Renormalons and the OPE

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Outline

- Renormalons and operator product expansion: basic relations
- Pole mass and τ decay
- OPE as a transseries: Operator product expansion at $\mathcal{O}(1/N)$ in the $O(N)$ σ-model

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Asymptotic behaviour

General structure of large-order behaviour is (believed to be) known.
Several components of factorial divergence.

$$R = \sum_n r_n \alpha_s(Q)^{n+1}, \qquad r_n \overset{n \gg 1}{=} K\, a^n \,\frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n} + O(1/n^2)\right)$$

Borel transform

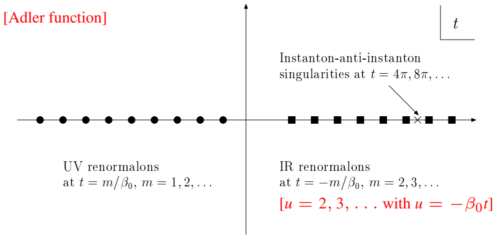
$$B[R](t) = \sum_{n=0}^{\infty} r_n \,\frac{t^n}{n!}, \qquad B[R](t) \overset{t \to 1/a}{=} \frac{K}{(1-at)^{1+b}}\,(1 + s_1(1-at) + \ldots)$$

Borel integral

$$R(\alpha_s) = \int_0^\infty dt\, \mathrm{e}^{-t/\alpha_s}\, B[R](t), \quad \frac{1}{\pi}\,\mathrm{Im}\, R = \pm \frac{K}{a\Gamma(1+b)}\, e^{-1/(a\alpha_s)}\, (a\alpha_s)^{-b}\, (1 + \#\, s_1 \alpha_s + \ldots)$$

... imaginary part (“ambiguity”) for positive a.

Renormalon (and instanton) singularities



- UV renormalons – from large loop momentum
  Sign-alternating, singularity structure related to higher-dim operators in the cut-off QCD Lagrangian [Parisi, 1977; MB, Kivel, Braun 1997]
- IR renormalons – from small loop momentum
  Fixed-sign, singularity structure related to higher-dim operators in the OPE [Gross, Neveu, 1974; Lautrup, 1977; 't Hooft, 1977; David, 1984; Mueller, 1985; Zakharov, 1992; MB, 1993]
- Instanton-anti-instanton – number of diagrams
  [Bogomolny, Fadeev, 1977; Balitsky, 1991]

Large momentum/small coupling expansion (e.g. Adler function)

$$D(Q^2) = Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \sum_{k=0} \underbrace{C_k(\alpha_s(\mu), \ln \frac{Q^2}{\mu^2})}_{\text{perturbative series}} \times \frac{1}{(Q^2)^k} \times \underbrace{\langle \mathcal{O}_k \rangle}_{\substack{\text{condensates } (k>0) \\ \text{"power corrections"}}}$$

$$= \sum_{k=0} \left[\frac{\langle \tilde{\mathcal{O}}_k \rangle}{\Lambda^{2k}} \right] \times \left[e^{-\frac{1}{(-\beta_0)\alpha_s(Q)}} \right]^k (-\beta_0 \alpha_s(Q))^{k\beta_1/\beta_0^2 - \gamma_{0,k}/\beta_0} \times \sum_{n=0} c_k^{(n)} \alpha_s(Q)^n$$

where $\langle \tilde{\mathcal{O}}_k \rangle / \Lambda^{2k}$ are pure numbers.

General belief: series $c_k^{(n)}$ divergent and not Borel-summable (IR renormalons), $\langle \tilde{\mathcal{O}}_k \rangle / \Lambda^{2k}$ ambiguous and values related to series summation prescription. Ambiguities cancel in the entire double ("transseries") expansion.

- perturbative/diagrammatic (large-β_0 $[-2n_f/3 \to b_0]$, subleading $1/n_f$ terms, stochastic lattice PT)
- Non-perturbative $1/N$ expansion (see part II)
- RGE and consistency assumption

Singularities fixed up to constants

Example: single operator at dimension d (e.g. gluon condensate for the Adler function)

$$\frac{1}{Q^d}\, C(Q^2/\mu^2,\alpha_s)\,\langle 0|\mathcal{O}|0\rangle(\mu) \;=\; \text{const}\,\cdot e^{d/(2\beta_0\alpha_s(Q))}\,(-\beta_0\alpha_s(Q))^{d\beta_1/(2\beta_0^2)}$$
$$\times\, F(\alpha_s(Q))^{d/2}\,\exp\left(-\int_{\alpha_0}^{\alpha_s(Q)} dx\,\frac{\gamma(x)}{2\beta(x)}\right) C(1,\alpha_s(Q))$$

implies

$$r_n \overset{n\gg 1}{=} K\,a^n\,\frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{s_1}{n}+O(1/n^2)\right)$$

with

$$a=-\frac{2\beta_0}{d} \qquad b=-\frac{d\beta_1}{2\beta_0^2}+\frac{\gamma_0}{2\beta_0} \qquad a s_1=\frac{c_1}{c_0}-\frac{\gamma_1}{2\beta_0}+\frac{\gamma_0\beta_1}{2\beta_0^2}+\frac{d\beta_2}{2\beta_0^2}-\frac{d\beta_1^2}{2\beta_0^3}$$

In general $s_n(c_{i\leq n},\gamma_{i\leq n},\beta_{i\leq n+1})$.

