Chiral Haldane-SPT phases of SU(N) quantum spin chains in the adjoint representation

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"Entanglement in Strongly Correlated Systems"

In collaboration with:
Kasper Duivenvoorden (arXiv:1206.2462 & 1208.0697)
Abhishek Roy (arXiv:1512.05229)

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– Introduction –
The Haldane phase of SU(2) quantum spin chains

Haldane's Conjecture

Setup and large-spin continuum limit

[Haldane]

Spin-s Heisenberg spin chain

$$H = \sum_{\langle kl \rangle} \vec{S}_k \vec{S}_l$$

O(3) non-linear σ-model (field $\vec{n} : S^2 \to S^2$)

$$\mathcal{S}[\vec{n}] = \mathcal{S}_{\text{kin}}[\vec{n}] + \underbrace{\theta \cdot \text{Winding}[\vec{n}]}_{\theta\text{-term: } \theta = 2\pi s}$$

Conjecture

[Haldane]

The physics depends crucially on the "parity" of the spin s

Half-integer: Gapless spin liquid $\to$ SU(2) WZW model

Integer: Gapped spin liquid $\to$ "Haldane phase"

Definition of a Haldane-SPT phase

What do I mean by a Haldane-SPT phase?

A 1D quantum spin system with continuous symmetry whose ground state

- is unique (no spontaneous symmetry breaking) $\rightarrow$ spin liquid
- is gapped
- exhibits symmetry fractionalization (realizes a non-trivial SPT phase)

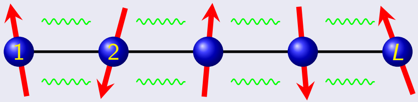
Warning

The definition of a Haldane phase may vary, in particular with regard to

- the nature of the symmetry (continuous vs. discrete)
- the exclusion of spontaneous symmetry breaking
- Historically, the term simply seems to be associated with the integer spin phase of the SU(2) Heisenberg model

Phases of SU(2) spin models

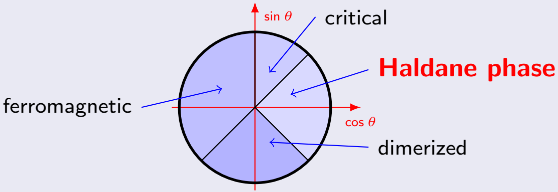
The bilinear-biquadratic SU(2) spin-1 chain



$$H = \sum_{\langle kl \rangle} \left[\cos\theta \, \vec{S}_k \vec{S}_l + \sin\theta \, (\vec{S}_k \vec{S}_l)^2 \right]$$

Phase diagram

[Läuchli,Schmid,Trebst]

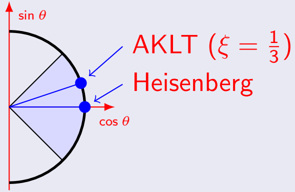


The Haldane phase of SU(2) spin models

The Haldane phase of spin-1 chains

- Unique ground state → SU(2) singlet
- Diluted anti-ferromagnetic order
- Symmetry protected topological phase

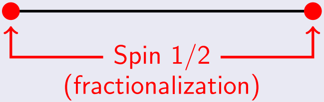
$$H = J \sum_{\langle kl \rangle} \Big[\vec{S}_k \vec{S}_l + \xi (\vec{S}_k \vec{S}_l)^2 \Big]$$



[Affleck,Kennedy,Lieb,Tasaki] [Den Nijs,Rommelse] [Gu,Wen] [Pollmann,Berg,Turner,Oshikawa]

Peculiar property: Emergent massless boundary modes

Open BC



Periodic BC

Virtual boundary modes
(through bipartite entanglement)

Outline of this talk

Goal: Discuss exotic SU(N) spin liquid phases

Why SU(N) systems?

- Open fundamental theoretical issues, e.g. Haldane's Conjecture
- Large number of exotic phases $\sim N$ (even in 1D)
- Experimental realization, numerical challenge, large-N considerations, ...

