Fractional quantum Hall effect Conformal Field Theory and Matrix Product States

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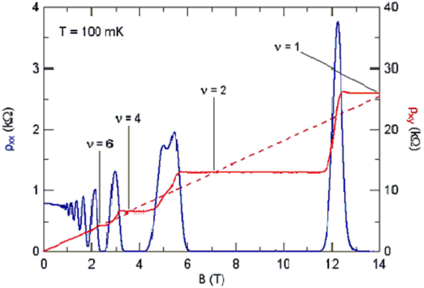
Entanglement in Strongly Correlated Systems

Benasque

1. Integer quantum Hall effect
   - Landau levels
2. Fractional quantum Hall effect
   - Laughlin state
3. The chiral boson
   - and the Laughlin state
4. Conformal field theory...
   - as an ansatz for FQH states
5. Matrix Product States
   - a powerful numerical method

Integer quantum Hall effect



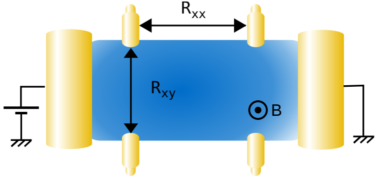
Landau levels

Classical Hall effect

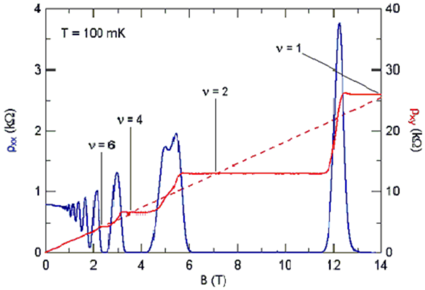
Hall effect : a 2D electron gas in a perpendicular magnetic field.

$\Rightarrow$ **current $\perp$ voltage**

$R_{xy} \propto B$



Integer Quantum Hall effect (IQHE)



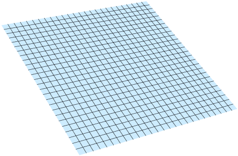
IQHE : von Klitzing (1980)

Quantized Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

ν is an integer up to $O(10^{-9})$
Used in metrology

A single electron in 2D and in a $\perp$ magnetic field B.



Uniform $\perp$ magnetic field : gauge choice

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2, \qquad \vec{A} = \frac{B}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

$$H = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} + \frac{eB}{2}y\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{eB}{2}x\right)^2$$

- **energy scale** cyclotron frequency $\omega_c = \frac{|eB|}{m}$,
- **length scale** : magnetic length $l_B = \sqrt{\frac{\hbar}{|eB|}}$

$$H = \frac{1}{2}\hbar\omega_c\left[\left(-il_B\frac{\partial}{\partial x} + \frac{y}{2l_B}\right)^2 + \left(-il_B\frac{\partial}{\partial y} - \frac{x}{2l_B}\right)^2\right]$$

Landau levels

In (dimensionless) complex coordinate $z = (x + iy)/l_B$, and setting

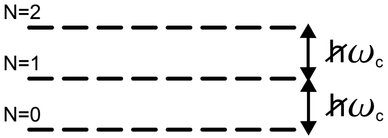
$$a = \sqrt{2}\left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2}\right), \qquad a^{\dagger} = -\sqrt{2}\left(\frac{\partial}{\partial z} - \frac{\bar{z}}{2}\right)$$

Familiar form of the Hamiltonian

$$H = \hbar\omega_c\left(a^{\dagger}a + \frac{1}{2}\right) \qquad [a, a^{\dagger}] = 1$$

$(N+1)^{\mathrm{th}}$ Landau level :

$$E_N = \hbar\omega_c\left(N + \frac{1}{2}\right)$$



Discrete spectrum, large **degeneracy**

Lowest Landau Level ($N = 0$)

Since $a = \sqrt{2}\left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2}\right)$, ground states are of the form

$$\Psi(z, \bar{z}) = f(z)\, e^{-z\bar{z}/4}$$

with $f(z)$ is any holomorphic function ($\partial_{\bar{z}} f = 0$).

$$\Rightarrow \textbf{chirality : } (x, y) \rightarrow z = (x + iy)$$

Ground states, a.k.a. Lowest Landau level (LLL) states

$$\Psi(x, y) = f(x + iy)\, e^{-(x^2+y^2)/4l_B^2}$$

Projection to the LLL : x and y no longer commute $[\hat{x}, \hat{y}] = i\, l_B^2$

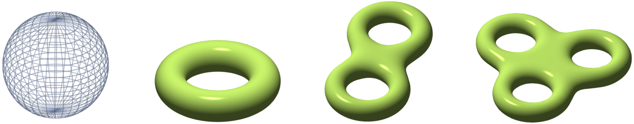
$$\Delta_x\, \Delta_y \geq l_B^2/2$$

$\Rightarrow$ **each electron occupies an area** $2\pi l_B^2$
magnetic flux through this area = quantum of flux $\Phi = h/e$

LLL degeneracy $\sim$ number N_Φ of flux quanta through the surface

Landau problem on arbitrary surfaces

Lowest Landau Level on arbitrary surface :



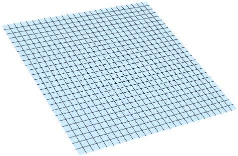
The magnetic flux has to be quantized $\int d^2x\, B = N_\Phi \frac{h}{e}$, with N_Φ integer.