Only K remains undetermined [Beneke (1993, 1994)]

I. Quark mass and τ decay

These two are unique for the study of large-order behaviour

- Perturbative series known to four loops ($n = 4$).
  [Baikov et al. (2008) – Adler function; Marquard et al. (2015) – pole mass]
- Leading IR renormalon determined by a single operator.
  Only one unknown constant.
- Perturbative series on the lattice to very high orders $n \sim 20$.
  [Bali et al. (2011-1014)]
- Large-β_0 model reasonably good (in particular for M).

The next IR poles and even the leading UV renormalon pole are already complicated – several operators with anomalous dimensions. The singularity at u_0 then has a multiple cut structure [MB, Kivel, Braun (1997) – UV renormalon]

$$B[R](u) \overset{u\to -1}{=} \sum_i \frac{K_i}{(1+u)^{1+\lambda_i}} \left[1 + \ldots\right]$$

- Pole closest to the origin at $u = 1/2$ ("dimension 1") [MB, Braun (1994), Bigi et al. (1994)]
  Expected to be important already at low orders in $\overline{\text{MS}}$.

- QCD self-energy near mass-shell related to UV properties to HQET self-energy. Unique operator $\bar{h}_v h_v$. No anomalous dimension, trivial coefficient function by RPI [MB (1994)]

$$r_n \overset{n\gg 1}{=} K\,(-2\beta_0)^n \,\frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{s_1}{n}+\frac{s_2}{n^2}+\ldots\right)$$

with

$$b = -\frac{\beta_1}{2\beta_0^2} \qquad (-2\beta_0)s_1 = \frac{\beta_2}{2\beta_0^2} - \frac{\beta_1^2}{2\beta_0^3} \qquad s_2 = \text{known}$$

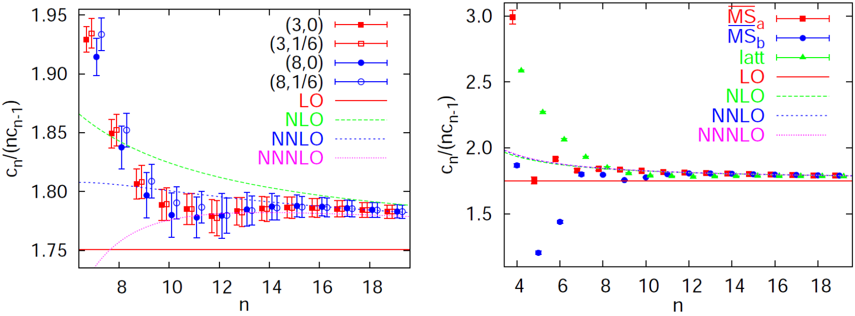
- Fit K to low orders (up to $n = 3$) [Pineda (2002); Lee (2002)]

$$K = 0.60(3) \quad [n_f = 0]$$ [Ayala et al. (2014)]

Predicted $n = 4$ correctly to 10%. [Note: not enough for quark mass determinations, since renormalon cancellation is already built in.]

- Perturbative expansion to $\mathcal{O}(\alpha_s^{20})$ in lattice scheme, $n_f = 0$ [Bali et al. (2011, 2013)]

$K = 0.60(3)$ low orders, continuum $\quad K = 0.66(6)$ lattice, numerical



[Figures from Bali et al. (2013)]

- Very consistent picture for the first pole mass IR renormalon.

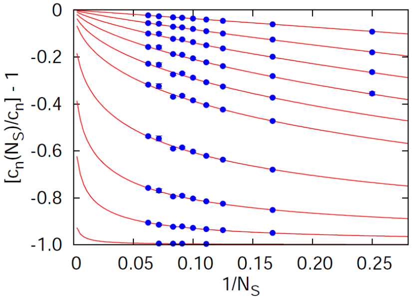
No renormalons on a finite volume lattice (neither UV nor IR)

$$r_n(N) = r_n - \frac{f_n(N)}{N} + \mathcal{O}(1/N^2) = \text{no factorial divergence}$$

IR renormalon at u_0 cut-off when

$$e^{n/(2u_0)} > \frac{L}{a} \equiv N$$

[Figures from Bali et al. (2013)]



[$N = 0, 1, 2, 3, 4, 5, 7, 9, 11, 15$ from top to bottom]

Result follows from a large extrapolation. What is the large-order behaviour at finite volume?

