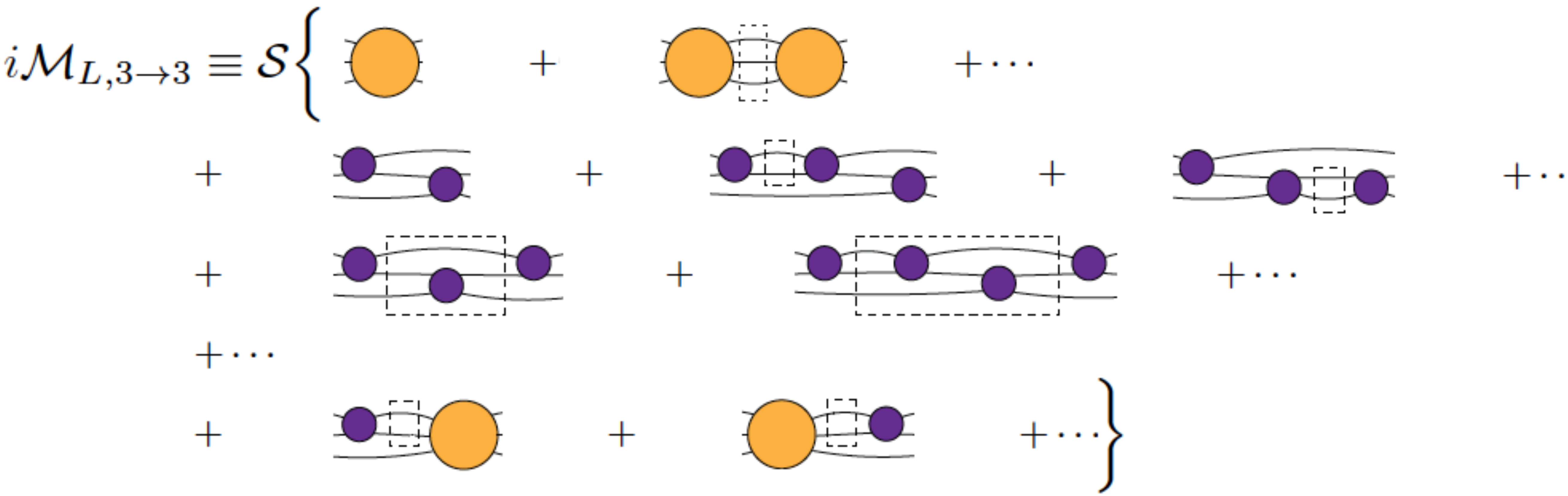
Modifying C_L to obtain $\mathcal{M}_{L,3}$

$$i\mathcal{M}_{L,3\to 3} \equiv \mathcal{S}$$



Step 3: Symmetrize

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{df,3}$



$$\boxed{i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L\ i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3\ i\mathcal{K}_{\mathrm{df},3\to3}}\ \mathcal{R}_L\right]}$$

$$i\mathcal{D}_L \equiv \mathcal{S}\left[\frac{1}{1 - i\mathcal{M}_{L,2\to2}\ iG}\ i\mathcal{M}_{L,2\to2}\ iG\ i\mathcal{M}_{L,2\to2}[2\omega L^3]\right]$$

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{df,3}$

$$\boxed{i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \; \mathcal{R}_L\right]}$$

$$i\mathcal{D}_L \equiv \mathcal{S}\left[\frac{1}{1 - i\mathcal{M}_{L,2\to2} \; iG} \; i\mathcal{M}_{L,2\to2} \; iG \; i\mathcal{M}_{L,2\to2}[2\omega L^3]\right]$$

- $\mathcal{L}_L$ and $\mathcal{R}_L$ depend only on $\mathcal{M}_{L,2}$, G and F_{PV}
- $\mathcal{M}_{L,2}$ is "finite volume two particle scattering amplitude"

$$i\mathcal{M}_{L,2\to2} \equiv i\mathcal{K}_{2\to2} \frac{1}{1 - i\underset{\mathrm{PV}}{\widetilde{F}} i\mathcal{K}_{2\to2}}$$

