Theoretical background for calculating hadronic light-by-light scattering contribution with **twisted boundary condition**

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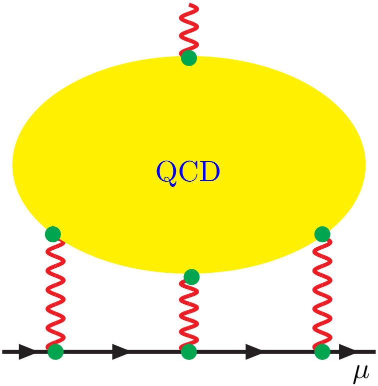
Background and motivation

- In many cases, we are interested in $A(Q=0)$, but its direct computation is difficult or impossible.
- The momentum components are discretized in unit of $\frac{2\pi}{L}$ in a periodic box.
- In most cases, what we can do at best is to measure $A(Q=0)$ with nonvanishing and discretized Q for various L and to make extrapolation to $Q=0$.
- The twisted boundary condition modifies $Q_\mu = \frac{1}{L_\mu}(2\pi n_\mu + \theta_\mu)$, and measure $A(Q)$ with $|Q| \ll 2\pi/L$.
- The application of the twisted boundary condition to the hadronic light-by-light scattering contribution (HLBL) to $F_2(Q)$ for Q closer to 0 may help to get more reliable estimate of $a_\mu(\mathrm{HLBL})$ (though essential "sea dynamics" cannot be explored).
- However, we will immediately confront with a difficulty.

Background and motivation



- Let's suppose that the external momentum has fractional number(s) as its component(s).
- The momentum conservation implies that at least one of the three virtual photons must carry fractional momentum.

Background and motivation

- Every method to compute $a_\mu(\mathrm{HLBL})$ developed thus far requires the photon propagators in the measurement.
- Recall that
  - The HVP contribution $a_\mu(\mathrm{HVP})$ is expressed as the integral in terms of the HVP function $\Pi(-Q^2)$.
  - The lattice study can be focused on the calculation of $\Pi(-Q^2)$ itself.
  - The twisted boundary condition can be applied to *guess* the shape of (the connected diagram contribution to) $\Pi(-Q^2)$ (C. Aubin, T. Blum, M. Golterman and S. Peris, Phys. Rev. D **88**, no. 7, 074505 (2013)).
- Therefore, we must replace, say, one of the three photons, by some massless vector boson that can carry fractional momentum.

Background and motivation

- We are tempted to use the following propagator for that vector boson

$$D_{\mu\nu}(x) = \frac{1}{V_4}\sum_n e^{\,\mathrm{i}p_n\cdot x}\widetilde{D}_{\mu\nu}(\widehat{p}_n)\,,$$

$$\widetilde{D}_{\mu\nu}(p) = \frac{\delta_{\mu\nu}}{p^2}\,,$$

$$\widehat{p} \equiv \frac{2}{a}\,\sin\left(\frac{ap_\mu}{2}\right)\,,\qquad (p_n)_\mu \equiv \frac{1}{L_\mu}\left(2\pi n_\mu + \theta_\mu\right)\,.$$

- I am tempted to ask the following question :
  - Does the result change for

$$\widetilde{D}_{\mu\nu}(p) = \frac{1}{p^2}\left(\delta_{\mu\nu} - (1-\lambda)\,\frac{p_\mu p_\nu}{p^2}\right)\,?$$

  - If this is actually the case, the gauge invariance at $\theta_\mu \in 2\pi\mathbb{Z}$ appears to be just accidental.

Background and motivation

What I'd like to discuss here is as follows :

- It is possible to find a theoretical foundation which realizes that vector field (w_μ) as a gauge field in some gauge theory (2013).
- Such a gauge theory explains why ψ_1 and ψ_2 in

$$\overline{\psi_1} w_\mu \gamma^\mu \psi_2 \, ,$$

  must have the same mass and the same $\mathrm{U}(1)_{\mathrm{em}}$ charge, because the gauge theory is an $\mathrm{SU}(2)$ gauge theory
    - with w as the off-diagonal gauge boson
    - and with (ψ_1, ψ_2) as a multiplet of its fundamental representation.
- Last week, I noticed that the theory is just a special case of known twisted Yang-Mills theory formulated by G. 't Hooft, Nucl. Phys. B 153 (1979).

