Non-perturbative tests of continuum HQET through small-volume two-flavour QCD

Patrick Fritzsch

<p.fritzsch@csic.es>

Instituto de Física Teórica (UAM/CSIC), Madrid

IN COLLABORATION WITH J.HEITGER

Instituto de Física Teórica UAM-CSIC

ALPHA Collaboration

EXCELENCIA SEVERO OCHOA

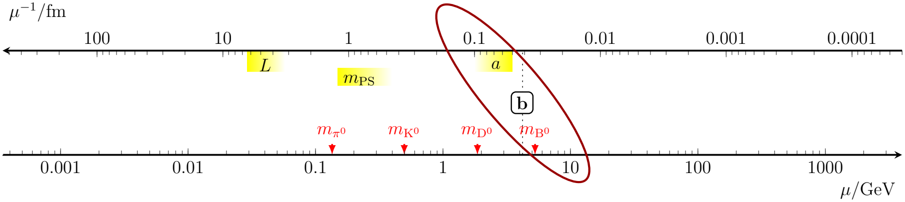
High-precision QCD at low energy

2015, Aug 02 – Aug 22

CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

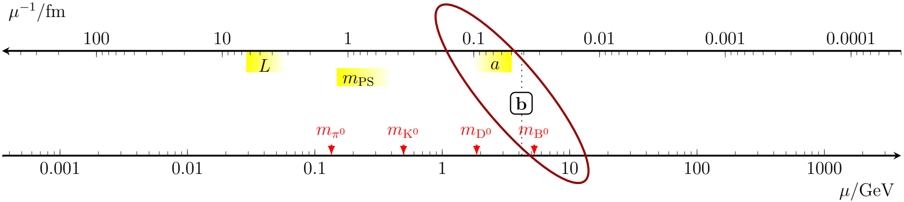
Why effective theory framework at all?

B-meson is to heavy to be resolved on LV lattices with $a \geq 0.05\,\mathrm{fm}$:



Why effective theory framework at all?

B-meson is to heavy to be resolved on LV lattices with $a \geq 0.05\,\mathrm{fm}$:

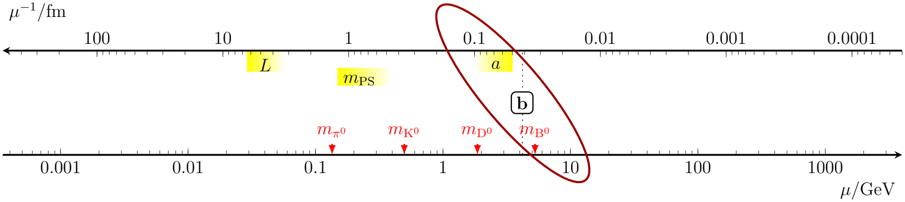


- relativistic b-quark impossible to incorporate using techniques of u/d-,s-,c-quark
- necessity of effective theory description remains in the near future

  NRQCD,[1, 2] Fermilab/RHQ,[3]–[5] HQET action[6]–[11]
- HQET cleanest way for heavy-light systems from field theoretic point of view
- Remark: *ALPHA collaboration could develop a full non-perturbative treatment of HQET incl.* $\mathrm{O}(1/m_{\mathrm{h}})$ *terms (renormalization & matching to QCD)* [12]–[20]

Why effective theory framework at all?

B-meson is to heavy to be resolved on LV lattices with $a \geq 0.05\,\mathrm{fm}$:

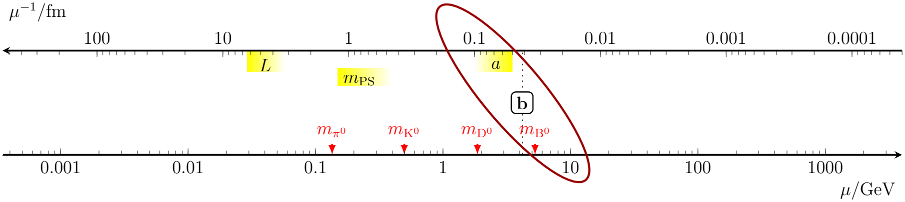


- relativistic b-quark impossible to incorporate using techniques of u/d-,s-,c-quark
- necessity of effective theory description remains in the near future
  NRQCD,[1, 2] Fermilab/RHQ,[3]–[5] HQET action[6]–[11]
- HQET cleanest way for heavy-light systems from field theoretic point of view
- Remark: *ALPHA collaboration could develop a full non-perturbative treatment of HQET incl.* $\mathrm{O}(1/m_{\mathrm{h}})$ *terms (renormalization & matching to QCD)* [12]–[20]

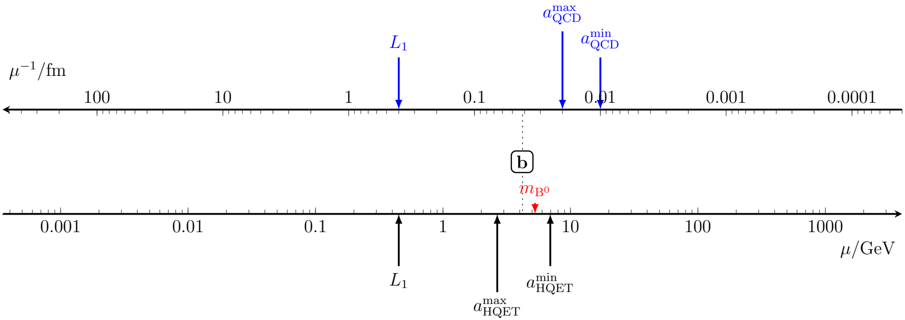
HERE: Probe predictions of HQET numerically using lattice QCD.

see[21, 22]

B-meson is to heavy to be resolved on LV lattices with $a \geq 0.05\,\mathrm{fm}$:



How is that possible? in small volume $L_1 \approx 0.4\,\mathrm{fm}$



OUR SETUP / FRAMEWORK

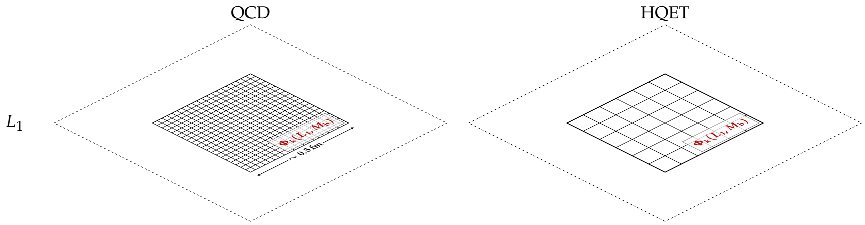
Schrödinger functional as finite-volume renormalization scheme