The ground state degeneracy on a surface of genus g is

$$N_\Phi + (1 - g)$$

- it depends on the topology (genus).
- it does NOT depend on the geometry (metric)

Back on flat space : magnetic translations



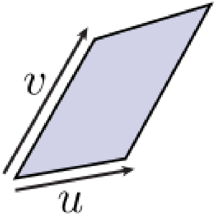
translation invariance : $\vec{x}$ and $\vec{x}+\vec{u}$ are gauge equivalent

$$\vec{A} = \frac{B}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

Magnetic translations $R_{\vec{u}} = e^{iq\vec{u}.\vec{A}} e^{\vec{u}.\vec{\nabla}}$

Aharonov-Bohm effect :

$$R_{\vec{u}} R_{\vec{v}} = e^{i\frac{qB}{\hbar}\vec{u}\wedge\vec{v}} R_{\vec{v}} R_{\vec{u}}$$

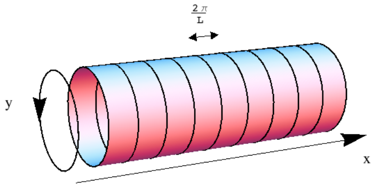


Infinitesimal generators of translations commute with H, but

$$[T_x, T_y] = -i \neq 0$$

Let us choose momentum along the y direction as a quantum number.

Cylinder with perimeter L (we identify $y \equiv y + L$)



Natural gauge choice : $\vec{A} = B \begin{pmatrix} 0 \\ x \end{pmatrix}$

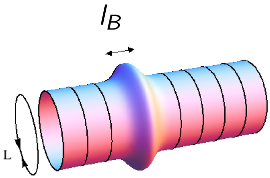
$$T_y |\Psi_k\rangle = k_y |\Psi_k\rangle, \qquad k_y = \frac{2\pi n}{L}$$

$$\Psi_{k_y}(x,y) = e^{iyk_y} e^{-\frac{(x-k_y)^2}{2}} \qquad (l_B = 1)$$

Momentum k_y and position x are locked :

$$x \sim l_B^2 k_y$$

- $[\hat{x}, \hat{y}] = i l_B^2$ implies that $\hbar \hat{x} = l_B^2 \hat{p}_y$.
- localized in $\hat{x}$ and delocalized in $\hat{y}$
- the interorbital distance is $\frac{2\pi}{L} l_B^2$



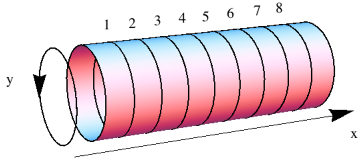
Density profile of the LLL orbital $\Psi_{k_y}(x,y)$.

Projection to the LLL : dimensional reduction

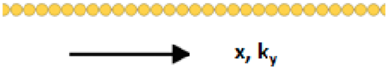
Projection to the LLL : x and y no longer commute $[\hat{x}, \hat{y}] = i\, l_B^2$ (link with non-commutative geometry).

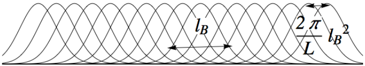
4 dimensional phase space $\Rightarrow$ 2 dimensional phase space

A **basis** of LLL states



looks like a one-dimensional chain



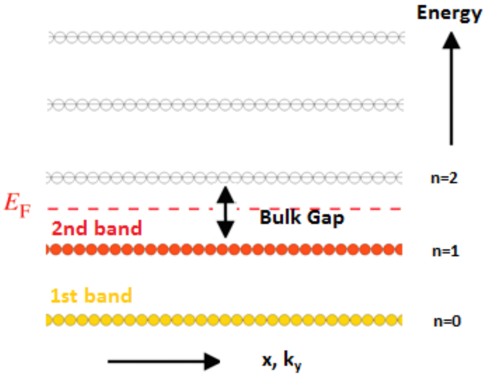


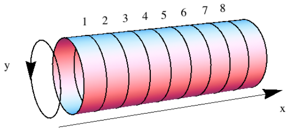
But !

Physical short range interactions become long range in this description (distance of order l_B means $\sim L/l_B$ sites).

The IQHE : bulk insulator

Cartoon picture : no interactions, no disorder

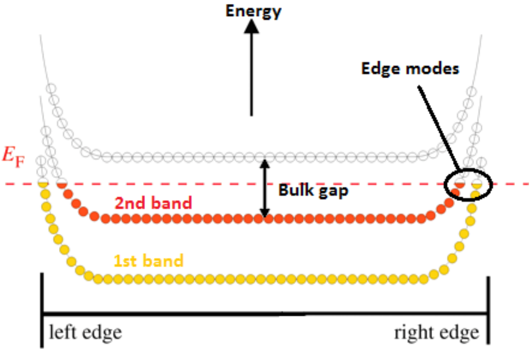




- Landau Levels = flat bands
- Integer filling with fermions $\Rightarrow$ **Bulk insulator.**

How come we have $I \propto V$ then ?

The IQHE : conducting edges



⇒ **Conducting edges**
each channel contributes e^2/h to the Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

Chiral (and therefore protected) massless edges

Topological insulator

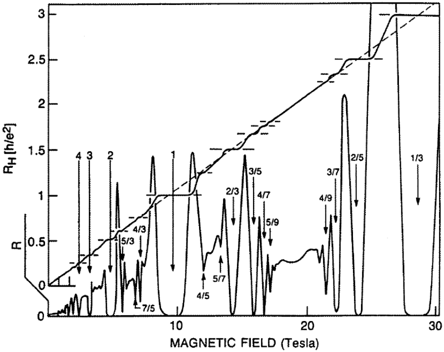
This quantization is insensitive to disorder or strong periodic potential :

topological invariant : the Chern number

Disclaimer : this is just a cartoon picture. Does not explain plateaux.