Twisted vector boson w_μ

- We'd like to identify $D_{\mu\nu}(x)$ as the free propagator of some field $w_\mu(x)$:

$$\langle w_\mu(x) w_\nu(y)^* \rangle = \frac{1}{V_4} \sum_n e^{\,\mathrm{i} p_n \cdot (x-y)}\, \widetilde{D}_{\mu\nu}(\widehat{p}_n)\,,$$

$$(p_n)_\nu \equiv \frac{1}{L_\nu}\,(2\pi\, n_\nu + \theta_\nu)\ .$$

  The right-hand side is complex unless θ_ν is a multiple of 2π.

- The following boundary condition for w_μ is compatible with the above :

$$w_\mu(x + L_\nu \widehat{\nu}) = e^{\,\mathrm{i}\theta_\nu}\, w_\mu(x) \quad (\text{no sum over } \nu)\,.$$

  In particular $w_\mu(x)$ is a complex field.

- We are tempted to examine whether it is possible to embed $w_\mu(x)$ into the off-diagonal part of the $\mathrm{SU}(2)$ gauge field.

Twisted vector boson w_μ

- We embed $w_\mu(x)$ into the off-diagonal part of the $\mathrm{SU}(2)$ gauge field $W_\mu(x)$

$$W_\mu(x) = \begin{pmatrix} \frac{w_\mu^3(x)}{2} & \frac{w_\mu(x)}{\sqrt{2}} \\ \frac{(w_\mu(x))^*}{\sqrt{2}} & -\frac{w_\mu^3(x)}{2} \end{pmatrix} .$$

- The boundary condition

$$W_\mu(x + L_\nu \widehat{\nu}) = e^{\,\mathrm{i}\frac{\tau_3}{2}\theta_\nu}\, W_\mu(x)\, e^{-\mathrm{i}\frac{\tau_3}{2}\theta_\nu} ,$$

corresponds to

$$w_\mu(x + L_\nu \widehat{\nu}) = e^{\,\mathrm{i}\theta_\nu}\, w_\mu(x) ,$$
$$w_\mu^3(x + L_\nu \widehat{\nu}) = w_\mu^3(x) .$$

Twisted vector boson w_μ

- The boundary condition for $W_\mu(x)$ is preserved under the transformation

$$W_\mu(x) \mapsto W'_\mu(x) = G(x) W_\mu(x) G(x)^{-1} - \mathrm{i}\, \partial_\mu G(x) G(x)^{-1} \,,$$

if $\mathrm{SU}(2)$-valued field $G(x)$ satisfies the boundary condition

$$G(x + L_\nu \widehat{\nu}) = e^{\mathrm{i} \frac{\tau_3}{2} \theta_\nu}\, G(x)\, e^{-\mathrm{i} \frac{\tau_3}{2} \theta_\nu} \,.$$

- The set of all such $G(x)$ forms a group, called gauge group.
- The action

$$S[W] = \int d^4x\, \frac{1}{2} \operatorname{tr} \left(\partial_\mu W_\nu - \partial_\nu W_\mu - \mathrm{i}\, e_{\mathrm{SU}(2)} \left[W_\mu,\, W_\nu \right] \right)^2 \,.$$

is gauge-invariant.
- The gauge-fixed action

$$S[W] + \int d^4x\, \frac{1}{2\lambda} \left(\sum_\mu \partial_\mu W_\mu(x) \right)^2 \,,$$

reproduces $D_{\mu\nu}(x)$.