- renormalization scale $\mu = 1/L_1$ set implicitly via non-perturbatively defined coupling, $\bar{g}^2_{\rm SF}(L_1)$
- massless scheme, $N_{\rm f} = 2$ degenerate quarks with $m_{\rm l} = 0$
- relativistic (quenched) heavy quark of mass $m_{\rm h}$ parameterized via RGI parameter $z = L_1 M$
- probe HQET on $T = L = L_1$ lattices with different kinematics using twisted b.c. for heavy-quark

$$\psi_{\rm h}(x + L\hat{k}) = {\rm e}^{{\rm i}\theta_k} \psi_{\rm h}(x) \ , \qquad k = 1, 2, 3$$

with $\theta_k \equiv \theta \in \{0, 0.5, 1\}$

Framework



Define renormalized trajectory in L_1 (line of constant physics)

'light' sector:

$$\bar{g}^2(L_1/2) \equiv 2.989\,, \qquad L_1 m_l \equiv 0 \qquad (L_1 \approx 0.4\,\mathrm{fm}) \qquad \Rightarrow \text{tuning of } (\beta, \kappa_l, L_1/a)$$

QCD: $L_1/a \in \{20, 24, 32, 40\}$ $\Rightarrow a \leq 0.02\,\mathrm{fm}$ $\rightsquigarrow$ relativistic b-quark

HQET: $L_1/a \in \{6, 8, 10, 12, 16\}$ coarser lattices sufficient

'heavy' sector in QCD: fix RGI heavy quark mass PF,Heitger,Tantalo[23]

$$z = L_1 M = L_1 Z_{\mathrm{M}}(1 + b_{\mathrm{m}}\, am_{\mathrm{q,h}}) am_{\mathrm{q,h}} + \mathrm{O}(a^2)\,, \qquad Z_{\mathrm{M}} = \frac{Z(g_0) Z_{\mathrm{A}}(g_0)}{Z_{\mathrm{P}}(\mu, g_0)} h(L_1/2)$$

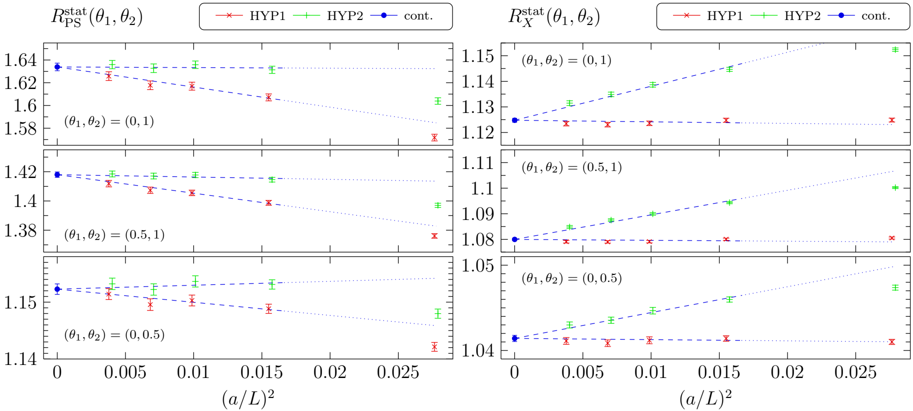
$$\in \{2, 2.7, 3, 3.3, 4, 6, 7, 9, 11, 13, 15, 18, 21\}$$

CONTINUUM EXTRAPOLATIONS

Example for continuum limit in HQET

- $L/a \in \{4, 6, 8, 10, 12\}$ with flavour-twisted mom. $(\theta_1, \theta_2) \in \{(0, 0.5), (0.5, 1), (0, 1)\}$
- 2 static actions (HYP1,HYP2) + universality of CL to reduce systematics in cont. extrapolation

$$\Omega_\delta^{\mathrm{HQET}}(L, a) = \Omega^{\mathrm{HQET}}(L) \left[1 + (a/L)^2 \cdot A_\delta\right] , \qquad \delta = 1, 2$$

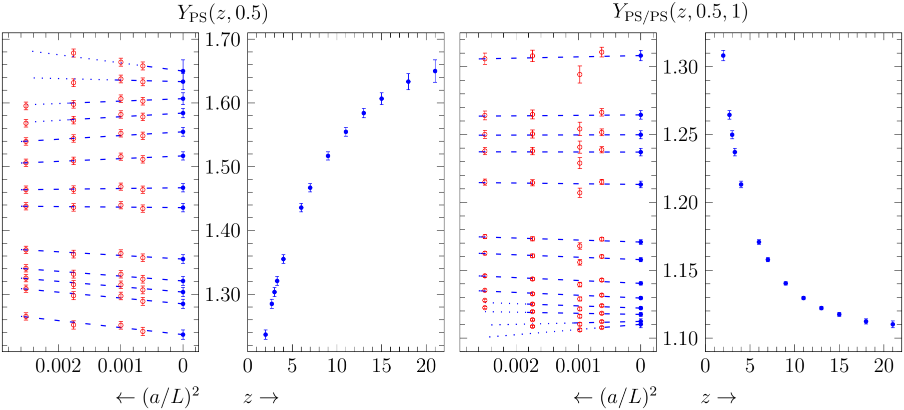


$$R_{\mathrm{PS}}^{\mathrm{stat}}(\theta_1, \theta_2) = \left[\frac{\langle 0|A_0|\mathrm{PS}(\theta_1)\rangle}{\langle 0|A_0|\mathrm{PS}(\theta_2)\rangle}\right]^{\mathrm{stat}} \qquad R_X^{\mathrm{stat}}(\theta_1, \theta_2) = \frac{F_{\mathrm{PS}}^{\mathrm{stat}}(\theta_1)}{F_{\mathrm{PS}}^{\mathrm{stat}}(\theta_2)}$$

Example for continuum limit in QCD

- $L/a \in \{20, 24, 32, 40\}$ and $z \in \{2, 2.7, 3, 3.3, 4, 6, 7, 9, 11, 13, 15, 18, 21\}$
- global fit ansatz taking mass-dep. cutoff effects into account

$$\Omega^{\mathrm{QCD}}(L,z,a) = \Omega^{\mathrm{QCD}}(L,z)\left[1 + (a/L)^2 \cdot \{\rho_0 + \rho_1 z + \rho_2 z^2\}\right]$$