Fractional filling

the many-body problem



FQHE trial wavefunctions

Fractional filling : the role of interactions

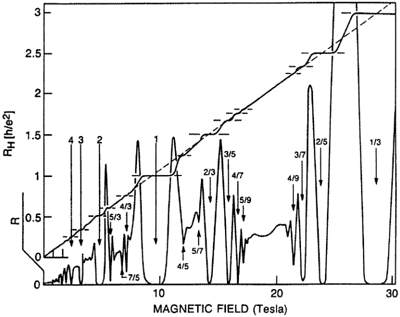
N fermions in N_Φ orbital/states (filling fraction $\nu = N/N_\phi < 1$)
(or N bosons at any filling fractions)

without interactions we would expect a **metallic bulk !**

Experimentally, emergence of exotic and non perturbative physics :

- insulating bulk,
- metallic chiral edge modes,
- excitations with fractional charges,

due to electron-electron interactions



Strongly correlated system, no small parameter. What can we do ?

- Exact diagonalization
- Effective field theories (theories of anyons)
- Trial wavefunctions

Trial wave functions

The $\nu = 1/3$ Laughlin state.

filling fraction $\nu = 1/3$ + short range model interaction

$\Rightarrow$ exact ground-state :

$$\Psi_{\frac{1}{3}} = \prod_{i<j} (z_i - z_j)^3$$

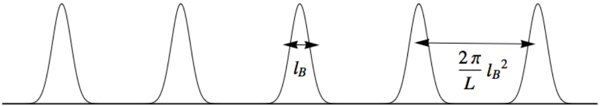
The model interaction is the short range part of Coulomb.

Extremely high overlap with Coulomb interaction !
(obtained by exact diagonalization)

First hints of a topological phase :

- excitations with fractional charge $e/3$
- topology dependent ground state degeneracy : 3^g exact ground states.

Cartoon picture : thin cylinder limit ($L \ll l_B$)



Very small cylinder perimeter L : **LLL orbitals no longer overlap**
1d problem

Laughlin's Hamiltonian → Haldane's exclusion statistics
no more than 1 particle in three orbitals

At filling fraction $\nu = 1/3$, we get three possible states

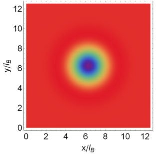
$$|\Psi_1\rangle = |\cdots 1\,0\,0\,1\,0\,0\,1\,0\,0 \cdots\rangle$$
$$|\Psi_2\rangle = |\cdots 0\,1\,0\,0\,1\,0\,0\,1\,0 \cdots\rangle$$
$$|\Psi_3\rangle = |\cdots 0\,0\,1\,0\,0\,1\,0\,0\,1 \cdots\rangle$$

3-fold degenerate ground state on the cylinder (and torus).

Bulk excitations/defects : anyons

Adiabatic insertion of a flux quantum at position w
creates a hole in the electronic liquid :

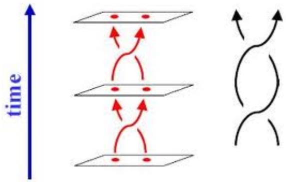
$$\Psi_w = \prod_i (w - z_i) \prod_{i<j} (z_i - z_j)^3$$



Electronic density around a quasihole
(N. Regnault)

Cartoon picture : $|\cdots 1\,0\,0\,1\,0\,0\,0\,1\,0\,0 \cdots\rangle$

fractionalization : the missing electronic charge is $e/3$
these excitations are called **quasi-holes**.



under adiabatic exchange of two quasi-holes

$\Rightarrow$ phase $e^{2i\pi/3}$
non trivial braiding !

$\Rightarrow$ **quasi-holes = abelian anyons**

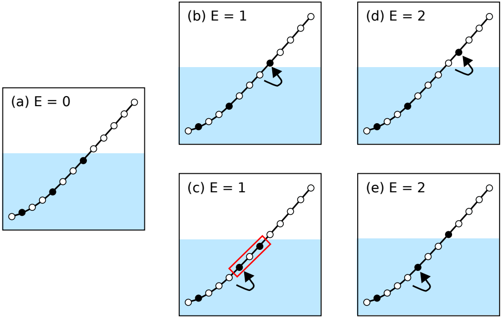
Massless edge modes

$$\Psi_u = P_u \prod_{i<j} (z_i - z_j)^3$$

where P_u is any symmetric, homogeneous polynomial.

Cartoon picture : no more than 1 electron in 3 orbitals.

- dispersion relation : $E \propto P$
  chiral and **gapless** edge
- Number of edge states :
  - $E = 0$: 1 state
  - $E = 1$: 1 state
  - $E = 2$: 2 states
  - $E = 3$: 3 states
  - $E = 4$: 5 states
  - $E = 5$: 7 states
  - ...



(cartoon picture)

spectrum of massless chiral boson.

Massless edge modes

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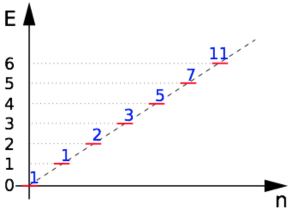
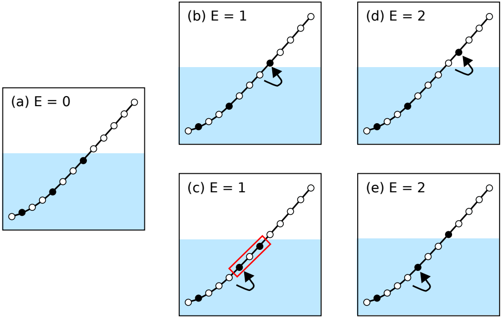




(cartoon picture)

spectrum of massless chiral boson.

