Numerical Stochastic Perturbation Theory (NSPT)

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High precision QCD at low energy

Benasque, 11-08-2015

Parma group introduced NSPT in the 90s. Since then also other people got interested in. My viewpoint (I confess) will be still Parma-centered...

Many collaborators over the years on the subject ...

E. Onofri
G. Marchesini

G. Burgio

L. Scorzato
M. Pepe

V. Miccio
C. Torrero
A. Mantovi
M. Brambilla
M. Hasegawa
D. Hesse

Y. Schroder
M. Laine

E.M. Ilgenfritz
H. Perlt
A. Schiller

G. Bali

S. Sint
M. Dalla Brida

Perturbation Theory (PT) is nothing less than ubiquitous in Field Theory. In principle the lattice is a regulator among the others ... in practice it is a dreadful one so that when it comes to compute something in Lattice Perturbation Theory (LPT) you will probably start to get nervous ...

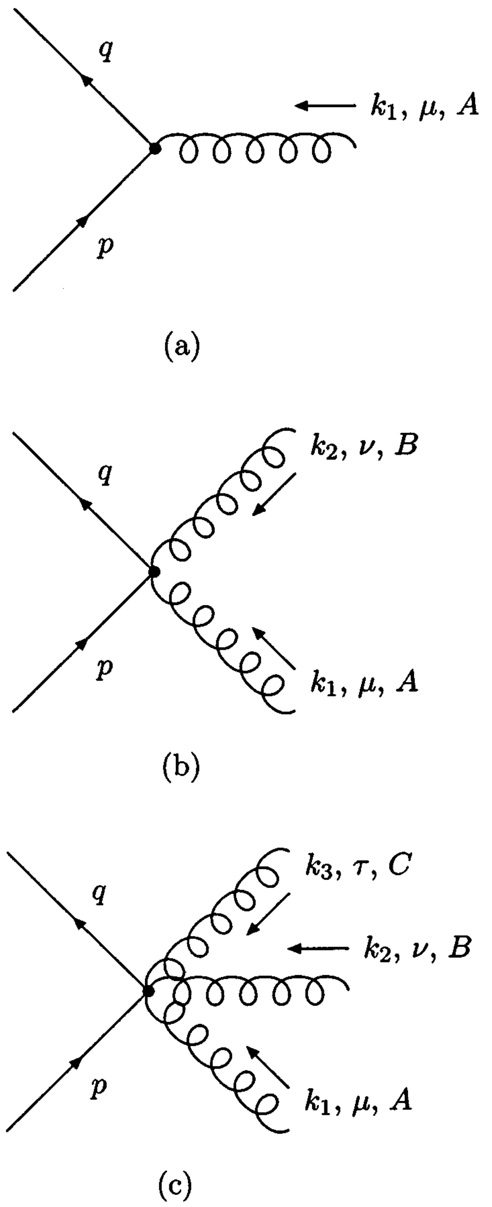

FIG. 1. Momentum assignments for the quark-antiquark-gluon vertices.

A lot of vertices (not given once and for all)
Sums and/or integrals ... a lot of trigonometrics ...
A variety of actions (both for glue and for quarks)

and as an extra bonus ... often bad convergence properties

$$V^{A}_{1\mu}(p,q) = -gT^A\left[i\gamma_\mu \cos\left(\frac{p_\mu+q_\mu}{2}\right) + r\sin\left(\frac{p_\mu+q_\mu}{2}\right)\right]$$

$$V^{A}_{c1\mu}(p,q) = -gT^A c_{\mathrm{SW}}\frac{r}{2}\sum_\nu \sigma_{\mu\nu} \times\cos\left(\frac{p_\mu-q_\mu}{2}\right)\sin(p_\nu-q_\nu)$$

$$V^{AB}_{2\mu\nu}(p,q) = \frac{a}{2}g^2\frac{1}{2}\{T^A,T^B\}\delta_{\mu\nu} \times\left[i\gamma_\mu \sin\left(\frac{p_\mu+q_\mu}{2}\right) - r\cos\left(\frac{p_\mu+q_\mu}{2}\right)\right]$$

$$V^{AB}_{c2\mu\nu}(p,q,k_1,k_2) = -\frac{a}{2}g^2 i f_{ABC}T^C c_{\mathrm{SW}}\frac{r}{4}\left\{\sigma_{\mu\nu}\left[4\cos\left(\frac{k_{1\nu}}{2}\right)\cos\left(\frac{k_{2\mu}}{2}\right)\cos\left(\frac{q_\mu-p_\mu}{2}\right) \times\cos\left(\frac{q_\nu-p_\nu}{2}\right) - 2\cos\left(\frac{k_{1\mu}}{2}\right)\cos\left(\frac{k_{2\nu}}{2}\right)\right] + \delta_{\mu\nu}\sum_\rho \sigma_{\mu\rho}\sin\left(\frac{q_\mu-p_\mu}{2}\right)[\sin(k_{2\rho})-\sin(k_{1\rho})]\right\}$$