Matching scheme

Our scheme to match QCD and HQET

in some renormalization scheme:

$$\Omega^{\mathrm{QCD}}(m_{\mathrm{h}}) = \widetilde{C}_{\Omega}(m_{\mathrm{h}}, \mu) \times \Omega^{\mathrm{HQET}}(\mu) + \mathrm{O}(1/m_{\mathrm{h}})$$

to eliminate scheme dependence, parameterize matching relation in terms of QCD RGIs Λ, M_{h}

$$\Omega^{\mathrm{QCD}}(M_{\mathrm{h}}) = C_{\Omega}(M_{\mathrm{h}}/\Lambda) \times \Omega^{\mathrm{HQET}}_{\mathrm{RGI}} + \mathrm{O}(1/M_{\mathrm{h}})$$

where

$$C_{\Omega}(M/\Lambda) = \exp\left\{\int^{g_*} \mathrm{d}x \frac{\gamma_{\mathrm{match}}(x)}{\beta(x)}\right\} , \qquad g_*^2 \equiv \bar{g}^2(m_*) , \qquad m_* = \overline{m}(m_*) , \qquad \mu = m_*$$

Example: static decay constant $F_{\mathrm{PS}}^{\mathrm{stat}}$

$$F_{\mathrm{PS}}\sqrt{m_{\mathrm{PS}}} = C_{\mathrm{PS}}(M/\Lambda) \times X_{\mathrm{RGI}} , \qquad\qquad X_{\mathrm{RGI}} \propto \langle 0|A_0^{\mathrm{stat}}|\mathrm{PS}\rangle$$

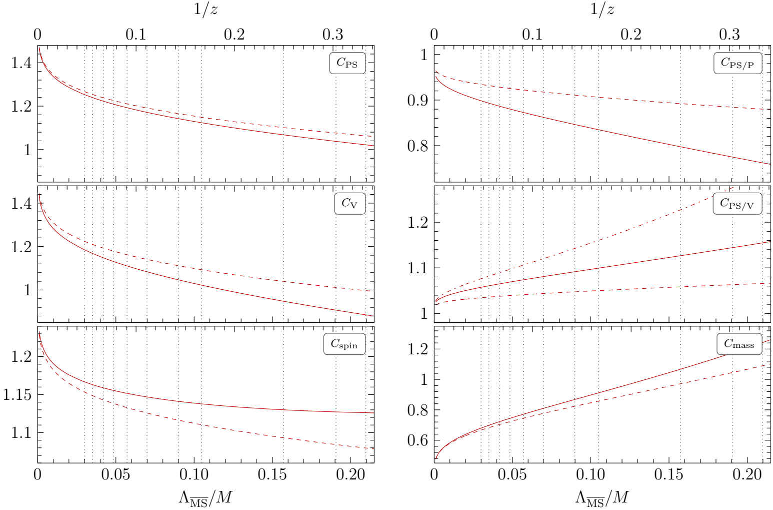
HERE: Matching/Conversion functions known in PT only!

for details see[13, 21, 24]

Conversion functions in the matching scheme

$C_{\rm X}^{\rm n-lp}$ for $n=2$ (dashed), $n=3$ (solid), $n=4$ (dash-dotted): $z = L_1 M$, $L_1\Lambda_{\overline{\rm MS}} = 0.629(36)$

$$C_{\rm X}^{n-{\rm lp}}(x) = x^{\gamma_0^{\rm X}/2b_0}\left[1+\sum_{i=1}^{n-1} c_i x^i\right] , \qquad x \equiv \frac{1}{\ln[M/\Lambda_{\overline{\rm MS}}]} = \frac{1}{\ln[z/L_1\Lambda_{\overline{\rm MS}}]}$$



Static extra-polations vs. HQET

2 sets of observables $\Omega^{\rm QCD}(L, M)$

A w/ conversion functions $C_\Omega(M/\Lambda)$

- effective masses Γ
- decay constants $Y_{\rm PS}, Y_V$
- spin splitting
- ratios of heavy-light currents

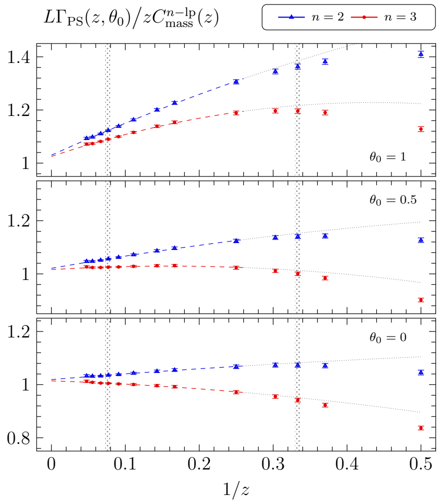
B w/o conversion functions

- ratios of same heavy-light currents with different kinematics

static extrapolations

- ¡unconstrained only!
- $\dfrac{\Omega^{\rm QCD}(z)}{z^\ell \cdot C_\Omega(z)} \sim \Omega^{\rm HQET} + {\rm O}(1/z)$
- heavy-quark spin symmetry $\leftrightarrow$ B, B^* degenerate

Extrapolating effective meson masses



- $\theta_0 \in \{0, 0.5, 1\}$
- data: statistical errors only!

Definition:

$$\Gamma_{\rm PS} = -\tilde{\partial}_0 \ln\left[\langle 0|A_0(x_0)|\mathrm{B}\rangle\right]_{x_0=\frac{T}{2}}$$

Static extrapolation:

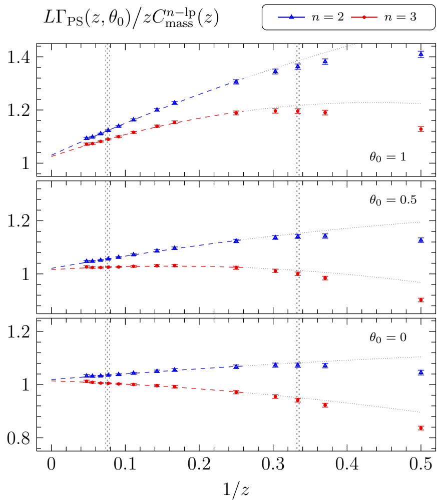
$$\frac{\Omega^{\rm QCD}(z,\theta_0)}{z\cdot C^{\rm n-lp}_{\rm mass}(z)} = \Omega^{[0]} + \Omega^{[1]}\frac{1}{z} + \Omega^{[2]}\frac{1}{z^2}$$

$$1/z \leq 0.26$$

HQET expectation:

$$1 + \mathrm{O}(1/z)$$

Extrapolating effective meson masses



- $\theta_0 \in \{0, 0.5, 1\}$
- data: statistical errors only!