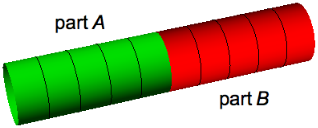
Entanglement entropy

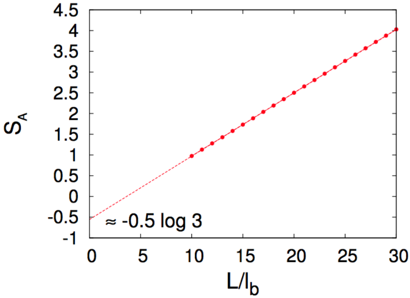
Cut the system in two parts A and B (the boundary has length L)

The **entanglement entropy** is

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

with ρ_A the reduced density matrix.





Entanglement entropy of the $\nu = 1/3$ Laughlin state as a function of the cylinder perimeter L (N. Regnault)

For a topological phase :

$$S_A \sim \alpha L - \log \mathcal{D}$$

where $\mathcal{D}$ is the quantum dimension.

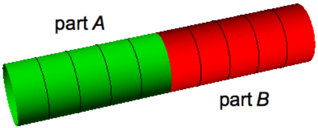
For $\nu = 1/3$ Laughlin : $\mathcal{D} = \sqrt{3}$

Entanglement spectrum

Schmidt decomposition

$$|\Psi\rangle = \sum_{\alpha} \exp(-\xi_\alpha/2)\, |A,\alpha\rangle \otimes |B,\alpha\rangle$$

$$\rho_a = \sum_{\alpha} \exp(-\xi_\alpha)\, |A,\alpha\rangle \langle A,\alpha|$$



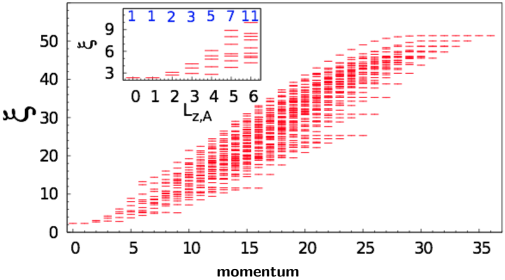
Entanglement spectrum

Li and Haldane (2008) :

spectrum of $\xi = -\log \rho_A$

(plot ξ vs momentum)

$\Rightarrow$ **Reproduces the physical edge spectrum !**

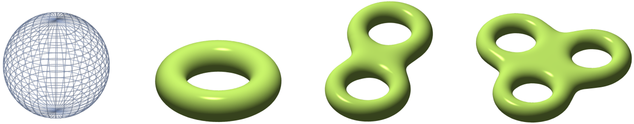


Entanglement spectrum of the $\nu = 1/3$ Laughlin state on the sphere

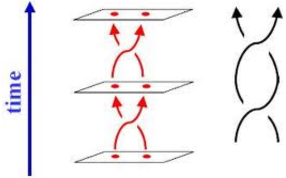
Topological phases

A system is in a topological phase if, at low energy, all observables are invariant under smooth deformation of the underlying space-time manifold, i.e. when its low energy effective field theory is a TQFT (with a gap).

- Ground state degeneracy depends on the genus



- Excitations ("quasi-holes") with fractional charges, possibly non-abelian anyons (non trivial action of the braid group)



Link between $2+1$ TQFT and $1+1$ CFT

Quasi-hole wavefunctions are conformal blocks.

- degeneracy = number of conformal blocks
- braiding = monodromies

Chiral boson and Laughlin

using the edge theory to describe the bulk

The free boson a.k.a. U(1) CFT

Massless gaussian field in $1+1$ dimensions

$$S = \int \mathrm{d}^2 z \, \partial\phi \, \bar{\partial}\phi$$

The mode decomposition of the **chiral** free boson is

$$\phi(z) = \mathbf{\Phi_0} - i\mathbf{a_0} \log(z) + i \sum_{n \neq 0} \frac{1}{n} \mathbf{a_n} z^{-n}$$

$$[\mathbf{a_n}, \mathbf{a_m}] = n\delta_{n+m,0}, \qquad [\mathbf{\Phi_0}, \mathbf{a_0}] = i$$

$U(1)$ symmetry : $\phi(z) \to \phi(z) + \theta$

conserved current :

$$J(z) = i\partial\phi(z) = \sum_n a_n z^{-n-1}$$

Vertex operators :

$$V_Q(z) =: e^{iQ\varphi(z)} :$$

Primary states/ vacua $|Q\rangle$ are defined by their U(1) charge Q

$$a_0|Q\rangle = Q|Q\rangle, \qquad a_n|Q\rangle = 0 \text{ for } n > 0$$

The Hilbert space is simply a Fock space

Descendants are obtained with the lowering operators $a_n^\dagger = a_{-n}$, $n > 0$

- $\Delta E = 0$: 1 state : $|Q\rangle$
- $\Delta E = 1$: 1 state : $a_{-1}|Q\rangle$
- $\Delta E = 2$: 2 states : $a_{-1}^2|Q\rangle$, $a_{-2}|Q\rangle$
- $\Delta E = 3$: 3 states : $a_{-1}^3|Q\rangle$, $a_{-2}a_{-1}|Q\rangle$, $a_{-3}|Q\rangle$
- $\Delta E = 4$: 5 states : $a_{-1}^4|Q\rangle$, $a_{-2}a_{-1}^2|Q\rangle$, $a_{-2}^2|Q\rangle$, $a_{-3}a_{-1}|Q\rangle$, $a_{-4}|Q\rangle$
- $\Delta E = 5$: 7 states : $\cdots$

The Laughlin state written in terms of a $U(1)$ CFT

Ground state wavefunction

$$\prod_{i<j}(z_i - z_j)^3 = \langle 0|\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle, \qquad V(z) =: e^{i\sqrt{3}\varphi(z)}:$$

where $\mathcal{O}_{\text{b.c.}} = e^{-i\sqrt{3}N\varphi_0}$ is just a neutralizing background charge.