$$V^{ABC}_{3\mu\nu\tau}(p,q) = \frac{a^2}{6}g^3\frac{1}{6}[T^A\{T^B,T^C\}+T^B\{T^C,T^A\} + T^C\{T^A,T^B\}]\delta_{\mu\nu}\delta_{\mu\tau}\left[i\gamma_\mu\cos\left(\frac{p_\mu+q_\mu}{2}\right) + r\sin\left(\frac{p_\mu+q_\mu}{2}\right)\right],$$

$$V^{ABC}_{c3\mu\nu\tau}(p,q,k_1,k_2,k_3) = -3ig^3\frac{a^2}{6}c_{\mathrm{SW}}r\left\{T^AT^BT^C\delta_{\mu\nu}\delta_{\mu\tau}\sum_\rho i\sigma_{\mu\rho}\left[-\frac{1}{6}\cos\left(\frac{q_\mu-p_\mu}{2}\right)\sin(q_\rho-p_\rho) + \cos\left(\frac{q_\mu-p_\mu}{2}\right)\cos\left(\frac{q_\rho-p_\rho}{2}\right)\cos\left(\frac{k_{3\rho}-k_{1\rho}}{2}\right)\sin\left(\frac{k_{2\rho}}{2}\right)\right] - \frac{1}{2}[T^AT^BT^C + T^CT^BT^A]i\sigma_{\mu\nu}\left[\delta_{\nu\tau}2\cos\left(\frac{q_\mu-p_\mu}{2}\right)\cos\left(\frac{q_\nu-p_\nu}{2}\right)\cos\left(\frac{k_{3\mu}+k_{2\mu}}{2}\right)\sin\left(\frac{k_{1\nu}}{2}\right) + \delta_{\nu\tau}\sin\left(\frac{k_{3\nu}+k_{2\nu}}{2}\right)\cos\left(\frac{k_{1\mu}}{2}+k_{2\mu}\right) + \delta_{\mu\tau}\sin\left(\frac{k_{1\mu}+2k_{2\mu}+k_{3\mu}}{2}\right)\cos\left(\frac{q_\nu-p_\nu}{2}\right)\cos\left(\frac{k_{3\nu}-k_{1\nu}}{2}\right)\right.$$

Despite this ...

ELSEVIER

Nuclear Physics B 457 (1995) 202–216

NUCLEAR PHYSICS B

Renormalons from eight-loop expansion of the gluon condensate in lattice gauge theory ★

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Compelling Evidence of Renormalons in QCD from High Order Perturbative Expansions

Clemens Bauer,[1] Gunnar S. Bali,[1] and Antonio Pineda[2]

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We compute the static self-energy of SU(3) gauge theory in four spacetime dimensions to order α^{20} in the strong coupling constant α. We employ lattice regularization to enable a numerical simulation within the framework of stochastic perturbation theory. We find perfect agreement with the factorial growth of high order coefficients predicted by the conjectured renormalon picture based on the operator product expansion.

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Agenda

- Basics of Stochastic Quantization and Stochastic Perturbation Theory
- From Stochastic Perturbation Theory to NSPT
- Stochastic Gauge Fixing
- Fermionic loops in NSPT
- A couple of (what I would regard as) relevant applications for this Workshop
- Conclusions

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$$\langle O[\phi] \rangle = \frac{\int D\phi \; O[\phi] \; e^{-S[\phi]}}{\int D\phi \; e^{-S[\phi]}}$$

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Parisi-Wu, Sci. Sinica 24 (1981) 35, Damgaard-Huffel, Phys Rept 152 (1987) 227

You now want an extra degree of freedom which you will think of as a stochastic time in which an evolution takes place according to the Langevin equation

$$\phi(x) \mapsto \phi_\eta(x;t) \qquad\qquad \frac{d\phi_\eta(x;t)}{dt} = -\frac{\partial S[\phi]}{\partial \phi_\eta(x;t)} + \eta(x;t)$$

The drift term is given by the equations of motion...