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{df,3}$

$$\boxed{i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L\ i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3\ i\mathcal{K}_{\mathrm{df},3\to3}}\ \mathcal{R}_L\right]}$$

$$i\mathcal{D}_L \equiv \mathcal{S}\left[\frac{1}{1 - i\mathcal{M}_{L,2\to2}\ iG}\ i\mathcal{M}_{L,2\to2}\ iG\ i\mathcal{M}_{L,2\to2}[2\omega L^3]\right]$$

- Key point: the *same (ugly)* $\mathcal{K}_{df,3}$ appears in $\mathcal{M}_{L,3}$ as in C_L

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A_3' iF_3 \frac{1}{1 - i\mathcal{K}_{\mathrm{df},3\to3}\ iF_3} A_3$$

- Can use $\mathcal{M}_{L,3}$ to derive quantization condition

Final step: taking L→∞

$$\boxed{i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \; \mathcal{R}_L\right]}$$

$$i\mathcal{D}_L \equiv \mathcal{S}\left[\frac{1}{1 - i\mathcal{M}_{L,2\to2} \; iG} \; i\mathcal{M}_{L,2\to2} \; iG \; i\mathcal{M}_{L,2\to2}[2\omega L^3]\right]$$

$$iF_3 \equiv \frac{iF_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to2} \; iG} \; i\mathcal{M}_{L,2\to2} \; iF_{\widetilde{\mathrm{PV}}}\right]$$

- All equations involve matrices with indices *k, l, m*

Spectator momentum
k =2 **n** π / L
Summed over **n**

Already in infinite
volume variables

Final step: taking L→∞

$$\boxed{i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L\ i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3\ i\mathcal{K}_{\mathrm{df},3\to3}}\ \mathcal{R}_L\right]}$$

$$i\mathcal{D}_L \equiv \mathcal{S}\left[\frac{1}{1 - i\mathcal{M}_{L,2\to2}\ iG}\ i\mathcal{M}_{L,2\to2}\ iG\ i\mathcal{M}_{L,2\to2}[2\omega L^3]\right]$$

$$iF_3 \equiv \frac{iF_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to2}\ iG}\ i\mathcal{M}_{L,2\to2}\ iF_{\widetilde{\mathrm{PV}}}\right]$$

- Sums over momenta → integrals (+ now irrelevant 1/L terms!)
- Must introduce pole prescription for sums to avoid singularities

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon}\ i\mathcal{M}_{L,3\to3}$$

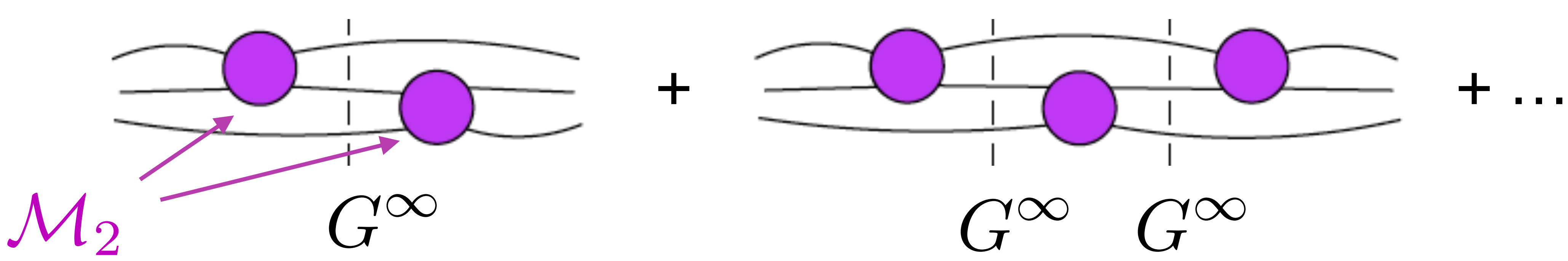
Final result: nested integral equations

(1) Obtain L→∞ limit of $\mathcal{D}_L$

$$iD^{(u,u)}(\vec{p},\vec{k}) = i\mathcal{M}_2(\vec{p})iG^\infty(\vec{p},\vec{k})i\mathcal{M}_2(\vec{k}) + \int_s \frac{1}{2\omega_s} i\mathcal{M}_2(\vec{p})iG^\infty(\vec{p},\vec{s})iD^{(u,u)}(\vec{s},\vec{k})$$

$$G^\infty_{\ell' m';\ell m}(\vec{p},\vec{k}) \equiv \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell' m'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{2\omega_{kp}(E-\omega_k-\omega_p-\omega_{kp}+i\epsilon)} \left(\frac{p^*}{q_k^*}\right)^{\ell}$$