Bulk excitations

Wavefunction for p quasiholes

$$\langle \mathcal{O}_{\text{b.c.}} V_{\text{qh}}(w_1)\cdots V_{\text{qh}}(w_p) V(z_1)\cdots V(z_N)\rangle$$

with

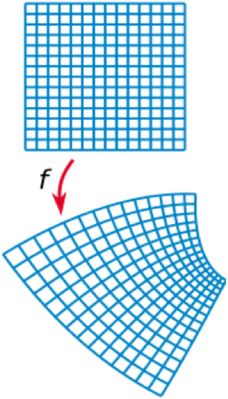
$$V_{\text{qh}}(w) =: e^{\frac{i}{\sqrt{3}}\varphi(w)}:$$

Edge excitations

$$\Psi_u = \langle u|\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle$$

- edge mode = CFT descendant
- we recover $1, 1, 2, 3, 5, 7, \cdots$

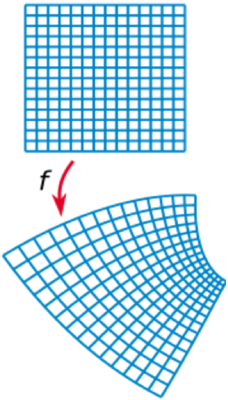
Conformal field theories (CFT)



CFT = Quantum Field Theory + conformal invariance

conformal = angle preserving

$$z \to f(z) = \sum_n f_n z^n$$



Symmetry generators $\{L_n,\ n \in \mathbb{Z}\}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

In particular L_0 generates dilatations.

conformal invariance comes from **criticality**.

- 2D classical stat mech models : **scale invariance**
- 1+1 quantum models : **masslessness**

Calculating in CFT : primary fields

Observables in a QFT = correlation functions

$$\langle \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\rangle$$

In a CFT fields ϕ_i (observables) have a scaling dimension Δ_i :

$$\phi_i(\lambda x) = \lambda^{\Delta_i}\phi_i(x), \qquad [L_0, \phi_i] = \Delta_i \phi_i$$

Fields fall into representations of the Virasoro algebra :

$$\begin{array}{cc}
\Delta_a & \Phi_a \\
\Delta_a + 1 & L_{-1}\Phi_a \\
\Delta_a + 2 & L_{-1}^2\Phi_a,\ L_{-2}\Phi_a \\
\Delta_a + 3 & L_{-1}^3\Phi_a,\ L_{-2}L_{-1}\Phi_a,\ L_{-3}\Phi_a \\
\vdots & \cdots
\end{array}$$

Finitely many primary fields Φ_a.

Calculating in CFT : Operator Product Expansion

$$\text{as } z \to w : \qquad \Phi_i(z)\Phi_j(w) \sim \sum_k F_{i\,j}^{k}(z,w)\Phi_k(w)$$

Conformal symmetry fixes everything, and OPEs are exact !

$$\Phi_a(z)\Phi_b(w) = \sum_c \frac{C_{a\,b}^{c}}{(z-w)^{\Delta_a+\Delta_b-\Delta_c}} \left(\Phi_c(w) + \gamma_{a\,b}^{c}(z-w)L_{-1}\Phi_c(w) + \cdots\right)$$

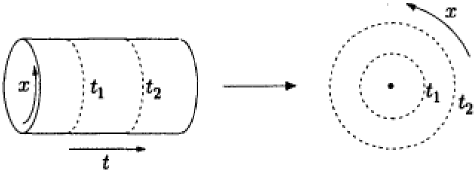
closely related to anyon models : fusion rules $\Phi_a \times \Phi_b = N_{a\,b}^{c}\Phi_c$

OPEs reduces n-point correlation functions to $(n-1)$-point ones !

$$\langle \underbrace{\phi_1(x_1)\phi_2(x_2)}_{OPE} \cdots \phi_n(x_n) \rangle$$

CFT : operator picture

From the $1+1$D perspective : cylinder of perimeter L.



$$\langle \phi_1(x_1,t_1)\phi_2(x_2,t_2)\cdots\phi_n(x_n,t_n)\rangle =$$
$$\langle 0|\hat{\phi}_n(x_n)\cdots\hat{\phi}_3(x_3)e^{-\hat{H}(t_3-t_2)}\hat{\phi}_2(x_2)e^{-\hat{H}(t_2-t_1)}\hat{\phi}_1(x_1)e^{-\hat{H}t_1}|0\rangle$$

Dilatations on the plane become translations in the *time* direction :

$$\hat{H} \sim \frac{2\pi}{L} L_0$$

CFT : Hilbert space

Product of **matrices** with **auxiliary space = CFT Hilbert space**.

$$\sum_{\alpha,\beta,\cdots} \langle 0|\hat{\phi}_n(x_n)\cdots|\beta\rangle e^{-\frac{2\pi}{L}(t_3-t_2)\Delta_\beta}\langle\beta|\hat{\phi}_2(x_2)|\alpha\rangle e^{-\frac{2\pi}{L}(t_2-t_1)\Delta_\alpha}\langle\alpha|\hat{\phi}_1(x_1)|0\rangle$$