... but this is a stochastic differential equation due to the presence of the gaussian noise

$$\eta(x;t) : \qquad \langle \eta(x,t) \;\, \eta(x',t') \rangle_\eta = 2\,\delta(x-x') \;\, \delta(t-t')$$

Noise expectation values are now naturally defined

$$\langle \ldots \rangle_\eta = \frac{\int D\eta(z,\tau) \;\, \ldots \; e^{-\frac{1}{4}\int dz d\tau \eta^2(z,\tau)}}{\int D\eta(z,\tau)\, e^{-\frac{1}{4}\int dz d\tau \eta^2(z,\tau)}}$$

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The key assertion of Stocastic Quantization can be now simply stated

$$\langle O[\phi_\eta(x_1;t)\ldots\phi_\eta(x_n;t)] \rangle_\eta \rightarrow_{t\rightarrow\infty} \langle O[\phi(x_1)\ldots\phi(x_n)] \rangle$$

A conceptually simple proof comes from the Fokker Planck equation formalism

$$\langle O[\phi_\eta(t)]\rangle_\eta = \frac{\int D\eta\, O[\phi_\eta(t)]\, e^{-\frac{1}{4}\int dz d\tau \eta^2(z,\tau)}}{\int D\eta\, e^{-\frac{1}{4}\int dz d\tau \eta^2(z,\tau)}} = \int D\phi\; O[\phi]\, P[\phi,t]$$

$$\dot{P}[\phi,t] = \int dx\, \frac{\delta}{\delta\phi(x)}\left(\frac{\delta S[\phi]}{\delta\phi(x)} + \frac{\delta}{\delta\phi(x)}\right) P[\phi,t]$$

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Floratos-Iliopoulos, Nucl.Phys. B 214 (1983) 392

for the solution of which we can introduce a perturbative expansion which generates a hierarchy of equations

$$P[\phi, t] = \sum_{k=0} g^k P_k[\phi, t]$$

Leading order is easy to solve and admits an infinite time (equilibrium) limit such that

$$P_0[\phi, t] \rightarrow_{t\rightarrow\infty} P_0^{eq}[\phi] = \frac{e^{-S_0[\phi]}}{Z_0}$$

In a convenient weak sense at every order one gets equilibrium $P_k[\phi, t] \rightarrow_{t\rightarrow\infty} P_k^{eq}[\phi]$

in terms of quantities which are interelated by a set of relations in which one recognizes the Schwinger-Dyson equations ... i.e. we are done!

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We want to go via another expansion, i.e. the expansion of the solution of Langevin equation in power of the coupling constant

$$\phi_\eta(x;t) = \phi_\eta^{(0)}(x;t) + \sum_{n>0} g^n \phi_\eta^{(n)}(x;t)$$

Parisi-Wu, Damgaard-Huffel

Langevin equation for the free scalar field (momentum space) $$\frac{\partial}{\partial t}\phi_\eta^{(0)}(k,t) = -(k^2+m^2)\phi_\eta^{(0)}(k,t) + \eta(k,t)$$

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i.e. $$G^{(0)}(k,t) = \theta(t)\,\exp\left(-(k^2+m^2)t\right)$$

$$\phi^{(0)}(k,t) = \phi^{(0)}(k,0)\exp\left(-(k^2+m^2)t\right) \;+\; \int_0^t d\tau \exp\left(-(k^2+m^2)(t-\tau)\right)\eta(k,\tau)$$

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Interacting case (cubic interaction in the following) is solved by superposition ...

$$\phi(k,t) = \int_0^t d\tau \exp -(k^2+m^2)(t-\tau)\left[\eta(k,\tau) - \frac{\lambda}{2!}\int \frac{dpdq}{(2\pi)^{2n}}\,\phi(p,\tau)\,\phi(q,\tau)\,\delta(k-p-q)\right]$$

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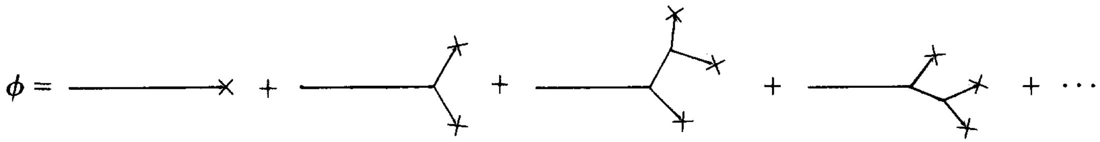
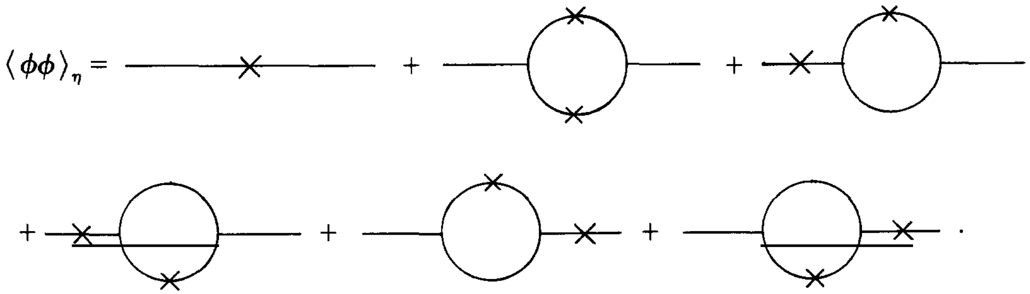
... which leaves the solution in a form which is ready for iteration. It is actually also ready for a graphical intepretation and for the formulation of a