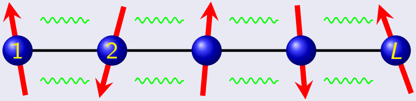
More specifically: SU(N) Haldane-SPT phases

- Classification (via AKLT states)
- Construction (of parent Hamiltonians)
- If time permits: Topological phase transitions

– Classification –
The Haldane-SPT phases of SU(N)

Results on anti-ferromagnetic gapped SU(N) spin chains

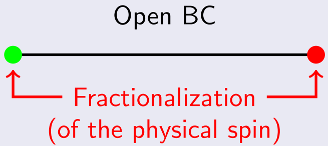
Anti-ferromagnetic SU(N) spin model in 1D



Spin operators: $\vec{S}_k \in su(N)$

Classification of gapped symmetry protected topological phases

[Duivenvoorden,TQ]



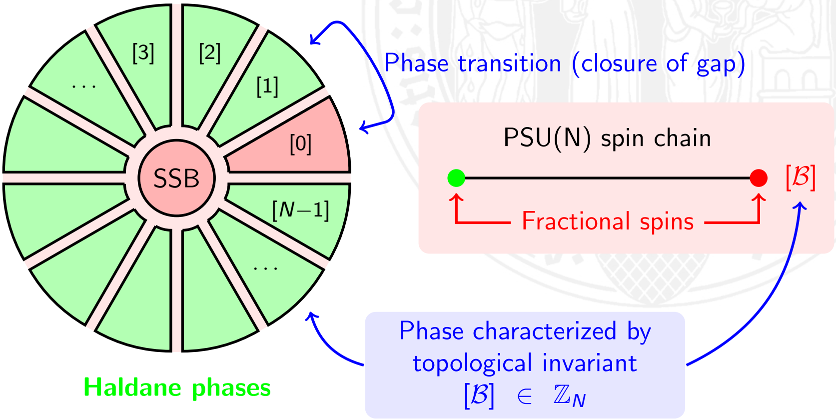
The symmetry can fractionalize in up to N topologically distinct ways.*

* Note: Similar results have been derived for all simple Lie groups G

Sketch of the physical situation

Space of gapped PSU(N) spin chains

[* PSU(N) = "variant of SU(N)"]



– Construction –
AKLT states and the design of entanglement properties

The AKLT construction

Basic idea: Realize physical spin $\mathcal{P}$ in terms of auxiliary spins $\mathcal{B}^*$ and $\mathcal{B}$...

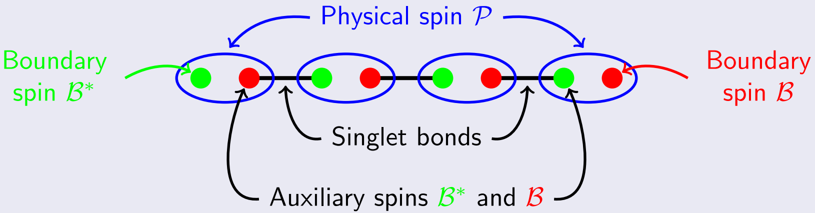
Auxiliary layer:

$$\mathcal{B}^* \otimes \mathcal{B} = 0 \oplus \mathcal{P} \oplus \cdots$$

Physical layer: ⟵ Projection to $\mathcal{P}$

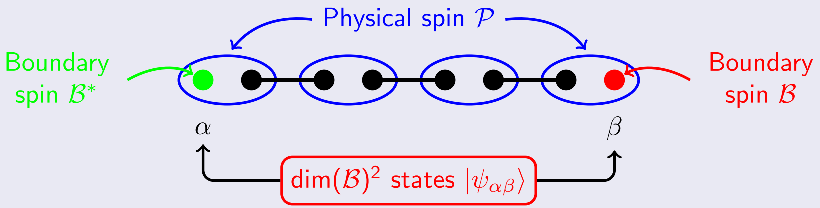
Fractionalized boundary spins from valence bonds

[Affleck,Kennedy,Lieb,Tasaki]



The AKLT states

Fractionalized boundary spins from valence bonds



Question

How to construct a Hamiltonian with $|\psi_{\alpha\beta}\rangle$ as exact ground states?

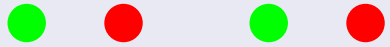
The AKLT Hamiltonian

Task

- Start with the AKLT states $|\psi_{\alpha\beta}\rangle$
- Construct a Hamiltonian which has $|\psi_{\alpha\beta}\rangle$ as its (unique) ground states

The two-site Hamiltonian as a projector

[Affleck,Kennedy,Lieb,Tasaki]

Auxiliary layer:  $\mathcal{B}^* \otimes \mathcal{B} \otimes \mathcal{B}^* \otimes \mathcal{B}$

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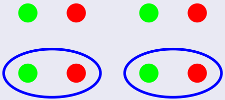

$\mathcal{B}^* \otimes \mathcal{B} \otimes \mathcal{B}^* \otimes \mathcal{B}$

$\mathcal{P} \otimes \mathcal{P} = \mathcal{B}^* \otimes \mathcal{B} \oplus \text{others}$

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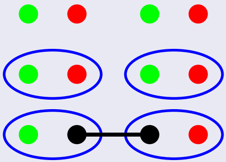
Auxiliary layer:

Physical layer:

Singlet bond:



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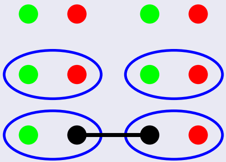


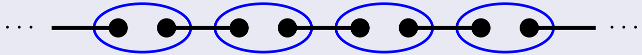
$$\mathcal{B}^* \otimes \mathcal{B} \otimes \mathcal{B}^* \otimes \mathcal{B}$$

$$\mathcal{P} \otimes \mathcal{P} = \boxed{\begin{matrix}\mathcal{B}^* \otimes \mathcal{B} \\ \mathcal{B}^* \otimes \mathcal{B}\end{matrix}} \oplus \text{others}$$

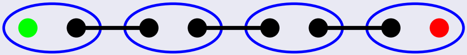
$H_{\text{2-site}}$ = "Projector onto $(\mathcal{B}^* \otimes \mathcal{B})^\perp$" = Function of $\vec{S}_1$ and $\vec{S}_2$

Ground states of the AKLT Hamiltonian

Periodic or infinite chain: Unique ground state (usually but not guaranteed)



Open chain: $(\dim \mathcal{B})^2$-fold ground state degeneracy (usually)



Protection by symmetry

[Pollmann,Berg,Turner,Oshikawa] [Chen,Gu,Wen] [Schuch,Perez-Garcia,Cirac]

The existence of boundary spins is a robust feature as long as

i) they are fractionalized

ii) the system remains gapped

Symmetry Fractionalization

Symmetry fractionalization

What is symmetry fractionalization?

Symmetry of emerging boundary spins

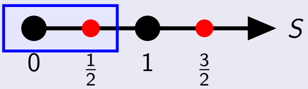
different from

Symmetry of physical spins

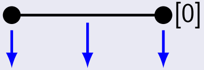
The case of SO(3)

[Chen,Gu,Wen] [Schuch,Perez-Garcia,Cirac]

Two types of su(2) reps

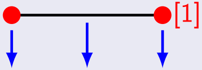


Trivial phase



same rep type

Haldane phase



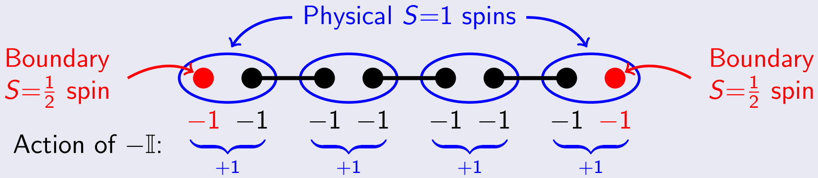
different rep type

Symmetry fractionalization in the SU(2) AKLT model

Basic idea

Physical and boundary spin behave differently under the action of the center $\mathbb{Z}_2 = \{\pm\mathbb{I}\} \subset \mathrm{SU}(2)$

Visualization of the action



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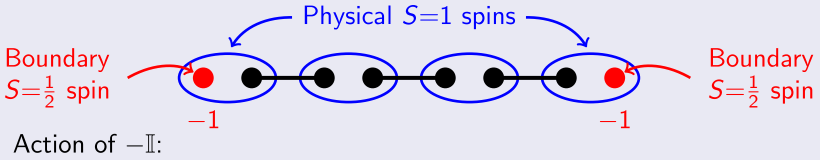
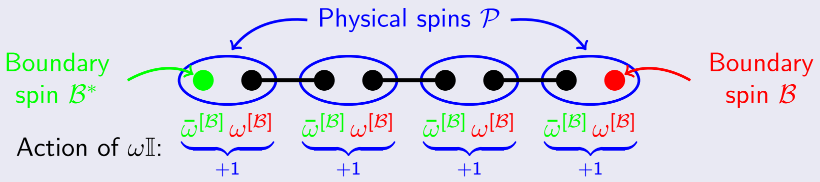


Symmetry fractionalization in the SU(N) AKLT model

Basic idea

Physical and boundary spin behave differently under the action of the center $\mathbb{Z}_N = \{\omega\mathbb{I} | \omega^N = 1\} \subset \mathrm{SU}(N)$

Visualization of the action



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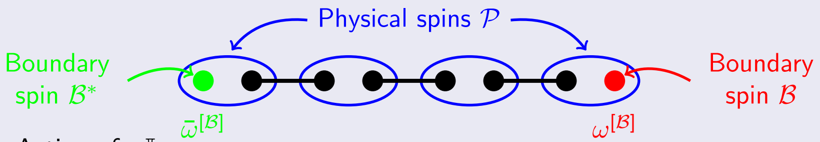
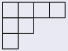