Twisted vector boson w_μ

- The BRS-invariant quantity is independent of the choice of gauge, λ.
- $W_\mu(x)$ couples to the matter field in the fundamental representation as

$$
\begin{aligned}
& e_{\mathrm{SU}(2)} \left(\; \overline{\psi}_1(x), \quad \overline{\psi}_2(x) \; \right) W_\mu(x) \gamma^\mu \left(\begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array} \right) \\
&= e_{\mathrm{SU}(2)} \left\{ \frac{w_\mu}{\sqrt{2}} \overline{\psi}_1 \gamma^\mu \psi_2 + \frac{(w_\mu)^*}{\sqrt{2}} \overline{\psi}_2 \gamma^\mu \psi_1 + w_\mu^3 \left(\cdots \right) \right\} ,
\end{aligned}
$$

where ψ_1 and ψ_2 have the same mass and EM charge, etc. For instance, we prepare $(u_1(x),\, u_2(x))^T$ for the up-quark field.

Twisted vector boson w_μ

- The boundary condition for ψ_1 and ψ_2 can be read off from the requirement that the Lagrangian density should be single-valued :

$$\left(\ \overline{\psi}_1(x'), \ \ \overline{\psi}_2(x') \ \right) W_\mu(x') \gamma^\mu \begin{pmatrix} \psi_1(x') \\ \psi_2(x') \end{pmatrix} \Bigg|_{x'=x+L_\nu \widehat{\nu}}$$

$$= \left(\ \overline{\psi}_1(x), \ \ \overline{\psi}_2(x) \ \right) W_\mu(x) \gamma^\mu \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} .$$

Thus,

$$W_\mu(x + L_\nu \widehat{\nu}) = e^{\,\mathrm{i}\frac{\tau_3}{2}\theta_\nu} \, W_\mu(x) \, e^{-\mathrm{i}\frac{\tau_3}{2}\theta_\nu} \, ,$$

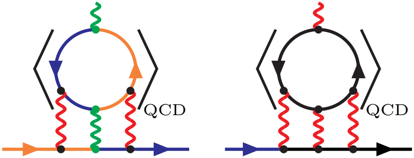
$$\begin{pmatrix} \psi_1(x + L_\nu \widehat{\nu}) \\ \psi_2(x + L_\nu \widehat{\nu}) \end{pmatrix} = e^{\,\mathrm{i}\frac{\tau_3}{2}\theta_\nu} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} .$$

or

$$w_\mu(x + L_\nu \widehat{\nu}) = e^{\,\mathrm{i}\theta_\nu} \, w_\mu(x) \, ,$$

$$\psi_1(x + L_\nu \widehat{\nu}) = e^{\,\mathrm{i}\frac{\theta_\nu}{2}} \, \psi_1(x) \, , \quad \psi_2(x + L_\nu \widehat{\nu}) = e^{-\mathrm{i}\frac{\theta_\nu}{2}} \, \psi_2(x) \, ,$$

Twisted vector boson w_μ



- In order for the result for the up-quark loop at $\theta_\mu = 0$ to match with the original one, $e_{\mathrm{SU}(2)}$ should be adjusted so that

$$\left(\frac{e^{(u)}_{\mathrm{SU}(2)}}{\sqrt{2}}\right)^3 = (Q_u\, e_{\mathrm{EM}})^2\,(Q_\mu e_{\mathrm{EM}})\ .$$

We must prepare another $\mathrm{SU}(2)_{(d)}$ gauge theory for the down-type quarks other than $\mathrm{SU}(2)_{(u)}$.

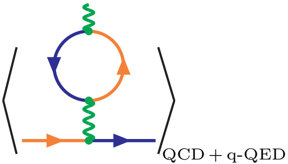
Twisted vector boson w_μ

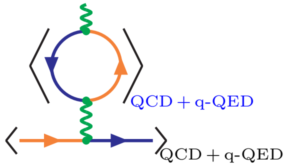
Species	$\mathrm{U}(1)_{\mathrm{em}}$	$\mathrm{SU}(2)_{(u)}$	$\mathrm{SU}(2)_{(d)}$
u	$\frac{2}{3}$	**2**	$-$
c	$\frac{2}{3}$	**2**	$-$
d	$-\frac{1}{3}$	$-$	**2**
s	$-\frac{1}{3}$	$-$	**2**
μ	-1	**2**	**2**