diagrammatic Stochastic Perturbation Theory

$$\phi = \int G\eta - \frac{\lambda}{3!}\int\int\int\int G(G\eta)(G\eta) + \ldots$$





The stochastic diagrams one obtains when averaging over the noise (contractions!) reconstruct, in a convenient infinite time limit, the contributions of the (topologically) correspondent Feynman diagrams ...

but we do not want to go this way ...

From Stochastic Perturbation Theory to NSPT

NSPT (directly in the LGT case) Di Renzo, Marchesini, Onofri 94

We now start with the Wilson action

$$S_G = -\frac{\beta}{2N_c}\sum_P \mathrm{Tr}\left(U_P + U_P^\dagger\right)$$

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We now start with the Wilson action
$$S_G = -\frac{\beta}{2N_c}\sum_P \mathrm{Tr}\left(U_P + U_P^\dagger\right)$$

We now deal with a theory formulated in terms of group variables and Langevin equation reads

$$U_{\mu x} = e^{A_\mu(x)}$$

$$\frac{\partial}{\partial t} U_{x\mu}(t;\eta) = \left(-i\nabla_{x\mu} S_G[U] - i\eta_{x\mu}(t)\right) U_{x\mu}(t;\eta)$$

where the Lie derivative is in place

$$\nabla_{x\mu} = T^a \nabla^a_{x\mu} = T^a \nabla^a_{U_{x\mu}} \qquad \nabla^a_V f(V) = \lim_{\alpha\to 0}\frac{1}{\alpha}\left(f\left(e^{i\alpha T^a} V\right) - f(V)\right)$$

This is again a stochastic differential equation with (gaussian) noise averages satisfying

$$\lim_{t\to\infty}\langle O[U(t;\eta)]\rangle_\eta = \frac{1}{Z}\int DU\, e^{-S_G[U]}\, O[U]$$

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In order to proceed we now need a (numerical) integration scheme, e.g. Euler

$$U_{x\mu}(n+1;\eta) = e^{-F_{x\mu}[U,\eta]}\,U_{x\mu}(n;\eta) \qquad F_{x\mu}[U,\eta] = \epsilon\nabla_{x\mu}S_G[U] + \sqrt{\epsilon}\,\eta_{x\mu}$$

Batrouni et al (Cornell group) PRD 32 (1985)

$$F_{x\mu}[U,\eta] = \frac{\epsilon\beta}{4N_c}\sum_{U_P\supset U_{x\mu}}\left[\left(U_P - U_P^\dagger\right) - \frac{1}{N_c}\mathrm{Tr}\left(U_P - U_P^\dagger\right)\right] + \sqrt{\epsilon}\,\eta_{x\mu}$$

$$\langle \eta_{i,k}(z)\,\eta_{l,m}(w)\rangle_\eta = \left[\delta_{il}\,\delta_{km} - \frac{1}{N_c}\delta_{ik}\,\delta_{lm}\right]\delta_{zw}$$

Now we look for a solution in the form of a perturbative expansion

$$U_{x\mu}(t;\eta) \to 1 + \sum_{k=1} \beta^{-k/2} U_{x\mu}^{(k)}(t;\eta)$$

then we plug it into the (numerical scheme!) Langevin equation and get a hierarchy of equations!

$$U^{(1)\,\prime} = U^{(1)} - F^{(1)}$$

$$U^{(2)\,\prime} = U^{(2)} - F^{(2)} + \frac{1}{2} F^{(1)\,2} - F^{(1)} U^{(1)}$$

$$U^{(3)\,\prime} = U^{(3)} - F^{(3)} + \frac{1}{2}(F^{(2)} F^{(1)} + F^{(1)} F^{(2)}) - \frac{1}{3!} F^{(1)\,3} - (F^{(2)} - \frac{1}{2} F^{(1)\,2})\, U^{(1)} - F^{(1)} U^{(2)}$$

$$\ldots$$

Now we look for a solution in the form of a perturbative expansion

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$$\cdots$$

In practice: we do not look closely at the (underlying) Stochastic Perturbation Theory because the computer is going to (numerically) take care of it and all that you are interested in are the observables, for which

$$\langle O[\sum_k g^k \phi_\eta^{(k)}(t)]\rangle_\eta = \sum_k g^k \langle O_k(t)\rangle_\eta \qquad \lim_{t\to\infty}\langle O_k(t)\rangle_\eta = \lim_{T\to\infty} 1/T \sum_{j=1}^{T} O_k(jn)$$