Leading Adler function renormalon(s)

- Pole closest to the origin at $u = -1$ (UV renormalon)
  Not seen in the first four coefficients.
  $\overline{\text{MS}}$ scheme seems to favour IR residues. In the large-β_0 model

$$K_{\text{UV}}/K_{\text{IR}} = \frac{4}{9}\, e^{-5}$$

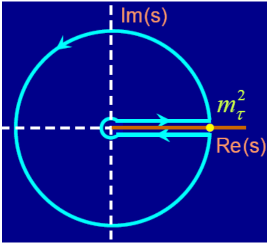
  In general for $\alpha_s \to \bar{\alpha}_s + \delta_1 \bar{\alpha}_s^2 + \ldots$, $K \to \bar{K} = K \exp(u_0 \delta_1/(-\beta_0))$.

- Leading IR renormalon at $u = 2$ related to unique $d = 4$ operator, $\alpha_s/\pi GG$.
  Precise singularity structure known inclduing the $1/n$ correction.

- Numerical perturbation theory for the plaquette up to $\mathcal{O}(\alpha_s^{35})$ [Bali et al. 2014]
  a^4 behaviour (corresponding to $u = 2$) seen after infinite-volume extrapolation and with $n \gtrsim 20$.

$$K = 0.61(25) \quad [n_f = 0]$$

  Central value significantly larger than expected. Important issue for τ decay and the QCD sum rule philosophy.



- Analyticity
- Condensate expansion
- Slightly Euclidean [$(1-x)^3$ suppression]
- $D^{(1+0)}(s) \equiv -s\,\frac{d}{ds}\left[\Pi^{(1+0)}(s)\right]$ (Adler fn)

$$
\begin{aligned}
R_\tau &= 12\pi\int_0^{M_\tau^2}\frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right)\mathrm{Im}\,\Pi^{(1)}(s)+\mathrm{Im}\,\Pi^{(0)}(s)\right] \\
&= -i\pi\oint_{|x|=1}\frac{dx}{x}\,(1-x)^3\left[3\,(1+x)\,D^{(1+0)}(M_\tau^2 x)+4\,D^{(0)}(M_\tau^2 x)\right] \\
&= N_c\,S_{\mathrm{EW}}\,|V_{ud}|^2\left[1+\delta^{(0)}+\delta'_{\mathrm{EW}}+\sum_{D\geq 2}\frac{C_D(s,\mu)\langle O_D(\mu)\rangle}{(-s)^{D/2}}\right]
\end{aligned}
$$

[Braaten, Narison, Pich, 1992]

First few terms, FOPT and CIPT

Numerical series expansions for $\alpha_s(M_\tau^2) = 0.34$.
(We will often use the estimate $c_{5,1} = 283 \pm 283$.)

$$
\begin{array}{lccccccc}
 & & \alpha_s^1 & \alpha_s^2 & \alpha_s^3 & \alpha_s^4 & \alpha_s^5 & \\
\delta_{\rm FO}^{(0)} & = & 0.1082 & +\,0.0609 & +\,0.0334 & +\,0.0174 & (+\,0.0088\,) & = \; 0.2200\;(0.2288) \\
\delta_{\rm CI}^{(0)} & = & 0.1479 & +\,0.0297 & +\,0.0122 & +\,0.0086 & (+\,0.0038\,) & = \; 0.1984\;(0.2021)
\end{array}
$$

- FO/CI difference *increases* by adding more orders. Systematic problem.
- Expansion of the running coupling on the circle as used in FO has only a finite radius of convergence [Le Diberder, Pich; 1992; Pivovarov] and actual $\alpha_s(M_\tau^2)$ is close.

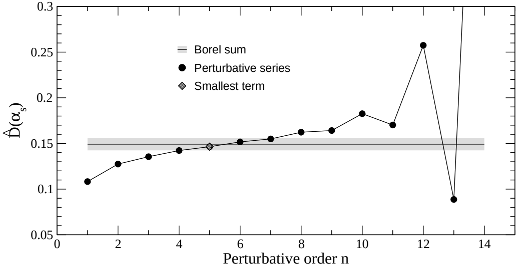
$$\alpha_s(M_\tau^2 e^{i\phi}) = \frac{\alpha_s(M_\tau^2)}{1 + \frac{\beta_0}{4\pi} i\phi\alpha_s(M_\tau^2)}$$

But: QCD perturbation expansions are only asymptotic anyway. *Zero* radius of convergence.