Action of $\omega\mathbb{I}$:

Representation types of SU(N) spins

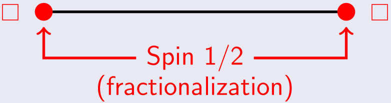
Representations of SU(N)

Young tableau		Representation type
$\lambda =$ 	$\longrightarrow$	$[\lambda] = \text{Boxes}(\lambda) \bmod N$

Remark: The center acts by $\rho_\lambda(\omega\mathbb{I}) = \omega^{[\lambda]}$

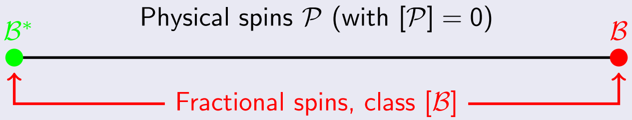
Symmetry fractionalization for SU(2) in terms of Young tableaux

$\square\square \cong$ Spin 1



Haldane phases of PSU(N) spin chains

Symmetry fractionalization for PSU(N) $= \mathrm{SU(N)}/\mathbb{Z}_N$



Classification

[Duivenvoorden,TQ]

PSU(N) chains admit $N-1$ distinct types of Haldane-SPT phases: $[\mathcal{B}] \in \mathbb{Z}_N \backslash \{0\}$

Topological invariant

[Duivenvoorden,TQ] [Duivenvoorden,TQ]

The representation type $[\mathcal{B}]$ of the boundary spin $\mathcal{B}$ is a topological invariant. It can be measured using a non-local string order parameter.

Chiral Haldane phases

Definition

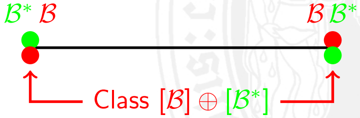
A Haldane-SPT phase is called chiral if $\mathcal{B} \neq \mathcal{B}^*$

SU(N) examples

- A necessary condition for $\mathcal{B} = \mathcal{B}^*$ is $[\mathcal{B}] \in \{[0], [\frac{N}{2}]\}$
- The Haldane-SPT phases of SU(2) chains are all non-chiral

SU(N)	N even	N odd
Chiral	$[\mathcal{B}] \notin \{[0], [\frac{N}{2}]\}$	$[\mathcal{B}] \neq [0]$
Non-chiral	$[\mathcal{B}] = [\frac{N}{2}]$	$\emptyset$

Combination with spontaneous symmetry breaking



Observation

Realizations of chiral Haldane-SPT phases require the use of chiral Hamiltonians

Previous considerations in the literature

"Naive" (non-chiral) AKLT Hamiltonians lead to a two-fold degenerate ground state

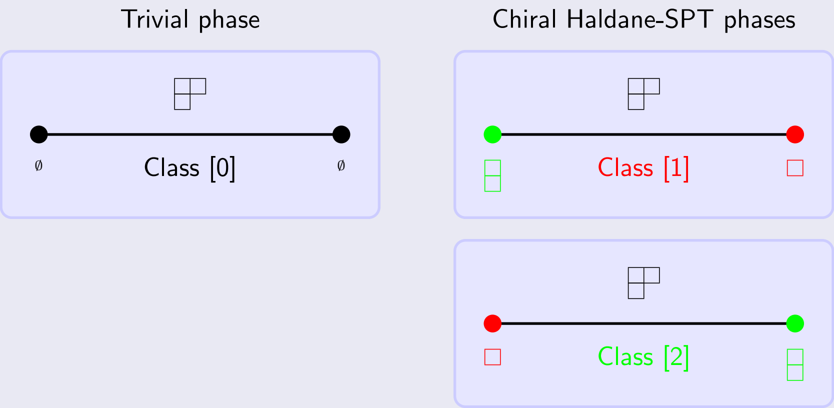
[Affleck,Arovas,Marston,Rabson] [Greiter,Rachel] [Rachel,Schuricht,Scharfenberger,Thomale,Greiter] [Morimoto,Ueda,Momoi,Furusaki]

Examples

Illustration for PSU(3) (adjoint representation)

Sketch of possible edge modes

[Duivenvoorden,TQ]



Sketch of possible edge modes

[Roy,TQ]