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$$\langle O[\sum_k g^k \phi^{(k)}_\eta(t)] \rangle_\eta = \sum_k g^k \langle O_k(t) \rangle_\eta \qquad \lim_{t\to\infty} \langle O_k(t) \rangle_\eta = \lim_{T\to\infty} 1/T \sum_{j=1}^{T} O_k(jn)$$

Beware! Lattice PT is (always!) a decompactification of lattice formulation, so that ultimately one should be able to make contact with the continuum Langevin equation, i.e.

$$\frac{\partial}{\partial t} A^a_\mu(\eta, x; t) \; = \; D^{ab}_\nu F^b_{\nu\mu}(\eta, x; t) + \eta^a_\mu(x; t)$$

Where has this gone?

We did not loose anything, since we can always think of all this in the algebra

$$A_{x\mu}(t;\eta) \to \sum_{k=1} \beta^{-k/2} A^{(k)}_{x\mu}(t;\eta)$$

$$A = \log(U) = \log\left(1 + \sum_{k>0} \beta^{-\frac{k}{2}} U^{(k)}\right)$$

$$= \frac{1}{\sqrt{\beta}} U^{(1)} + \frac{1}{\beta}\left(U^{(2)} - \frac{1}{2} U^{(1)\,2}\right) + \left(\frac{1}{\beta}\right)^{\frac{3}{2}} \left(U^{(3)} - \frac{1}{2}\left(U^{(1)}U^{(2)} + U^{(2)}U^{(1)}\right) + \frac{1}{3} U^{(1)\,3}\right) + \dots$$

$$= \frac{1}{\sqrt{\beta}} A^{(1)} + \frac{1}{\beta} A^{(2)} + \left(\frac{1}{\beta}\right)^{\frac{3}{2}} A^{(3)} + \dots \qquad A^{(k)\,\dagger} = -A^{(k)} \qquad \mathrm{Tr} A^{(k)} = 0 \qquad \forall k$$

and the (expanded) Langevin equation now reads

$$A^{(1)\prime} = A^{(1)} - F^{(1)}$$

$$A^{(2)\prime} = A^{(2)} - F^{(2)} - \frac{1}{2}\left[F^{(1)}, A^{(1)}\right]$$

$$A^{(3)\prime} = A^{(3)} - F^{(3)} - \frac{1}{2}\left[F^{(1)}, A^{(2)}\right] - \frac{1}{2}\left[F^{(2)}, A^{(1)}\right] + \frac{1}{12}\left[F^{(1)}, \left[F^{(1)}, A^{(1)}\right]\right] + \frac{1}{12}\left[A^{(1)}, \left[F^{(1)}, A^{(1)}\right]\right]$$

... which I wanted to specify because it is an effective way of preparing for the fact that this is not the end of the story! Problems are going to pop out which we have to take care of ...

Stochastic Gauge Fixing

Stochastic Gauge Fixing D. Zwanziger, Nucl.Phys. B 192 (1981) 259

Let's go back to the continuum

$$\frac{\partial}{\partial t} A^a_\mu(\eta, x; t) \;=\; D^{ab}_\nu F^b_{\nu\mu}(\eta, x; t) + \eta^a_\mu(x; t)$$

whose expanded version has a (momentum space) solution

$$A^{(n)a}_\mu(k; t) \;=\; T^{ab}_{\mu\nu} \int_0^t ds\, e^{-k^2(t-s)} f^{(n)b}_\nu(k, s) + L^{ab}_{\mu\nu} \int_0^t ds\, f^{(n)b}_\nu(k, s)$$

in which vertices pop in (as they should ...)

$$f^{(0)a}_\nu(k; t) = \eta_\nu(k; t)^a \qquad\qquad f^{(n)a}_\nu(k; t) = gI^{(3)(n-1)a}_\mu(k; t) + g^2 I^{(4)(n-2)a}_\mu(k; t)$$

$$gI^{(3)a}_\mu(k; t) \;=\; \frac{igf^{abc}}{2(2\pi)^n} \int dpdq\, \delta(k+p+q)\, A^b_\nu(-p; t)\, A^c_\sigma(-q; t)\, v^{(3)}_{\mu\nu\sigma}(k, p, q)$$

$$v^{(3)}_{\mu\nu\sigma}(k, p, q) \;=\; \delta_{\mu\nu}(k-p)_\sigma + \text{cyclic permutations}$$