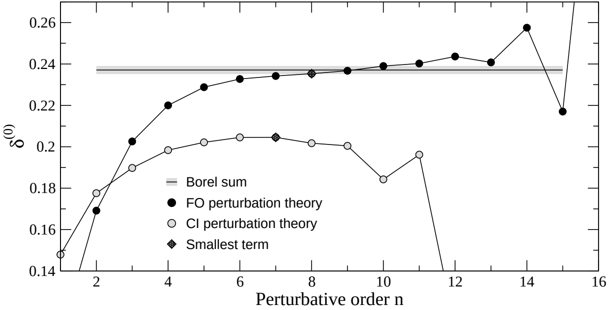
Ansatz

$$B[D](u) \;=\; B[D_1^{\rm UV}](u) + B[D_2^{\rm IR}](u) + B[D_3^{\rm IR}](u) + d_0^{\rm PO} + d_1^{\rm PO}u$$ [MB, Jamin (2008)]

- Ansatz for the Adler function that reproduces known $c_{4,1}$ and $c_{5,1} = 283$.
- Fit constants c_p for $u = -1, 2, 3$ to $c_{3,1}$, $c_{4,1}$ and $c_{5,1}$, and adjust $d_{0,1}^{\rm PO}$ to reproduce $c_{1,1}$ and $c_{2,1}$.
- Find $d_1^{\rm UV} = -1.56 \cdot 10^{-2}$, $d_2^{\rm IR} = 3.16$, $d_3^{\rm IR} = -13.5$, $d_0^{\rm PO} = 0.781$, $d_1^{\rm PO} = 7.66 \cdot 10^{-3}$.
- Pole ansatz works well already at $n = 2$ ($d_1^{\rm PO}$ small).



Convergence of R_τ



- FO converges to Borel sum
- CI more quickly than FO at low orders, but never reaches the Borel sum.
- At $n = 4, 5$ FO is close to the true result, CI too small $\Rightarrow$ α_s from CI too large.
  (Similar result in the large-β_0 approximation [Ball, MB, Braun, 1995].)

Cancellations at large orders and CIPT

$$\delta^{(0)}_{\rm FO} = \sum_{n=1}^{\infty} \left[c_{n,1} + g_n\right] a(M_\tau^2)^n \qquad g_n = \sum_{k=2}^{n} k\, c_{n,k} J_{k-1}$$

- g_n from integration on the circle, $c_{n,1}$ from Adler function
- For the leading IR contribution ($u = 2$) there are *large cancellations:*

$$\frac{c_{n,1} + g_n}{c_{n,1}} \propto 1/n^2$$

- $c_{n,k}$ depends on $c_{m,1}$, β_m up to $m = n - k + 1$, e.g. $c_{4,2} = -\frac{1}{4}\left(\beta_3\, c_{1,1} + 2\beta_2\, c_{2,1} + 3\beta_1\, c_{3,1}\right)$

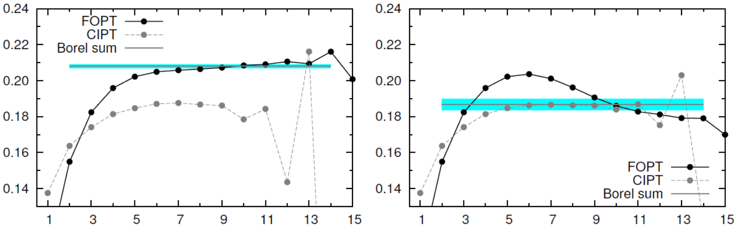
α_s^n	$c_{1,1}$	$c_{2,1}$	$c_{3,1}$	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	g_n	$c_{n,1} + g_n$
2	3.56	1.64							3.56	5.20
4	−20.6	30.5	68.1	49.1					78.0	127.1
6	−2924	−2858	−2280	2214	5041	3275			−807	2468
8	14652	−29552	−145846	−502719	−393887	260511	467787	388442	−329054	59388

CI at order n sums the first n columns to all orders. Destroys cancellations, running coupling effects are only dominant at $n < 5$, then factorial behaviour is more important.

Crucial role of the leading IR singularity

[MB, Boito, Jamin (2012)]

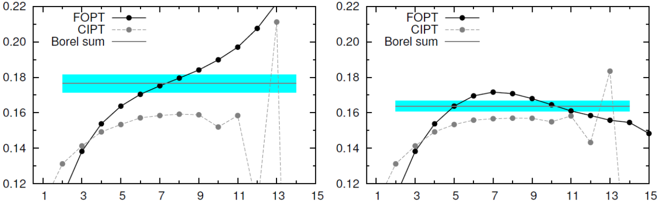
- FOPT preferred, if the $u = 2$ singularity is dominant and cancellations are important (reference model).
- CIPT preferred, if $u = 2$ singularity artificially set to zero or suppressed. No cancellations, factorial behaviour suppressed relative to running coupling-effect (alternative model)
- Models with (unnaturally) large cancellations between $u = 2$ and $u = 3$ and/or polynomial coefficients. Anything can happen.