Definition:

$$\Gamma_{\rm PS} = -\tilde{\partial}_0 \ln \left[\langle 0|A_0(x_0)|{\rm B}\rangle\right]_{x_0=\frac{T}{2}}$$

Static extrapolation:

$$\frac{\Omega^{\rm QCD}(z,\theta_0)}{z\cdot C_{\rm mass}^{\rm n-lp}(z)} = \Omega^{[0]} + \Omega^{[1]}\frac{1}{z} + \Omega^{[2]}\frac{1}{z^2}$$

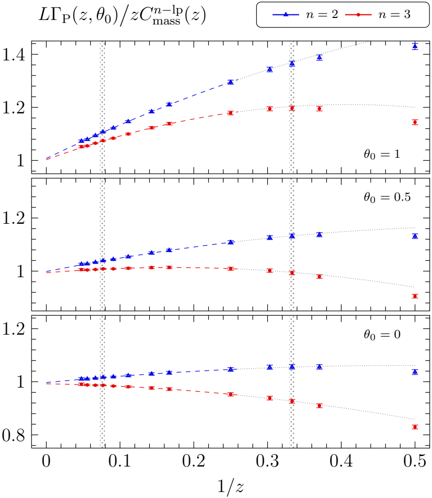
$$1/z \lesssim 0.26$$

HQET expectation:

$$1 + {\rm O}(1/z)$$

...verified within total error budget

Extrapolating effective meson masses



- $\theta_0 \in \{0, 0.5, 1\}$
- data: statistical errors only!

Definition:

$$\Gamma_P = -\tilde{\partial}_0 \ln\left[\langle 0|P(x_0)|B\rangle\right]_{x_0=\frac{T}{2}}$$

Static extrapolation:

$$\frac{\Omega^{QCD}(z,\theta_0)}{z\cdot C_{mass}^{n-lp}(z)} = \Omega^{[0]} + \Omega^{[1]}\frac{1}{z} + \Omega^{[2]}\frac{1}{z^2}$$

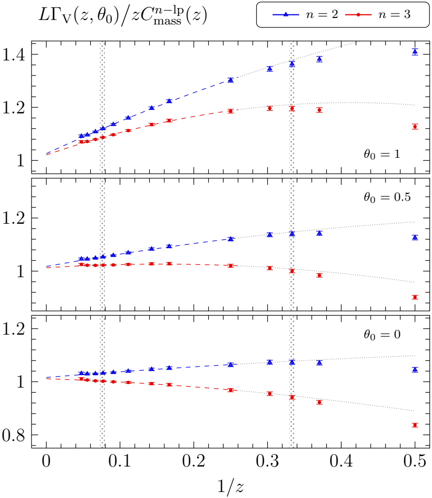
$$1/z \le 0.26$$

HQET expectation:

$$1 + O(1/z)$$

...verified within total error budget

Extrapolating effective meson masses



- $\theta_0 \in \{0, 0.5, 1\}$
- data: statistical errors only!

Definition:

$$\Gamma_{\rm V} = -\tilde{\partial}_0 \ln\left[\langle 0|V_k(x_0)|{\rm B}^*\rangle\right]_{x_0=\frac{T}{2}}$$

Static extrapolation:

$$\frac{\Omega^{\rm QCD}(z,\theta_0)}{z\cdot C^{\rm n-lp}_{\rm mass}(z)} = \Omega^{[0]} + \Omega^{[1]}\frac{1}{z} + \Omega^{[2]}\frac{1}{z^2}$$

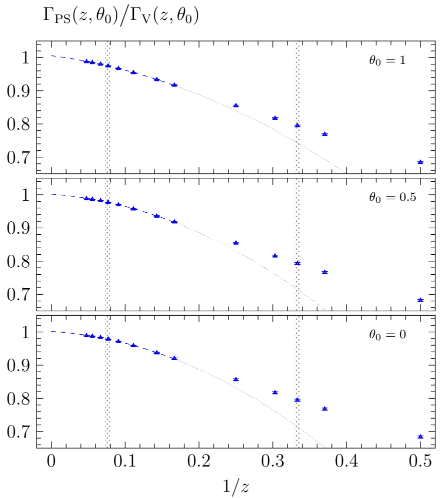
$$1/z \leq 0.26$$

HQET expectation:

$$1 + {\rm O}(1/z)$$

...verified within total error budget

Extrapolating effective meson masses



- $\theta_0 \in \{0, 0.5, 1\}$
- data: statistical errors only!

Definition:

$$\Omega^{\mathrm{QCD}}(z,\theta_0) = \frac{\Gamma_{\mathrm{PS}}(z,\theta_0)}{\Gamma_{\mathrm{V}}(z,\theta_0)}$$

Static extrapolation:

$$\Omega^{\mathrm{QCD}}(z,\theta_0) = \Omega^{[0]} + \Omega^{[1]}\frac{1}{z} + \Omega^{[2]}\frac{1}{z^2}$$

$$1/z \leq 0.2$$

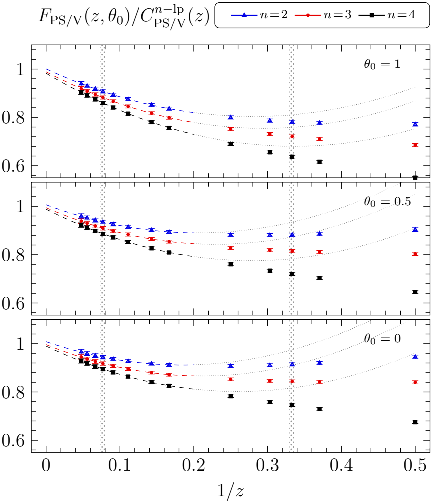
HQET expectation:

$$1 + \mathrm{O}(1/z)$$

. . . verified within total error budget

WITHOUT CONVERSION FUNCTIONS

Ratio pseudoscalar/vector decay constants



$$\lim_{L\to\infty} \frac{Y_{\mathrm{PS}}(z,\theta_0)}{Y_{\mathrm{V}}(z,\theta_0)}\Bigg|^{z=z_{\mathrm{b}}}_{\theta_0=0} = \frac{f_{\mathrm{B}}}{f_{\mathrm{B}^*}}\frac{\sqrt{m_{\mathrm{B}}}}{\sqrt{m_{\mathrm{B}^*}}}$$

Definition:

$$F_{\mathrm{PS/V}}(z,\theta_0) = \frac{Y_{\mathrm{PS}}}{Y_{\mathrm{V}}}\left(\frac{\Gamma_{\mathrm{PS}}}{\Gamma_{\mathrm{V}}}\right)^{-\frac{1}{2}}$$

Static extrapolation:

$$\frac{\Omega^{\mathrm{QCD}}(z,\theta_0)}{C^{n-\mathrm{lp}}_{\mathrm{PS/V}}(z)} = \Omega^{[0]} + \Omega^{[1]}\frac{1}{z} + \Omega^{[2]}\frac{1}{z^2}$$