Stochastic Gauge Fixing D. Zwanziger, Nucl.Phys. B 192 (1981) 259

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$$v^{(3)}_{\mu\nu\sigma}(k, p, q) \; = \; \delta_{\mu\nu}(k - p)_\sigma + \text{cyclic permutations}$$

Remember the scalar case ... $\phi(k,t) = \int_0^t d\tau \exp -(k^2 + m^2)(t - \tau) \left[\eta(k, \tau) - \frac{\lambda}{3!} \int \frac{dp dq ds}{(2\pi)^{2n}} \, \phi(p, \tau) \, \phi(q, \tau) \, \phi(s, \tau) \, \delta(k - p - q - s) \right]$

Stochastic Gauge Fixing D. Zwanziger, Nucl.Phys. B 192 (1981) 259

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BUT ALL THIS IS GOING TO BE ONLY FORMAL ... WE WILL NOT OBTAIN LONG TIME CONVERGENCE BECAUSE OF THE LOSS OF DAMPING IN THE LONGITUDINAL (NON-gauge-invariant) SECTOR

SOLUTION: add an extra piece $$\dot{A}^a_\mu(x;t) = -\frac{\delta S[A]}{\delta A^a_\mu(x;t)} + D^{ab}_\mu V^b[A,t] + \eta^a_\mu(x;t)$$

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but GAUGE INVARIANT ones are such that

$$D^{ab}_\mu \frac{\delta F[A]}{\delta A^b_\mu(x)} = 0$$

and thus physics is unaffected! (integration by parts ...) ... while if we make a convenient choice for the extra term we have new damping factors in place!

$$-D^{ab}_\mu V^b = \frac{1}{\alpha} D^{ab}_\mu \partial_\nu A^b_\nu \qquad A^{a(n)}_\mu(k;t) = T_{\mu\nu} \int_0^t ds \, e^{-k^2(t-s)} f^{a(n)}_\nu(k,s) + L_{\mu\nu} \int_0^t ds \, e^{-\frac{k^2}{\alpha}(t-s)} f^{a(n)}_\nu(k,s)$$

SOLUTION: add an extra piece $$\dot{A}^a_\mu(x;t) = -\frac{\delta S[A]}{\delta A^a_\mu(x;t)} - D^{ab}_\mu V^b[A,t] + \eta^a_\mu(x;t)$$

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On the lattice we interleave a gauge fixing step to the Langevin evolution

$$U'_{x\mu} = e^{-F_{x\mu}[U,\eta]} \, U_{x\mu}(n)$$

$$U_{x\mu}(n+1) = e^{w_x[U']} \, U'_{x\mu} \, e^{-w_{x+\hat{\mu}}[U']}$$

which has by the way an obvious interpretation

$$U_{x\mu}(n+1) = e^{-F_{x\mu}[U^G, G\eta G^\dagger]} \, U^G_{x\mu}(n)$$

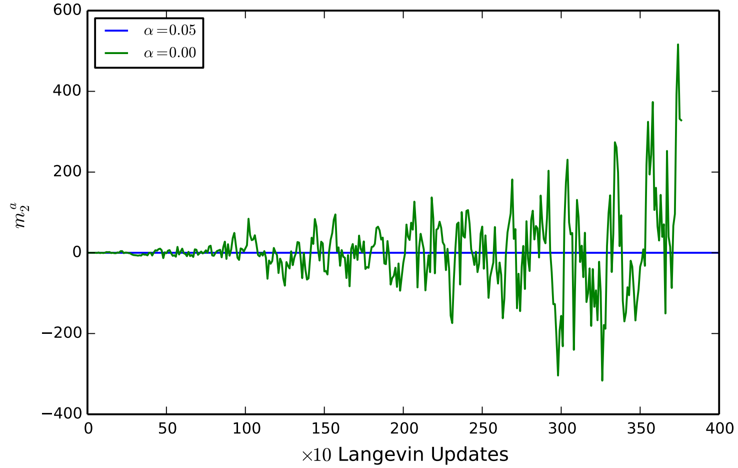

Figure 1. The effect of stochastic gauge fixing.