(a) $w_\tau = (1-x)^2(1+2x)$, reference model (b) $w_\tau = (1-x)^2(1+2x)$, alternative model

Crucial role of the leading IR singularity

- Also for different moments/weight functions. Moments that have an x term and do not suppress the $u = 2$ singularity, have generally less stable perturbative behaviour.



(c) $w_{13} = (1-x)^3(1+2x)$, reference model

(d) $w_{13} = (1-x)^3(1+2x)$, alternative model

Important input could come from a reliable determination of the residue parameter for $u = 2$, related to the ambiguity of the gluon condensate. Would provide one extra input to

$$B[D](u) \;=\; B[D_1^{\rm UV}](u) + B[D_2^{\rm IR}](u) + B[D_3^{\rm IR}](u) + d_0^{\rm PO} + d_1^{\rm PO} u$$

II. OPE to all orders in the $O(N)$ σ model

MB, V.M. Braun and N. Kivel, Phys.Lett. B443 (1998) 308-316 [hep-ph/9809287]

OPE and momentum analyticity

$$D(Q^2) = Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \sum_{k=0} \underbrace{C_k(\alpha_s(\mu), \ln \frac{Q^2}{\mu^2})}_{\text{perturbative series}} \times \frac{1}{(Q^2)^k} \times \underbrace{\langle \mathcal{O}_k \rangle}_{\substack{\text{condensates } (k>0) \\ \text{“power corrections”}}}$$

$$= \sum_{k=0} \left[\frac{\langle \tilde{\mathcal{O}}_k \rangle}{\Lambda^{2k}} \right] \times \left[e^{-\frac{1}{(-\beta_0)\alpha_s(Q)}} \right]^k (-\beta_0 \alpha_s(Q))^{k\beta_1/\beta_0^2 - \gamma_{0,k}/\beta_0} \times \sum_{n=0} c_k^{(n)} \alpha_s(Q)^n$$

Trade $\alpha_s(Q)$ for Q/Λ

$$(-\beta_0) \ln \frac{Q^2}{\Lambda^2} = \frac{1}{\alpha_s(Q)} + \frac{\beta_1}{\beta_0} \ln(-\beta_0 \alpha_s(Q)) + \int_0^{\alpha_s(Q)} dx \left(\frac{1}{x^2} - \frac{\beta_1}{\beta_0} \frac{1}{x} - \frac{\beta_0}{\beta(x)} \right)$$

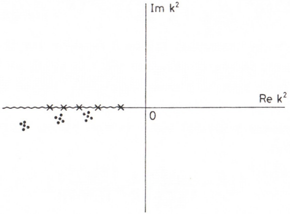
$$\beta(x) = \frac{\beta_0 x^2}{1 - \frac{\beta_1}{\beta_0} x} \qquad \Longrightarrow \qquad \frac{\Lambda^2}{Q^2} = e^{-\frac{1}{(-\beta_0)\alpha_s(Q)}} (-\beta_0 \alpha_s(Q))^{\beta_1/\beta_0^2}$$

Coupling constant analyticity $\leftrightarrow$ momentum analyticity (known non-perturbatively)

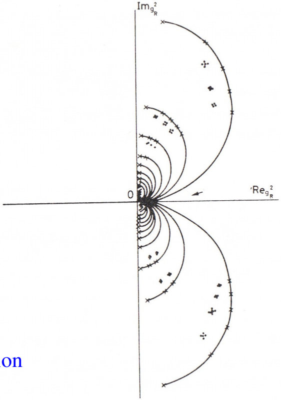
Coupling constant analyticity

't Hooft cuts ['t Hooft (1977), Khuri (1981)]

$$\frac{1}{\alpha_s(Q)} = (-\beta_0)\left[\ln\frac{|Q^2|}{\Lambda^2} + (2n+1)\pi i\right] + \beta_1 - \text{corre}$$



$\Rightarrow$



- How is the analytic structure of the exact correlation function recovered?
  How does the Landau pole of the running coupling disappear?
- Are the Borel integrals convergent at infinity?
- Is the OPE convergent or divergent?

The $O(N)$ non-linear σ-model

Two dimensions, $d = 2 - \epsilon$

$$S = \frac{1}{2}\int d^d x\, \partial_\mu \sigma^a \partial_\mu \sigma^a, \qquad \sigma^a \sigma^a = N/g, \qquad a = 1, \ldots N$$

Perturbation theory: solve for σ_N

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu \sigma^i \partial_\mu \sigma^i + \sum_{a=0}^{\infty}\left(\frac{g}{N}\sigma^i\sigma^i\right)^a \sigma^k \partial_\mu \sigma^k \sigma^l \partial_\mu \sigma^l\right\}, \qquad i, k, l = 1, \ldots N-1$$