Twisted nonperturbative QED method

The nonperturbative QED method for the connected-type diagram (M.H., T.Blum, T.Izubuchi and N.Yamada, PoS LAT **2005**, 353 (2006))

$$\frac{1}{3}\left(\mathcal{M}_C - \mathcal{S}_C\right) \,,$$

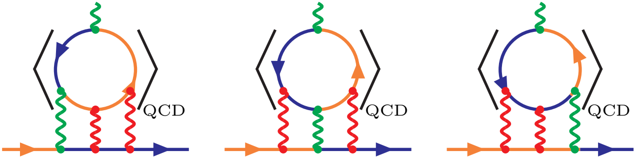
$\mathcal{M}_C =$ 

$\mathcal{S}_C =$ 

- Green wavy line denotes propagation of the twisted boson w_μ.
- Each line in the quark loop part denotes the inverse of the quark Dirac operator, $D_m[U^{\mathrm{QCD}}, A^{\mathrm{QED}}]^{-1}$.
- Each line in the muon part denotes the inverse of muon Dirac operator, $D_\mu[A^{\mathrm{QED}}]^{-1}$.

Twisted nonperturbative QED method

Here, we *define* the contribution of the connected-type diagram to $F_2[\mathrm{HLBL}](Q)$ to be $\frac{1}{3}$ of the sum of



Connection to the twisted Yang-Mills theory by 't Hooft

The setup of the twisted Yang-Mills theory is as follows (borrowed from M. Garcia Perez, A. Gonzalez-Arroyo and M. Okawa, [arXiv:1406.5655 [hep-th]]) :

- The twisted boundary condition is the periodicity up to the $\mathrm{SU}(N)$ gauge transformation Ω_ν

$$W_\mu\,(x + L_\nu \widehat{\nu}) = \Omega_\nu(x)\, W_\mu\,(x)\, \Omega_\nu(x)^\dagger - \mathrm{i}\, \partial_\mu \Omega_\nu(x)\, \Omega_\nu(x)^\dagger\,,$$

with a cocycle condition

$$\Omega_\mu(x + L_\nu \widehat{\nu})\, \Omega_\nu(x) = Z_{\mu\nu}\, \Omega_\nu(x + L_\mu \widehat{\mu})\, \Omega_\mu(x)\,.$$

where $Z_{\mu\nu}$ are the elements of the center C_N of $\mathrm{SU}(N)$

$$Z_{\mu\nu} = \exp\left(2\pi\,\mathrm{i}\,\frac{n_{\mu\nu}}{N}\right) \mathbb{I}_N\,,$$

with integers $n_{\mu\nu}$ modulo N.

Connection to the twisted Yang-Mills theory by 't Hooft

- The gauge transformation $G(x)$ is to transform the pair (?) (W_μ, Ω_μ) as

$$W_\mu(x) \mapsto W'_\mu(x) = G(x)W_\mu(x)G(x)^\dagger - \mathrm{i}\,\partial_\mu G(x)\,G(x)^\dagger\,,$$
$$\Omega_\mu(x) \mapsto \Omega'_\mu(x) = G(x+L_\mu\widehat{\mu})\,\Omega_\mu(x)\,G(x)^\dagger\,.$$

(If $G(x)$ obeys the boundary condition

$$G(x+L_\mu\widehat{\mu}) = \Omega_\mu(x)G(x)\Omega_\mu(x)^\dagger\,,$$

$\Omega_\mu(x)$ is left invariant; $\Omega'_\mu(x) = \Omega_\mu(x)$.)

- The theory developed thus far corresponds to $N=2$ and

$$\Omega_\nu(x) = \exp\left(\mathrm{i}\,\frac{\tau_3}{2}\,\theta_\nu\right)\,.$$

The cocycle condition is satisfied with $Z_{\mu\nu} = \mathbb{I}_2$.

Summary

- The lattice version of the twisted $\mathrm{SU}(2)$ gauge theory with the matter fields can be constructed using the compact variables, whose lowest order of $e_{\mathrm{SU}(2),\,(u)}$, $e_{\mathrm{SU}(2),\,(d)}$ gives the necessary and sufficient hadronic light-by-light diagrams.