Fermionic loops in NSPT

FERMIONIC LOOPS in NSPT Di Renzo, Scorzato 2001

Let's add fermions (Wilson fermions, in this case) in the Langevin equation

$$
\begin{aligned}
S_F^{(W)} &= \sum_{xy} \bar{\psi}_x \, M_{xy}[U] \, \psi_y \\
&= \sum_x (m+4) \, \bar{\psi}_x \, \psi_x - \frac{1}{2} \sum_{x\mu} \left(\bar{\psi}_{x+\hat{\mu}} \, (1+\gamma_\mu) \, U^\dagger_{x\mu} \, \psi_x \, + \, \bar{\psi}_x \, (1-\gamma_\mu) \, U_{x\mu} \, \psi_{x+\hat{\mu}} \right)
\end{aligned}
$$

From the point of view of the functional integral measure $e^{-S_G} \det M = e^{-S_{eff}} = e^{-(S_G - Tr \ln M)}$

and in turns $\nabla^a_{x\mu} S_G \mapsto \nabla^a_{x\mu} S_{eff} = \nabla^a_{x\mu} S_G - \nabla^a_{x\mu} \mathrm{Tr} \, \ln M = \nabla^a_{x\mu} S_G - \mathrm{Tr} \left((\nabla^a_{x\mu} M) M^{-1} \right)$

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Batrouni et al (Cornell group) PRD 32 (1985)

In $U_{x\mu}(n+1;\eta) = e^{-F_{x\mu}[U,\eta]} \, U_{x\mu}(n;\eta)$ we now write

$$F = T^a(\epsilon \Phi^a + \sqrt{\epsilon} \eta^a) \qquad \Phi^a = \left[\nabla^a_{x\mu} S_G - \mathrm{Re} \left({\xi_k}^\dagger (\nabla^a_{x\mu} M)_{kl} (M^{-1})_{ln} \xi_n \right) \right]$$

where $\langle \xi_i \xi_j \rangle_\xi = \delta_{ij}$ or (this is what we always do)

$$\Phi^a = \left[\nabla^a_{x\mu} S_G - \mathrm{Re} \left({\xi_l}^\dagger (\nabla^a_{x\mu} M)_{ln} \psi_n \right) \right] \qquad M_{kl} \psi_l = \xi_k$$

From a numerical point of view this boils down to the (technically challenging) problem of inverting the Dirac operator efficiently. This is a heavy task, making unquenched simulations much more demanding in terms of computer time.

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But we have not put our expansion in the coupling in place! Once we do it, we find much less problems than expected from the non-perturbative simulations point of view!

$$M = M^{(0)} + \sum_{k>0} \beta^{-k/2} M^{(k)} \qquad M^{-1} = {M^{(0)}}^{-1} + \sum_{k>0} \beta^{-k/2} {M^{-1}}^{(k)}$$

In NSPT we have to deal with only one inverse (known once and for all: the Feynman free propagator) plus a tower of recursive relations

$$\begin{aligned}
{M^{-1}}^{(1)} &= -{M^{(0)}}^{-1} M^{(1)} {M^{(0)}}^{-1} \\
{M^{-1}}^{(2)} &= -{M^{(0)}}^{-1} M^{(2)} {M^{(0)}}^{-1} - {M^{(0)}}^{-1} M^{(1)} {M^{-1}}^{(1)} \\
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\end{aligned}$$

i.e.

$$ {M^{-1}}^{(n)} = -{M^{(0)}}^{-1} \sum_{j=0}^{n-1} M^{(n-j)} {M^{(j)}}^{-1}$$

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\end{aligned}$$

i.e.

$${M^{-1}}^{(n)} = -{M^{(0)}}^{-1} \sum_{j=0}^{n-1} M^{(n-j)} {M^{(j)}}^{-1}$$

This has a direct counterpart in the solution of the linear system we have to face, which is also translated into a perturbative version (beware! the noise source is 0-th order)

$$\psi^{(j)} \equiv {M^{-1}}^{(j)} \xi$$

$$\begin{aligned}
\psi^{(0)} &= {M^{(0)}}^{-1} \xi \\
\psi^{(1)} &= -{M^{(0)}}^{-1} M^{(1)} \psi^{(0)} \\
\psi^{(2)} &= -{M^{(0)}}^{-1} \left[M^{(2)} \psi^{(0)} + M^{(1)} \psi^{(1)} \right] \\
\psi^{(3)} &= -{M^{(0)}}^{-1} \left[M^{(3)} \psi^{(0)} + M^{(2)} \psi^{(1)} + M^{(1)} \psi^{(2)} \right]
\end{aligned}$$

i.e.