- $[\sigma] = 0$, infinitely many renormalizable interactions
- Spontaneous symmetry breaking to $O(N-1)$
- Only massless particles in the spectrum ($N-1$ Goldstone bosons)
- Asymptotically free

$$\beta(g) = -\frac{g^2}{4\pi} + \ldots$$

Non-perturbatively: symmetry restoration, mass gap, no massless excitations

The $1/N$ expansion [Bardeen, Lee, Schrock (1976)]

Introduce Lagrange multiplier field $\alpha(x)$

$$Z[J] = \int \mathcal{D}[\sigma]\mathcal{D}[\alpha]\ \exp\left(-S[\sigma,\alpha] + \int d^d x\, J^a(x)\sigma^a(x)\right)$$

$$S[\sigma,\alpha] = \frac{1}{2}\int d^d x\,\left\{\partial_\mu\sigma^a\partial_\mu\sigma^a + \frac{\alpha}{\sqrt{N}}\left(\sigma^a\sigma^a - \frac{N}{g}\right)\right\}$$

Integrate over σ-fields, $1/N$-expansion corresponds to saddle-point expansion of α functional integral. Saddle point

$$\bar{\alpha} = \sqrt{N}\left(g_0\mu^\epsilon\Gamma\left(\frac{\epsilon}{2}\right)(4\pi)^{\frac{\epsilon-2}{2}}\right)^{2/\epsilon} \underbrace{\overset{d\to 2}{=}}_{g_0^{-1}=Z\,g^{-1}(\mu)} \sqrt{N}\mu^2\, e^{-4\pi/g(\mu)} \equiv \sqrt{N}m^2$$

- m^2 is exponentially small in $g(\mu)$, analogous to Λ^2
- Equation of motion

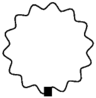
$$\alpha = -\frac{g}{\sqrt{N}}\partial_\mu\sigma^a\partial_\mu\sigma^a \qquad \Rightarrow \qquad \sigma \text{ field gets mass } m$$

[In the following rescale g by factor 4π, so $m^2 = \mu^2 e^{-1/g(\mu)}$]

Condensate ambiguities at order $\mathcal{O}(1/N)$

Define condensates with a momentum cut-off $\mu_f \gg m$ [Novikov, Shifman, Vainshtein, Zakharov (1984) – cut-off scheme; David (1982, 1984, 1986) – dim reg]

$$\langle \alpha^2 \rangle(\mu_f, m) \;=$$



$$= \int\limits_{p^2<\mu^2} \frac{d^2p}{(2\pi)^2}\, D_\alpha(p)$$

$$= \; m^4 \left[F(\ln A) + F(-\ln A) - 2\gamma_E\right]$$

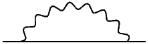
$$A = \left(\sqrt{1+\frac{\mu_f^2}{4m^2}} + \sqrt{\frac{\mu_f^2}{4m^2}}\right)^4, \qquad \ln A \overset{\mu_f \gg m}{\rightarrow} \frac{2}{g(\mu_f)} \gg 1$$

$F(x) = \mathrm{Ei}(-x) - \ln x$ has an essential singularity at $x = 0$ but no discontinuity. Asymptotic expansion for large x has a Stokes discsontinuity at negative arguments

$$F(-x) = e^x \left[\sum_{n=0}^{\infty} \frac{n!}{x^{n+1}} - e^{-x}\left(\ln x \mp i\pi\right)\right]$$

$$\langle \alpha^2 \rangle = \mu^4 \sum_{n=0}^{\infty} \left(\frac{g}{2}\right)^{n+1} n! \, + \, 2g\,\mu^2 m^2 \, + \, m^4 \left[-2\ln\frac{2}{g} \pm i\pi \, - 2\gamma_E - 4g + \frac{g^2}{2}\right] + \mathcal{O}\left(\frac{m^2}{\mu^2}\right)$$

OPE of the σ self-energy at order $\mathcal{O}(1/N)$

$\Sigma(p) =$  $= \frac{1}{\pi N} \int d^2k \, \frac{k^2 + 2m^2}{(p+k)^2 + m^2} \, g_{\text{eff}}(k) \; + \; \text{subtractions}$

$$g_{\text{eff}}(k) = \frac{\sqrt{k^2(k^2+4m^2)}}{k^2+2m^2} \left[\ln \frac{\sqrt{k^2+4m^2} + \sqrt{k^2}}{\sqrt{k^2+4m^2} - \sqrt{k^2}} \right]^{-1} = \frac{1}{\ln(k^2/m^2)} + \ldots$$

Similar to large-n_f expansion, but now the effective coupling is defined non-perturbatively (and has no Landau pole).

Defines an analytic function with a cut for $-p^2 > 9m^2$.

Angular integral trivial. Want to write the remaining integral as resurgent OPE expansion in $g(p)$ and $\exp(-1/g(p))$.