So how does the CFT Hilbert space looks like ?

state-operator correspondence : $|a\rangle = \Phi_a(0)|0\rangle$

Δ_a	$|a\rangle$
$\Delta_a + 1$	$L_{-1}|a\rangle$
$\Delta_a + 2$	$L_{-1}^2|a\rangle,\ L_{-2}|a\rangle$
$\Delta_a + 3$	$L_{-1}^3|a\rangle,\ L_{-2}L_{-1}|a\rangle,\ L_{-3}|a\rangle$
$\vdots$	$\cdots$

Truncated CFT : efficient way to approximate correlation functions.

FQH trial wave-function from CFT

Moore and Read (1990) proposed to write
FQH Trial wavefunctions as **CFT correlators**

$$\Psi(z_1, \cdots, z_N) = \langle u | \mathcal{O}_{\text{b.c.}} V(z_1) \cdots V(z_N) | v \rangle$$

- **Operator** $V(z) = \sum_n z^n V_n$
- **Infinite dimensional Hilbert space (graded by momentum/conformal dimension)**

Why is this ansatz sensible ?

- correct entanglement behavior (area law and counting)
- yields a consistent anyon model (pentagon and hexagon equations)
- Laughlin state is of this form

Beyond Laughlin (for bosons)

- $U(1)$ <u>$\nu = 1/r$ **Laughlin state**</u> $V(z) =: e^{i\sqrt{r}\varphi(z)}:$

$$\Psi_{\text{ground-state}} = \prod_{i<j} (z_i - z_j)^r$$

- $\mathrm{SU}(2)_2$ <u>**Moore-Read state**</u> $V(z) = \Psi(z) \otimes : e^{i\varphi(z)}:$

$$\Psi_{\text{ground-state}} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)$$

- $\mathrm{SU}(2)_k$ <u>**Read-Rezayi state**</u>

$$V(z) = J^+(z) = \Psi_1(z) \otimes : e^{i\sqrt{2/k}\varphi(z)}:$$

What about quasi-hole operators ?

$$V_{qh}(w) = \sigma_1(w) \otimes e^{i\sqrt{1/2k}\varphi(w)} : \Rightarrow \text{non-Abelian anyons}$$

Trial wavefunctions from CFT

Extrapolating the **thermodynamic limit** of these trial states is difficult.

- Gapped ?
- Well-defined quasi-holes ?
- Non-Abelian braiding ?
- Area law for the entanglement entropy ?
- Entanglement spectrum ?
- Quantum dimensions ?
- etc...

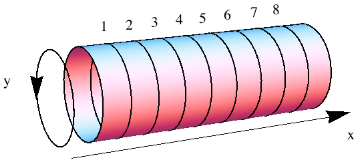
The natural conjecture is that they are described by the **anyon model** (TQFT) corresponding to the underlying CFT.

Matrix Product State (MPS)

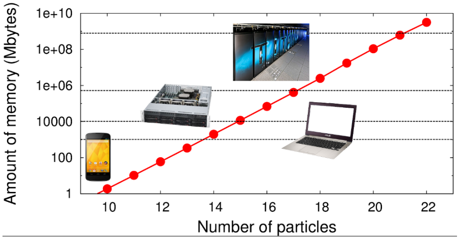
Limitations of exact diagonalizations and trial wf

$\rightarrow$ decomposition of a state $|\Psi\rangle$ on a convenient occupation basis

$$|\Psi\rangle = \sum_{\{m_i\}} c_{\{m_i\}} \, |m_1, \ldots, m_{N_\Phi}\rangle$$



What is the amount of memory needed to store the Laughlin state ?

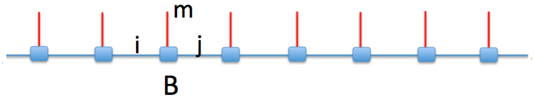


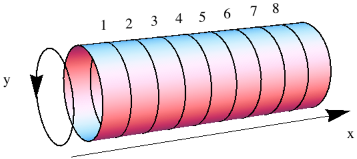
Can't store more than 21 particles !

Matrix Product State : more compact and computationally friendly

Matrix Product States

$$|\Psi\rangle = \sum_{\{m_i\}} \left(\langle u| \, B^{[m_1]} \dots B^{[m_n]} \, |v\rangle \right) \, |m_1, ..., m_n\rangle$$





Why is this formalism interesting ?

Many quantities (correlation functions, entanglement spectrum, . . .) can be computed in the (relatively small) auxiliary space.

The CFT ansatz $\Psi(z_1, \cdots, z_n) = \langle u | V(z_1) \cdots V(z_n) | v \rangle$
is a continuous MPS

Dubail, Read, Rezayi (2012)

Translation invariant MPS

$$|\Psi\rangle = \sum_{\{m_i\}} \left(\langle u| \, B^{m_1} B^{m_2} \cdots B^{m_n} \, |v\rangle \right) |m_1 \cdots m_n\rangle$$

Zaletel, Mong (2012)

- the matrices B^m are operators in the underlying CFT
- the auxiliary space is the (infinite dimensional) CFT Hilbert space ...
- ... which can be truncated while keeping arbitrary large precision

Where does this MPS structure come from ?