$$\psi^{(n)} = -{M^{(0)}}^{-1} \sum_{j=0}^{n-1} M^{(n-j)} \psi^{(j)}$$

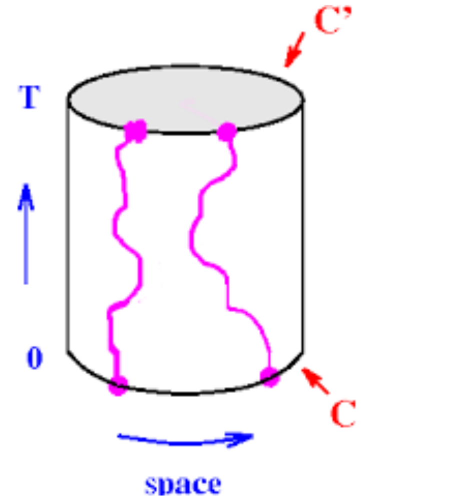
which is particularly nice, since it can be solved by going back and forth from momentum to coordinate representation!

A couple of (what I would regard as) relevant applications for this Workshop

The Schrodinger Functional (SF) in NSPT

Brambilla, Dalla Brida, Di Renzo, Hesse, Sint

The SF is a perfect framework framework for NSPT!



$$U_k(x)|_{x_0=0} = e^{aC_k}\,, \quad U_k(x)|_{x_0=T} = e^{aC'_k}$$

$$U_\mu(x) = e^{a\, g_0\, q_\mu(x)} V_\mu(x)$$

Alpha Collaboration

$$V_\mu(x) = e^{a\, B_\mu(x)}\,, \quad B_0 = 0\,, \quad B_k(x) = \frac{1}{T}\left[x_0 C'_k + (T - x_0) C_k\right]$$

You will probably hear from Stefan Sint some news concerning the SF coupling

$$\Gamma = -\ln\left[\int D[U] e^{-S[U]}\right] \qquad \frac{k}{\bar{g}^2} = \left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=\nu=0} \qquad \bar{g}^2 = g_0^2\,(1 + m_1\, g_0^2 + m_2\, g_0^4 + \ldots)$$

We now have a working implementation of the SF in NSPT!

ALPHA Collaboration / Nuclear Physics B 713 (2005) 378–406

2.4. Discretization effects

The influence of the underlying space–time lattice on the evolution of the coupling can be estimated perturbatively [29], by generalizing Symanzik's discussion [36–38] to the present case. Close to the continuum limit we expect that the relative deviation

$$\delta(u, a/L) = \frac{\Sigma(u, a/L) - \sigma(u)}{\sigma(u)} = \delta_1(a/L)u + \delta_2(a/L)u^2 + \cdots \qquad (2.30)$$

Fermionic loops and renormalons

This is a trivial observation: renormalons have been till now tested in the quenched approximation.

Since we have been knowing for a while how to treat fermionic loops, I have always been thinking that it would be a good idea to repeat for the unquenched case: fermions are actually a terrific handle on the control of the asysmptotics of the expansions!

This is so because in the end the coefficients entering the renormalon behavior are fixed by beta-function ...

A basic, but important technical point: we have to go for staggered quarks, not for the Wilson ones. Otherwise there would be an overwhelming amount of computations to fix the critical mass counterterms ... Needless to say, even for staggerd quarks the computation is nevertheless more demanding than the quenched case ...

Resurgence, trans-series and all that

From Mitat Unsal's presentation at LATTICE2015

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ, Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series following resurgence terminology.

Actually this "Resurgence people" have quite a number of predictions for perturbative expansions and they went many steps further than the typical claim for QM cases (double-well potential ...)

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle.

There quite a lot to study! ... and I admit my ignorance! Good point is that there are both low order/high order and low order/low order relations. I have already been asked by a few people (A. Gonzalez-Arroyo, A. Ramos; M. Unsal himself): why don't we give it a try by NSPT? My answer was (of course...) YES! We will see ..

Conclusions

- NSPT has been around for roughly 20 years, but it is never too late to have a closer look at it!

- I think there are always applications on our list, but now in particular maybe new working grounds are there in Resurgence.

Conclusions

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- Let's now listen to Antonio!

N-H
ELSEVIER
Nuclear Physics B 457 (1995) 202–216
NUCLEAR PHYSICS B

Renormalons from eight-loop expansion of the gluon condensate in lattice gauge theory ★

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Compelling Evidence of Renormalons in QCD from High Order Perturbative Expansions

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We compute the static self-energy of SU(3) gauge theory in four spacetime dimensions to order α^{20} in the strong coupling constant α. We employ lattice regularization to enable a numerical simulation within the framework of stochastic perturbation theory. We find perfect agreement with the factorial growth of high order coefficients predicted by the conjectured renormalon picture based on the operator product expansion.

DOI: 10.1103/PhysRevLett.108.242002 PACS numbers: 12.38.Cy, 11.10.Jj, 11.15.Bt, 12.38.Bx