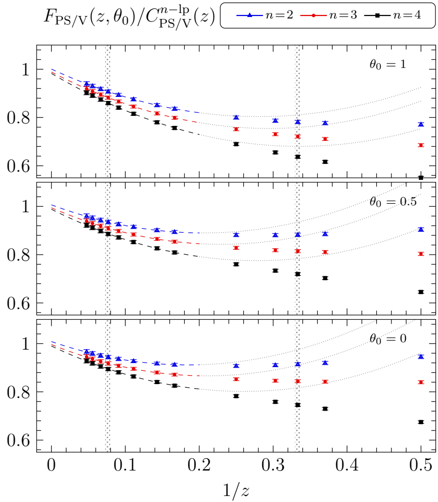
$$1/z \le 0.2$$

HQET expectation:

$$1 + \mathrm{O}(1/z)$$

- $\theta_0 \in \{0, 0.5, 1\}$
- data: statistical errors only!
- $C^{n-\mathrm{lp}}_{\mathrm{PS/V}}(z)$ known to $n=4$ loops

Notice: bad convergence[24, 25]



- $\theta_0 \in \{0, 0.5, 1\}$
- data: statistical errors only!
- $C^{n-\mathrm{lp}}_{\mathrm{PS/V}}(z)$ known to $n = 4$ loops

$$\lim_{L\to\infty} \frac{Y_{\mathrm{PS}}(z,\theta_0)}{Y_{\mathrm{V}}(z,\theta_0)}\Bigg|_{\theta_0=0}^{z=z_\mathrm{b}} = \frac{f_\mathrm{B}}{f_{\mathrm{B}^*}}\frac{\sqrt{m_\mathrm{B}}}{\sqrt{m_{\mathrm{B}^*}}}$$

Definition:

$$F_{\mathrm{PS/V}}(z,\theta_0) = \frac{Y_{\mathrm{PS}}}{Y_{\mathrm{V}}}\left(\frac{\Gamma_{\mathrm{PS}}}{\Gamma_{\mathrm{V}}}\right)^{-\frac{1}{2}}$$

Static extrapolation:

$$\frac{\Omega^{\mathrm{QCD}}(z,\theta_0)}{C^{n-\mathrm{lp}}_{\mathrm{PS/V}}(z)} = \Omega^{[0]} + \Omega^{[1]}\frac{1}{z} + \Omega^{[2]}\frac{1}{z^2}$$

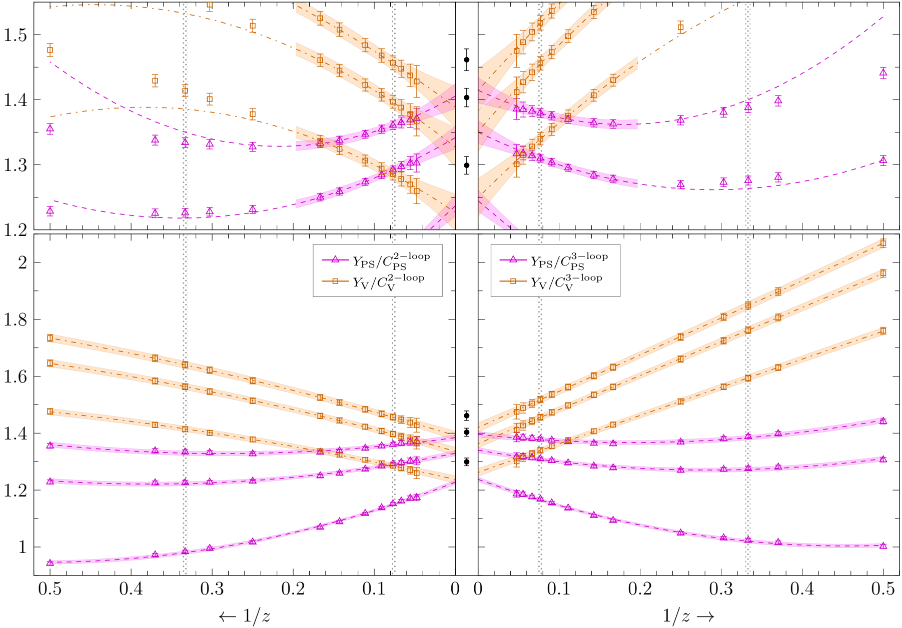
$$1/z \leq 0.2$$

HQET expectation:

$$1 + \mathrm{O}(1/z)$$

...verified within total error budget

Notice: bad convergence[24, 25]


Y_{PS}/C_{PS}^{2-loop}
Y_V/C_V^{2-loop}
Y_{PS}/C_{PS}^{3-loop}
Y_V/C_V^{3-loop}
1.5
1.4
1.3
1.2
2
1.8
1.6
1.4
1.2
1
0.5 0.4 0.3 0.2 0.1 0
0 0.1 0.2 0.3 0.4 0.5
← 1/z
1/z →

Eff. pseudoscalar & vector decay constants II

Observations:

- heavy-quark spin symmetry at work, $\lim_{1/z\to 0}[Y_{\rm PS}/C_{\rm PS}] = \lim_{1/z\to 0}[Y_{\rm V}/C_{\rm V}]$
- all data points $(2 \leq z \leq 21)$ well represented by quadratic function (resulting in smaller error in static extrapolation which may be misleading)
- but no reason to believe that $\mathrm{O}(1/z^3)$ terms do not contribute at $z \lesssim z_{\rm c}$
- systematic discrepancy between static order non-perturbative HQET prediction $X_{\rm RGI}(\theta_0)$ and static extrapolated QCD data at $1-2\,\sigma$ level

$$\frac{Y_{\rm PS}(z,\theta_0)}{C_{\rm PS}(z)} = X_{\rm RGI}(\theta_0)[1+\mathrm{O}(1/z)]\,, \qquad \frac{Y_{\rm V}(z,\theta_0)}{C_{\rm V}(z)} = X_{\rm RGI}(\theta_0)[1+\mathrm{O}(1/z)]$$

- difference slightly descreases for $C^{2-\rm loop} \to C^{3-\rm loop}$

2 explanations: statistical effect / matching functions $C_{\rm X}$ only perturbatively known

Eff. pseudoscalar & vector decay constants II

Observations:

- heavy-quark spin symmetry at work, $\lim_{1/z\to 0}[Y_{\rm PS}/C_{\rm PS}] = \lim_{1/z\to 0}[Y_{\rm V}/C_{\rm V}]$
- all data points $(2 \le z \le 21)$ well represented by quadratic function (resulting in smaller error in static extrapolation which may be misleading)
- but no reason to believe that $\mathrm{O}(1/z^3)$ terms do not contribute at $z \lesssim z_{\rm c}$
- systematic discrepancy between static order non-perturbative HQET prediction $X_{\rm RGI}(\theta_0)$ and static extrapolated QCD data at $1-2\,\sigma$ level

$$\frac{Y_{\rm PS}(z,\theta_0)}{C_{\rm PS}(z)} = X_{\rm RGI}(\theta_0)[1+\mathrm{O}(1/z)]\,, \qquad \frac{Y_{\rm V}(z,\theta_0)}{C_{\rm V}(z)} = X_{\rm RGI}(\theta_0)[1+\mathrm{O}(1/z)]$$