1. $k \gg m,\; p+k \gg m$ – expand both propagators at large momentum, $C_n^{(1)} \times \langle O_n \rangle^{(0)}$
2. $k \sim m,\; p+k \gg m$ – expand σ propagator, $C_n^{(0)} \times \langle O_n \rangle^{(1)}$ with $O_n \sim \langle \alpha \partial^{2n} \alpha \rangle$.
3. $k \gg m,\; p+k \sim m$ – expand effective coupling, $C_n^{(1)} \times \langle O_n \rangle^{(0)}$ with $O_n \sim \langle \sigma \partial^{2n} \sigma \rangle$

OPE of the σ self-energy at order $\mathcal{O}(1/N)$

Simpler to use Mellin-Barnes representation

$$\left[\ln \frac{\sqrt{k^2+4m^2}+\sqrt{k^2}}{\sqrt{k^2+4m^2}-\sqrt{k^2}}\right]^{-1} = \int_0^\infty dt \left[\frac{A-1}{A+1}\right]^t, \qquad A\left[\frac{A-1}{A+1}\right]^t = \int \frac{ds}{2\pi i} K(s,t)\left(\frac{m^2}{k^2}\right)^{-s}$$

$$A=(1+4m^2/k^2)^{1/2}, \qquad K(s,t) = \frac{\Gamma(t+s)\,\Gamma(-1-2s)}{\Gamma(t-1-s)}\left[1+\frac{2\,(t+s)}{t-s-1}+\frac{(t+s)(t+s+1)}{(t-s-1)(t-s)}\right].$$

Perform k-integral (standard loop integral) and s-integral (residues of poles).
Borel representation

$$\Sigma(p) = \frac{p^2}{N}\int_0^\infty dt \sum_{n=0}^\infty \left(-\frac{m^2}{p^2}\right)^n \left\{e^{-t/g(p)}\left[F_{\rm p}^{(n)}[t]\frac{1}{g(p)}+G_{\rm p}^{(n)}[t]\right]-H_{\rm np}^{(n)}[t]\right\}.$$

$H_{\rm np}^{(n)}[t]$ arises from (2)

(1) and (3) give $F_{\rm p}^{(n)}[t]\,\ln\frac{p^2}{\mu^2}+G_{\rm p}^{(n,1)}[t]$ and $F_{\rm p}^{(n)}[t]\,\ln\frac{\mu^2}{m^2}+G_{\rm p}^{(n,3)}[t]$, respectively, with μ an intermediate factorization scale.

OPE of the σ self-energy at order $\mathcal{O}(1/N)$

$$\Sigma(p) = \frac{p^2}{N}\int_0^\infty dt \sum_{n=0}^{\infty}\left(-\frac{m^2}{p^2}\right)^n \left\{ e^{-t/g(p)}\left[F_{\rm p}^{(n)}[t]\,\frac{1}{g(p)} + G_{\rm p}^{(n)}[t]\right] - H_{\rm np}^{(n)}[t]\right\}.$$

$$F_{\rm p}^{(0)}[t] = 1, \qquad G_{\rm p}^{(0)}[t] = \frac{1}{t} + \frac{1}{t-1} - \psi(1+t) - \psi(2-t) - 2\gamma_E, \qquad H_{\rm np}^{(0)}[t] = \frac{1}{t} + B_1(t).$$

$$F_{\rm p}^{(1)}[t] = t^2 - 1, \qquad G_{\rm p}^{(1)}[t] = -\frac{1}{t} + 2t^2 - 4t + 1 + (1-t^2)\Big[\psi(1+t) + \psi(2-t) + 2\gamma_E\Big]$$

$$H_{\rm np}^{(1)}[t] = -\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t+1} - B_0(t)\right),$$

$$B_0(t) = 3J(t,1), \qquad B_1(t) = 6J(t,2) - 6J(t,3) + 6J(1+t,3) - 7J(1+t,2), \qquad J(t,m) = \frac{1}{t+1}\,{}_3F_2\left[\begin{matrix} 1, m, t+1 \\ 1+\frac{t}{2}, \frac{3}{2}+\frac{t}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right]$$

$$F_{\rm p}^{(n)}[t] = \sum_{k=0}^{n} K_{\rm p}(n-k,t)\,\frac{(n-k-1+t)_k^2}{k!k!}, \qquad H_{\rm np}^{(n)}[t] = \sum_{k=0}^{n-2} K_{\rm np}(n-k-2,t)\,\frac{(n-k-1)_k^2}{k!k!}$$

$$G_{\rm p}^{(n)}[t] = \sum_{k=0}^{n} K_{\rm p}(n-k,t)\,\frac{(n-k-1+t)_k^2}{k!k!}\Big[\psi(t) - \psi(1-t) + 2\psi(k+1) - 2\psi(n+t-1)\Big]$$

$$K_{\rm p}(n,t) = \ldots, \qquad K_{\rm np}(n,t) = \frac{\Gamma(-t, 1+t, 2n+3)}{\Gamma(t+n+1, 3+n-t)}\left[1 - 2\,\frac{n+2-t}{t+n+1} + \frac{(t-2-n)(t-1-n)}{(t+n+1)(t+n+2)}\right]$$

OPE of the σ self-energy at order $\mathcal{O}(1/N)$

Each term in the sum has renormalon singularities.