Starting from a trial wavefunction given by a CFT correlator

$$\Psi(z_1, \cdots, z_N) = \langle u|\mathcal{O}_{\rm b.c.} V(z_1) \cdots V(z_N)|v\rangle$$

and expanding $V(z) = \sum_n V_{-n} z^n$, one finds (up to orbital normalization)

$$c_{(m_1,\cdots,m_n)} = \langle u|\, \mathcal{O}_{\rm b.c.} \frac{1}{\sqrt{m_1!}} V_{-1}^{m_1} \frac{1}{\sqrt{m_2!}} V_{-2}^{m_2} \cdots \frac{1}{\sqrt{m_n!}} V_{-n}^{m_n}\, |v\rangle$$

This is a site/orbital dependent MPS

$$c_{(m_1,\cdots,m_n)} = \langle u|\, \mathcal{O}_{\rm b.c.} B^{m_1}[1] B^{m_2}[2] \cdots B^{m_n}[n]\, |v\rangle$$

with matrices at site/orbital j :

$$\boxed{B^m[j] = \frac{1}{\sqrt{m!}} (V_{-j})^m}$$

Spreading the background charge

The background charge (for n orbitals) is a (non-local) operator

$$\mathcal{O}_{\text{b.c.}} = e^{-\frac{i}{\sqrt{\nu}} n\varphi_0} = \left(e^{-\frac{i}{\sqrt{\nu}}\varphi_0}\right)^n$$

where φ_0 is the bosonic zero mode.

$$c_{(m_1,\cdots,m_n)} = \langle u| \left(e^{-\frac{i}{\sqrt{\nu}}\varphi_0}\right)^n \frac{1}{\sqrt{m_1!}} {V_{-1}}^{m_1} \frac{1}{\sqrt{m_2!}} {V_{-2}}^{m_2} \cdots \frac{1}{\sqrt{m_n!}} {V_{-n}}^{m_n} |v\rangle$$

From $e^{-\frac{i}{\sqrt{\nu}}\varphi_0} V_{-j} = V_{-j+1} e^{-\frac{i}{\sqrt{\nu}}\varphi_0}$, so we get a site independent MPS

$$\langle u| \frac{1}{\sqrt{m_1!}} {V_0}^{m_1} e^{-\frac{i}{\sqrt{\nu}}\varphi_0} \frac{1}{\sqrt{m_2!}} {V_0}^{m_2} e^{-\frac{i}{\sqrt{\nu}}\varphi_0} \cdots \frac{1}{\sqrt{m_n!}} {V_0}^{m_n} e^{-\frac{i}{\sqrt{\nu}}\varphi_0} |v\rangle$$

Translation invariant MPS on the cylinder

Uniform background charge $\Rightarrow$ site independant MPS

$$B^m[j] = \frac{1}{\sqrt{m!}} (V_{-j})^m \qquad \Rightarrow \qquad B^m = \frac{1}{\sqrt{m!}} (V_0)^m \, e^{-\frac{i}{\sqrt{\nu}}\varphi_0}$$

Taking into account the orbital normalization on the cylinder :

$$B^m = \frac{1}{\sqrt{m!}} (V_0)^m \, e^{-\frac{i}{\sqrt{\nu}}\varphi_0} e^{-\frac{2\pi}{L} H}$$

where

- φ_0 is the bosonic zero mode (B_0 shifts the electric charge by ν)
- H is the cylinder Hamiltonian : $H = \frac{2\pi}{L} L_0$
- V_0 is the zero mode of $V(z)$

auxiliary space = CFT Hilbert space

infinite bond dimension :/

Truncation of the auxiliary space

The auxiliary space (i.e. the CFT Hilbert space) basis is graded by the conformal dimension Δ_α.

$$L_0\,|\alpha\rangle = \Delta_\alpha|\alpha\rangle$$

But in the MPS matrices we have a term

$$B^m = \frac{1}{\sqrt{m!}}\,(V_0)^m\, e^{-\frac{i}{\sqrt{\nu}}\varphi_0} e^{-\left(\frac{2\pi}{L}\right)^2 L_0}$$

The conformal dimension provides a natural cut-off.
Truncation parameter P : keep only states with $\Delta_\alpha \leq P$.

- $P = 0$ recovers the thin-cylinder limit $|\cdots 100100100\cdots\rangle$
- The correct 2d physics requires $L \gg$ bulk correlation length ζ
- For a cylinder perimeter L, we must take $P \sim L^2$
- Bond dimensions $\chi \sim e^{\alpha L}$ $\cdots$ of course ! since $S_A \sim \alpha L$.

What about the torus ?

CFT ansatz : ground state $|\Psi\rangle_a$

$$\Psi_a(z_1, \cdots, z_N) = \mathrm{Tr}_a \left(e^{i2\pi\tau L_0 - i\sqrt{\nu} n \varphi_0} V(z_1) \cdots V(z_N) \right)$$

becomes

$$|\Psi\rangle_a = \sum_{\{m_i\}} \mathrm{Tr}_a \left(e^{i\pi(N-1)\sqrt{\nu} a_0} B^{m_n} \dots B^{m_1} \right) |m_1, \cdots, m_n\rangle$$

where the blue term is only present for fermions (ensures antisymmetry).
The MPS matrices are

$$B^m = q^{\frac{L_0}{2n}} e^{-i\frac{\sqrt{\nu}}{2}\varphi_0} \frac{1}{\sqrt{m!}} V_0^m e^{-i\frac{\sqrt{\nu}}{2}\varphi_0} q^{\frac{L_0}{2n}}, \qquad q = e^{2i\pi\tau}$$

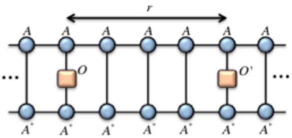
Again χ grows exponentially with torus thickness.