- difference slightly descreases for $C^{2-\rm loop} \to C^{3-\rm loop}$

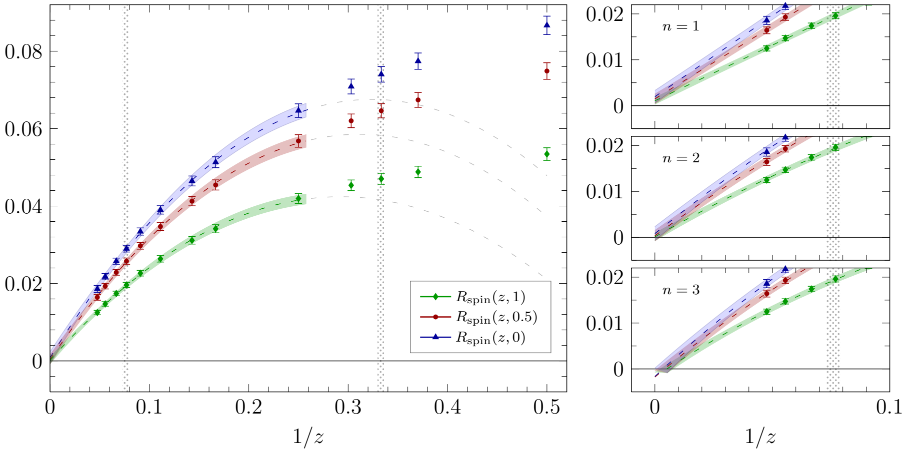
2 explanations: statistical effect / matching functions $C_{\rm X}$ only perturbatively known

NP data vs. PT conversion functions

$$X_{\rm RGI} = \left[\frac{Z^{\rm stat}_{\rm A,RGI}}{Z^{\rm stat}_{\rm A}(\mu)}\right]_{\rm cont.} \times X_{\rm R}(\mu)\,, \qquad \left[\frac{Z^{\rm stat}_{\rm A,RGI}}{Z^{\rm stat}_{\rm A}(\mu)}\right]^{\rm NP}_{\rm cont.} = 0.875(7) \text{ at } \mu = 1/L_1$$

- for PT running $[Z^{\rm stat}_{\rm A,RGI}/Z^{\rm stat}_{\rm A}(\mu)]$ (3/2-loop) $\Rightarrow$ 4% downward shift of static results
- PT evaluated matching functions $C_{\rm PS}$ insufficient for consistent NP treatment ?!?

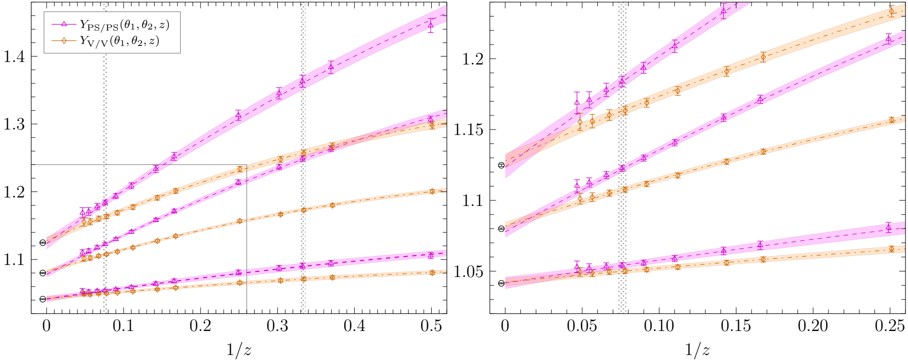
$B^* - B$ mass splitting



- HQET expectation: $0 + \mathrm{O}(1/z)$
- linear, quadratic & cubic extrapolation in $1/z$ for $z \geq 13$, $z \geq 4$ & $z \geq 3$ resp.
- subleading effect $(\propto 1/z)$
- logarithmic corrections $C_{\rm spin}$ not canceled
- but still, fit ansaetze describe data well

Ratios w/o conversion functions I

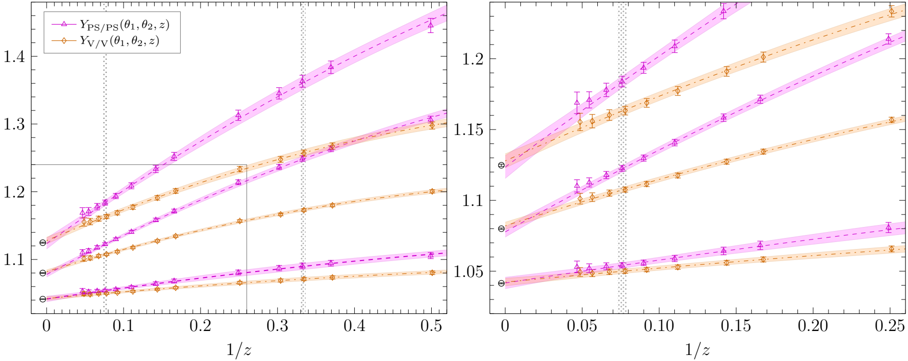
- exhibit unambiguous static extrapolation compatible with their heavy quark mass expansion
- data typically very well described by $\Omega(z) = v_0 + v_1 z^{-1} + v_2 z^{-2}, \forall z$
- Note: non-trivial, non-perturbative static HQET prediction (black points)



$$Y_{\mathrm{PS/PS}}(z, \theta_1, \theta_2) \equiv \frac{Y_{\mathrm{PS}}(z, \theta_1)}{Y_{\mathrm{PS}}(z, \theta_2)} \sim R_{\mathrm{X}}^{\mathrm{stat}}(\theta_1, \theta_2) + \mathrm{O}(1/z) \sim \frac{Y_{\mathrm{V}}(z, \theta_1)}{Y_{\mathrm{V}}(z, \theta_2)} \equiv Y_{\mathrm{V/V}}(z, \theta_1, \theta_2)$$

Ratios w/o conversion functions I

- exhibit unambiguous static extrapolation compatible with their heavy quark mass expansion
- data typically very well described by $\Omega(z) = v_0 + v_1 z^{-1} + v_2 z^{-2}, \forall z$
- Note: non-trivial, non-perturbative static HQET prediction (black points)