IR renormalon at $t = t_0$ in coefficient function $G_{\rm p}^{(n)}[t]$ cancels with UV renormalon in $H_{\rm np}^{(n+t_0)}[t]$.
With a prescription for avoiding the poles pull out the sum:

$$\Sigma(p) = \frac{p^2}{N}\sum_{n=0}^{\infty}\left(-\frac{m^2}{p^2}\right)^n \int_0^\infty dt\,\left\{e^{-t/g(p)}\left[F_{\rm p}^{(n)}[t]\,\frac{1}{g(p)} + G_{\rm p}^{(n)}[t]\right] - H_{\rm np}^{(n)}[t]\right\}.$$

Exact result of a resurgent expansion for a Green function to all orders in logs and powers (powers and exponentials).

$n = 0$ term:

$$\Sigma(p) = \frac{p^2}{N}\left[\ln g(p) + 1.887537 - 2\,g(p) + \sum_{n=1}^{\infty} g^{n+1}(p)\,n!\,\{[1+(-1)^n]\zeta(n+1) - 2\}\right]$$

Truncation of the OPE at order n destroys the cancellation. For $t_0 = 1$, it leaves a pole of the form

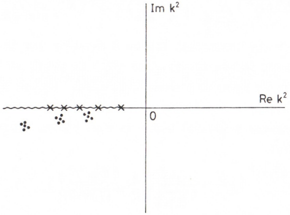
$$const \times \left(-\frac{9m^2}{p^2}\right)^{n+1}\frac{1}{1-t}$$

Convergence of the OPE

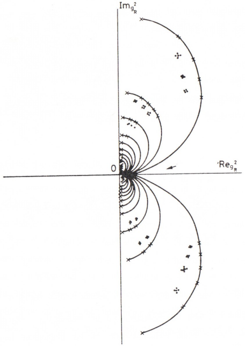
For fixed t

$$\frac{9^{5/4}\,m^4}{\pi\,p^2}\sum_{n\gg t}\frac{1}{n^2}\left(-\frac{9m^2}{p^2}\right)^n\left\{\left(\frac{9m^2}{p^2}\right)^t\left[-\ln(9m^2/p^2)+\psi(t)-\psi(1-t)\right]-\Gamma(t)\Gamma(1-t)\right\}.$$

Converges for $|p^2| > 9m^2$ (both, OPE and Borel integral). This is precisely the location of the physical threshold of the three σ-particle cut.
At order $1/N^k$ expect $|p^2| > (2k+1)^2 m^2$.
At finite N no convergence.



$\Rightarrow$



Convergence of the OPE

Conjectures:

- 't Hooft cuts are related to the divergence/convergence of the OPE, not the Borel integral of the perturbative series at $t = \infty$, since correct analytic properties are recovered only when all terms of the OPE are summed. [Ball, MB, Braun (1995)]
- The OPE is itself a divergent series, due to multi-particle thresholds at arbitrarily large energies.

Large-N_c QCD inspired model [Shifman (1994, 2000)]

$$\Pi(q^2) = -\frac{N_c\sigma}{12\pi^2}\sum_{n=0}^{\infty}\frac{1}{q^2 - M_n^2}, \qquad M_n^2 = M_0^2 + \sigma n$$

$$\Pi(q^2) - \Pi(0) = -\frac{N_c\sigma}{12\pi^2}\,\psi\left(\frac{Q^2 + M_0^2}{\sigma}\right)$$

OPE diverges as $(-1)^k k!\left(\frac{\sigma}{Q^2}\right)^k$.

However, if these is only a finite number of resonances with largest mass M_N, the series behaves as $\left(-M_N^2/Q^2\right)^k$ and is convergent for $|Q^2| > M_N^2$.

Summary

(I) OPE in QCD has every feature of a resurgent expansion.
Existence of renormalons is not rigorously established, but hard to see how it could be otherwise.
Assuming resurgence (“cancellation of ambiguities”) determines singularity structure of the Borel plane up to non-perutrbative constants.
Practical relevance for heavy quark masses and hadronic τ decay

(II) OPE of the self-energy in the large-N σ-model provides a non-trivial example of a resurgent series where all components are known exactly to all orders.

(III) Analyticity of the exact correlation functions is recovered only after summing the OPE series. At every finite order, there is an unphysical Landau pole.

Convergence/divergence of the OPE seems to be related to physical thresholds. Multi-particle production causes divergence.

Known examples are sign-alternating in the Euclidean region. Is the OPE summable? Are there effects not captured by the summed (and analytically continued OPE)? (“Duality violations”)