Matrix Product States :
a powerful numerical method

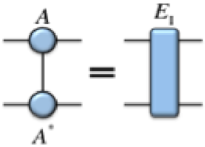
plots from collaborations with :
Y-L. Wu, Z. Papic, N. Regnault, B. A. Bernevig

Infinitely long cylinder, bulk correlation length

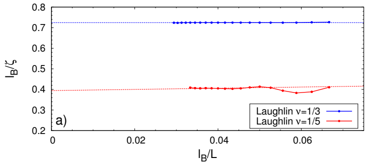
$$\langle O(0)O'(r)\rangle \sim \exp(-r/\zeta)$$

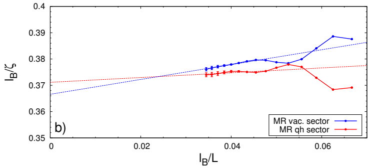


The **transfer matrix** $E_1 = \sum_m A_m \otimes A_m^*$



$\Rightarrow$ correlation length $\zeta^{-1} \propto \log(\lambda_1/\lambda_2)$





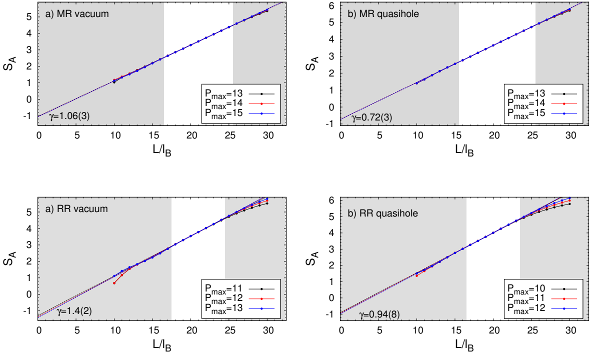
Model state	Laughlin 1/3	Laughlin 1/5	MR vac.	MR qh
ζ/l_B	1.381(1)	2.53(7)	2.73(1)	2.69(1)

Entanglement entropy (orbital cut)

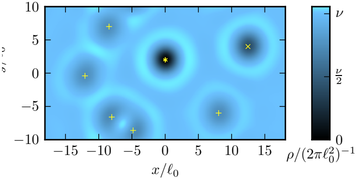
Area law $S_A = \alpha L - \gamma$, where the subleading term γ is universal

$$\gamma = \log \mathcal{D}/d_a$$

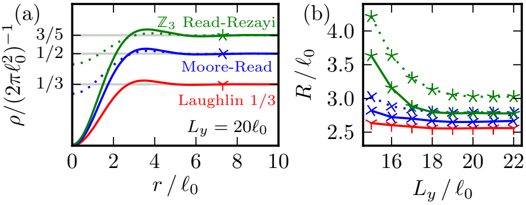
Model state	$\gamma_{\rm vac}$	$\gamma_{\rm qh}$	$\mathcal{D}$
MR	1.04	0.69	$2\sqrt{2}$
$\mathbb{Z}_3$ RR	1.45	0.97	$\frac{5}{2\sin\left(\frac{\pi}{5}\right)}$



Quasi-hole excitations

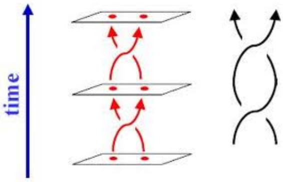


- Insert quasi-holes in the MPS
- Compute the density profile
- Measure the radius of the quasi-hole



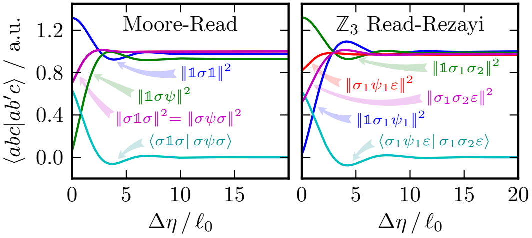
	ν	R/ℓ_0	
Laughlin	$\frac{1}{3}$	$\frac{e}{3}$: 2.6	
Moore-Read	$\frac{1}{2}$	$\frac{e}{4}$: 2.8	$\frac{e}{2}$: 2.7
$\mathbb{Z}_3$ Read-Rezayi	$\frac{3}{5}$	$\frac{e}{5}$: 3.0	$\frac{3e}{5}$: 2.8

Braiding non-Abelian quasi-holes



Instead of computing the Berry phase,
$\Rightarrow$ check the behavior of conformal block overlaps

$$\langle \Psi_a | \Psi_b \rangle = C_a \delta_{a\,b} + O\left(e^{-|\Delta\eta|/\xi_{ab}}\right)$$



Microscopic, quantitative verification of the non-Abelian braiding.

Conclusion

Conclusion

FQH trial wavefunctions have been used for more than 20 years :

They are nothing but Matrix Product States in disguise

Numerically powerful

- **Bulk correlation length** ζ (or equivalently bulk gap)
- precision computation of the **topological entanglement entropy** γ (and the **quantum dimensions** d_a)
- Non-Abelian quasihole radius and **braiding**

CFT/MPS provide a strong link between microscopics and 3d TQFT

As conjectured by Moore and Read

Model states $\Rightarrow$ (non-Abelian) chiral topological phases.

Limitations : at the end of the day these states are model states
with the anyon data as an input. Similar to quantum-double models.

- Are they in the same universality class as the experimental states ?
- DMRG methods might help answer this question.