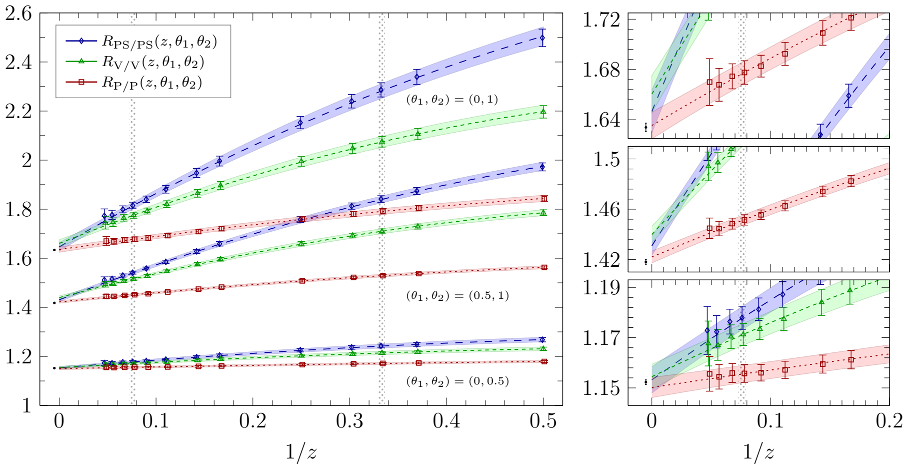
$$Y_{\mathrm{PS/PS}}(z,\theta_1,\theta_2) \equiv \frac{Y_{\mathrm{PS}}(z,\theta_1)}{Y_{\mathrm{PS}}(z,\theta_2)} \sim R_{\mathrm{X}}^{\mathrm{stat}}(\theta_1,\theta_2) + \mathrm{O}(1/z) \sim \frac{Y_{\mathrm{V}}(z,\theta_1)}{Y_{\mathrm{V}}(z,\theta_2)} \equiv Y_{\mathrm{V/V}}(z,\theta_1,\theta_2)$$

excellent agreement strongly advocates such observables (free of PT imperfections)

Ratios w/o conversion functions II

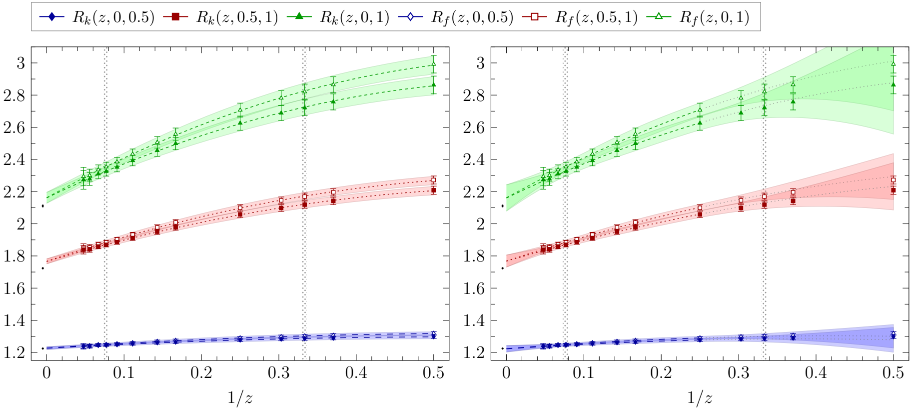
- $(\theta_1, \theta_2) \in \{(0, 0.5), (0.5, 1), (0, 1)\}$

$$R_{X/X}(z, \theta_1, \theta_2) \equiv \frac{\langle 0|X|X(z,\theta_1)\rangle}{\langle 0|X|X(z,\theta_2)\rangle} \sim R_{PS}^{stat}(\theta_1, \theta_2) + O(1/z) , \qquad X \in \{PS, P, V\}$$



Ratios w/o conversion functions III

$$R_f(z,\theta_1,\theta_2) \equiv \frac{\langle \mathrm{PS}|\mathrm{PS}(z,\theta_1)\rangle}{\langle \mathrm{PS}|\mathrm{PS}(z,\theta_2)\rangle} \sim R_f^{\mathrm{stat}}(\theta_1,\theta_2) + \mathrm{O}(1/z)\ , \qquad k \leftrightarrow \mathrm{V}$$



- data well represented by quadratic interpolation
- may lead to underestimated errors in static limit of heavy-quark expansion not respected

Summary

we have studied continuum, large-mass asymptotics of heavy-light meson observables

($N_{\mathrm{f}} = 2$ massless QCD in small volume, $L_1 \approx 0.4\,\mathrm{fm}$)

- strong numerical evidence that a universal CL of static effective theory exists
- HQET predictions consistently modeled by linear functions in $1/z$ ($1/z \leq 1/10$)
  - numerically demonstrates the applicability domain of HQET incl. $\mathrm{O}(1/m_{\mathrm{h}})$ at m_{b}
  - key ingredient of ALPHA Collaboration's B-physics programme
- static extrapolations via $\sum_{i=0}^{n} v_i z^{-i}$ generally stable for data in $[0, n \cdot \Delta z]$, $\Delta z \approx 0.13$
- heavy-light axial and vector meson decay constant fail to meet static HQET prediction
  - safer *not* to use PT conversion functions $C_{\mathrm{PS}}, C_{\mathrm{V}}$
  - otherwise inconsistencies might arise at non-perturbative level
  - logarithmic heavy-mass dependence vs. $1/z$ power corrections
  - avoided by non-perturbative finite-volume matching of QCD & HQET à la ALPHA
- exploring size of higher-order correction helps to improve HQET-QCD matching relations by minimizing $\mathrm{O}(1/m_{\mathrm{h}}^2)$ contributions

presented results will be published soon [22]

Thank you for your attention!

Bibliography I

[1] B. Thacker and G. P. Lepage, *Heavy quark bound states in lattice QCD*, *Phys.Rev.* **D43** (1991) 196–208.

[2] G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea, and K. Hornbostel, *Improved nonrelativistic QCD for heavy quark physics*, *Phys. Rev.* **D46** (1992) 4052–4067, [hep-lat/9205007].

[3] A. X. El-Khadra, A. S. Kronfeld, and P. B. Mackenzie, *Massive fermions in lattice gauge theory*, *Phys.Rev.* **D55** (1997) 3933–3957, [hep-lat/9604004].

[4] H.-W. Lin and N. Christ, *Non-perturbatively Determined Relativistic Heavy Quark Action*, *Phys. Rev.* **D76** (2007) 074506, [hep-lat/0608005].

[5] N. H. Christ, M. Li, and H.-W. Lin, *Relativistic Heavy Quark Effective Action*, *Phys. Rev.* **D76** (2007) 074505, [hep-lat/0608006].

[6] E. Eichten, *Heavy Quarks on the Lattice*, *Nucl.Phys.Proc.Suppl.* **4** (1988) 170.

[7] E. Eichten and B. R. Hill, *Renormalization of heavy-light bilinears and f_B for Wilson fermions*, *Phys.Lett.* **B240** (1990) 193.

[8] E. Eichten and B. R. Hill, *An Effective Field Theory for the Calculation of Matrix Elements Involving Heavy Quarks*, *Phys.Lett.* **B234** (1990) 511.