- Quantities are still matrices in *l,m* space
- Presence of cut-off function means that integrals have finite range
- $\mathcal{D}^{(u,u)}$ sums geometric series which gives physical divergences in $\mathcal{M}_3$



Final result: nested integral equations

(2) Sum geometric series involving $\mathcal{K}_{\mathrm{df},3}$

$$i\mathcal{T}(\vec{p},\vec{k}) = i\mathcal{K}_{\mathrm{df},3}(\vec{p},\vec{k}) + \int_s \int_r i\mathcal{K}_{\mathrm{df},3}(\vec{p},\vec{s}\,)\frac{i\rho(\vec{s}\,)}{2\omega_s} i\mathcal{L}^{(u,u)}(\vec{s},\vec{r}\,) i\mathcal{T}(\vec{r},\vec{k})\,,$$

$$\mathcal{L}^{(u,u)}(\vec{p},\vec{k}) = \left(\frac{1}{3} + i\mathcal{M}_2(\vec{p}\,)i\rho(\vec{p}\,)\right)(2\pi)^3\delta^3(\vec{p}-\vec{k}) + i\mathcal{D}^{(u,u)}(\vec{p},\vec{k})\frac{i\rho(\vec{k})}{2\omega_k}\,,$$

- $\rho(\mathbf{k})$ is a phase space factor (analytically continued when below threshold)
- Requires $\mathcal{D}^{(u,u)}$ and $\mathcal{M}_2$
- Corresponds to summing the core geometric series, i.e.

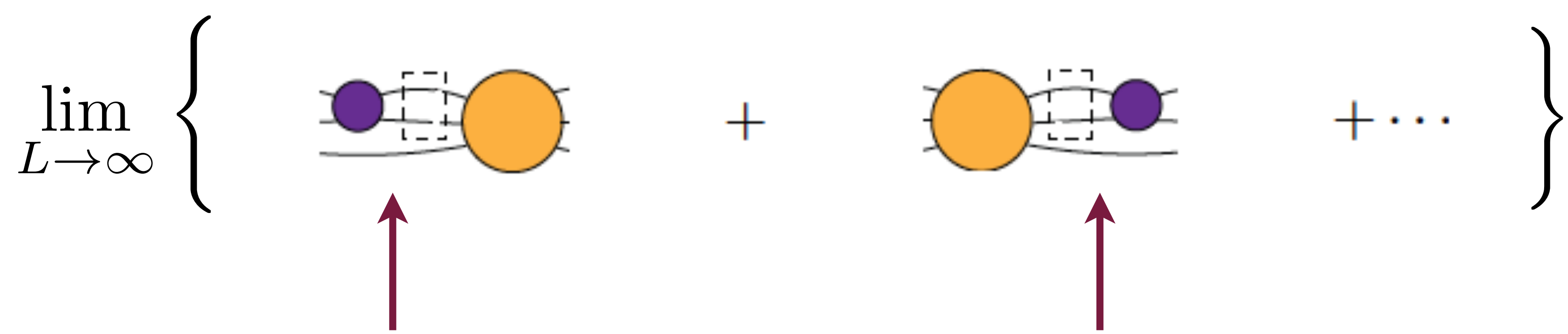
$$i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3\; i\mathcal{K}_{\mathrm{df},3\to3}}$$

Final result: nested integral equations

(3) Add in effects of external 2→2 scattering:

$$\underbrace{\mathcal{M}_3(\vec{p},\vec{k}) - \mathcal{S}\left\{\mathcal{D}^{(u,u)}(\vec{p},\vec{k})\right\}}_{\mathcal{M}_{df,3}} = -\mathcal{S}\left\{\int_s \int_r \mathcal{L}^{(u,u)}(\vec{p},\vec{s})\mathcal{T}(\vec{s},\vec{r})\mathcal{R}^{(u,u)}(\vec{r},\vec{k})\right\}$$

- Sums geometric series on "outside" of $\mathcal{K}_{\text{df},3}$'s



- Can also invert and determine $\mathcal{K}_{\text{df},3}$ given $\mathcal{M}_3$ and $\mathcal{M}_2$