[9] E. Eichten and B. R. Hill, *Static effective field theory: 1/m corrections*, *Phys.Lett.* **B243** (1990) 427–431.

[10] H. Georgi, *An effective field theory for heavy quarks at low energies*, *Phys.Lett.* **B240** (1990) 447–450.

[11] N. Isgur and M. B. Wise, *Weak Decays of Heavy Mesons in the Static Quark Approximation*, *Phys.Lett.* **B232** (1989) 113.

[12] J. Heitger and R. Sommer, *Nonperturbative heavy quark effective theory*, *JHEP* **0402** (2004) 022, [hep-lat/0310035].

[13] R. Sommer, *Non-perturbative Heavy Quark Effective Theory: Introduction and Status*, in *Proceedings, Advances in Computational Particle Physics: Final Meeting (SFB-TR-9)*, vol. 261-262, pp. 338–367, 2015. arXiv:1501.03060.

[14] B. Blossier, M. Della Morte, N. Garron, and R. Sommer, *HQET at order $1/m$: I. Non-perturbative parameters in the quenched approximation*, *JHEP* **1006** (2010) 002, [arXiv:1001.4783].

[15] B. Blossier, M. Della Morte, N. Garron, G. von Hippel, T. Mendes, et al., *HQET at order $1/m$: II. Spectroscopy in the quenched approximation*, *JHEP* **1005** (2010) 074, [arXiv:1004.2661].

[16] B. Blossier, M. Della Morte, N. Garron, G. von Hippel, T. Mendes, et al., *HQET at order $1/m$: III. Decay constants in the quenched approximation*, *JHEP* **1012** (2010) 039, [arXiv:1006.5816].

[17] B. Blossier, M. Della Morte, P. Fritzsch, N. Garron, J. Heitger, et al., *Parameters of Heavy Quark Effective Theory from $N_\mathrm{f} = 2$ lattice QCD*, *JHEP* **1209** (2012) 132, [arXiv:1203.6516].

[18] F. Bernardoni, B. Blossier, J. Bulava, M. Della Morte, P. Fritzsch, et al., *The b-quark mass from non-perturbative $N_\mathrm{f} = 2$ Heavy Quark Effective Theory at $O(1/m_\mathrm{h})$*, *Phys.Lett.* **B730** (2014) 171–177, [arXiv:1311.5498].

[19] F. Bernardoni, B. Blossier, J. Bulava, M. Della Morte, P. Fritzsch, et al., *Decay constants of B-mesons from non-perturbative HQET with two light dynamical quarks*, *Phys.Lett.* **B735** (2014) 349–356, [arXiv:1404.3590].

[20] F. Bernardoni, B. Blossier, J. Bulava, M. Della Morte, P. Fritzsch, et al., *B-meson spectroscopy in HQET at order 1/m*, arXiv:1505.03360. accepted by PRD.

[21] J. Heitger, A. Jüttner, R. Sommer, and J. Wennekers, *Non-perturbative tests of heavy quark effective theory*, *JHEP* **0411** (2004) 048, [hep-ph/0407227].

[22] P. Fritzsch, N. Garron, and J. Heitger, *Non-perturbative tests of continuum HQET through small-volume two-flavour QCD*, . to be submitted.

[23] P. Fritzsch, J. Heitger, and N. Tantalo, *Non-perturbative improvement of quark mass renormalization in two-flavour lattice QCD*, *JHEP* **1008** (2010) 074, [arXiv:1004.3978].

Bibliography III

[24] R. Sommer, *Introduction to Non-perturbative Heavy Quark Effective Theory*, in *Modern perspectives in lattice QCD: Quantum field theory and high performance computing. Proceedings, International School, 93rd Session, Les Houches, France, August 3-28, 2009*, pp. 517–590, 2010. arXiv:1008.0710.

[25] S. Bekavac, A. Grozin, P. Marquard, J. Piclum, D. Seidel, et al., *Matching QCD and HQET heavy-light currents at three loops*, *Nucl.Phys.* **B833** (2010) 46–63, [arXiv:0911.3356].

[26] M. Lüscher, R. Narayanan, P. Weisz, and U. Wolff, *The Schrödinger functional: A Renormalizable probe for non-Abelian gauge theories*, *Nucl.Phys.* **B384** (1992) 168–228, [hep-lat/9207009].

[27] M. Lüscher, R. Sommer, P. Weisz, and U. Wolff, *A precise determination of the running coupling in the SU(3) Yang-Mills theory*, *Nucl.Phys.* **B413** (1994) 481–502, [hep-lat/9309005].

[28] S. Sint, *On the Schrödinger functional in QCD*, *Nucl.Phys.* **B421** (1994) 135–158, [hep-lat/9312079].

[29] S. Sint, *One loop renormalization of the QCD Schrödinger functional*, *Nucl.Phys.* **B451** (1995) 416–444, [hep-lat/9504005].

[30] M. Della Morte et al., *Computation of the strong coupling in QCD with two dynamical flavors*, *Nucl.Phys.* **B713** (2005) 378–406, [hep-lat/0411025].

[31] R. Sommer and U. Wolff, *Non-perturbative computation of the strong coupling constant on the lattice*, in *Proceedings, Advances in Computational Particle Physics: Final Meeting (SFB-TR-9)*, vol. 261-262, pp. 155–184, 2015. arXiv:1501.01861.

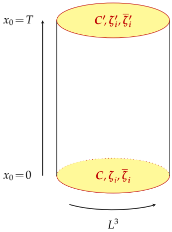
BACKUP SLIDES

The Schrödinger functional coupling

- Euclidean partition function

$$\mathcal{Z} \equiv \int_{T\times L^3} \mathcal{D}[U,\overline{\psi},\psi]\; \mathrm{e}^{-S[U,\overline{\psi},\psi]} = \langle 0|\, \mathrm{e}^{-TH} P|0\rangle$$

  with *periodic BC in L^3*
  and *Dirichlet BC in T* (breaking translational inv. in time)
- renormalization scale $\mu \propto L^{-1}$ (for step-scaling)
- mass-independent scheme, . . .



Abelian boundary fields: $C_k = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix}$; $C_k' = \begin{pmatrix} \phi_1' & 0 & 0 \\ 0 & \phi_2' & 0 \\ 0 & 0 & \phi_3' \end{pmatrix}$

SF coupling

defined as variation of effective action $\Gamma = -\ln \mathcal{Z}[C, C']$,

$$\left.\frac{\partial \Gamma}{\partial \eta}\right|_{\eta=0} = \frac{\mathrm{const}}{\bar{g}^2_{\mathrm{SF}}(L)}$$

for non-vanishing boundary gauge fields $C_k \neq 0 \neq C_k'$

for details see[26]–[31]