Trivial phase | Chiral Haldane-SPT phases

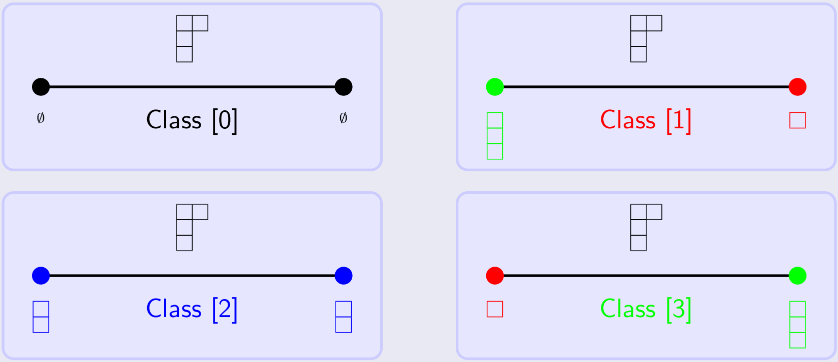
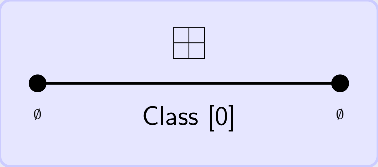


Illustration for PSU(4) (self-dual representation)

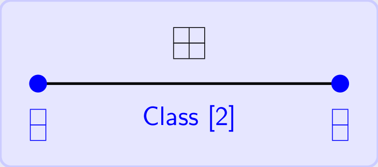
Sketch of possible edge modes

[Nonne,Molinet,Capponi,Lecheminant,Totsuka]

Trivial phase



Non-chiral Haldane-SPT phase

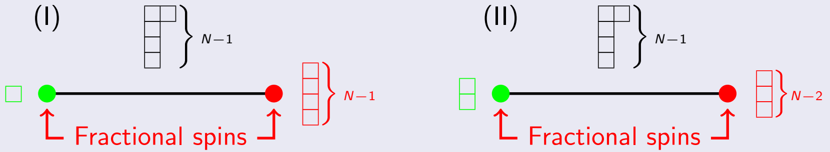


Technical details

Realization of non-trivial topological phases (arbitrary N)

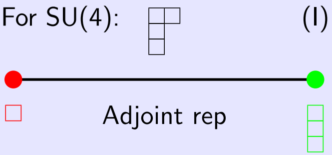
Potential symmetry fractionalizations

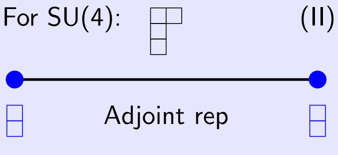
[Roy,TQ]



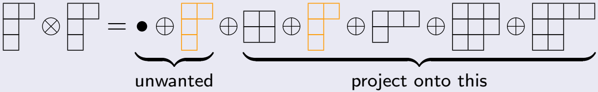
Symmetry group	SU(2)	SU(3)	SU(4)	SU(6)	SU(8)	SU(N)
Dim of physical spins	3	8	15	35	63	N^2-1
Dim of boundary spins (I)	2	3	4	6	8	N
Dim of boundary spins (II)	$\emptyset$	3	6	15	28	$\frac{1}{2}N(N-1)$

Sketch of the AKLT construction

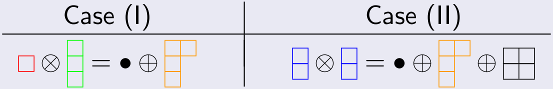




Two-site Hilbert space and Hamiltonian



Auxiliary layer

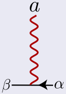


How to construct the projectors?

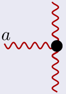
Introducing birdtracks

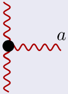
[Cvitanovic]

Fundamental rep

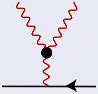
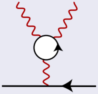
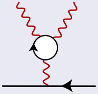
$(T^a)_{\alpha\beta} =$ 

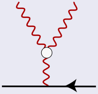
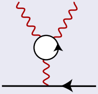
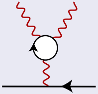
Adjoint rep

$S^a =$ 

$= -$ 

Graphical representation of structure constants

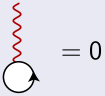
$f^{ab}{}_c\, T^c =$  $=$  $-$ 

$d^{ab}{}_c\, T^c =$  $=$  $+$ 

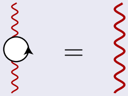
How to construct the projectors?

Graphical representation of some properties

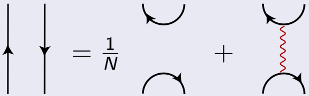
Tracelessness



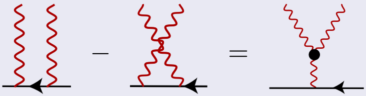
Normalization



Decomposition of unity



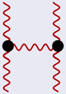
Commutation relations



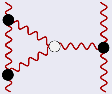
How to construct the projectors?

Graphical representation of 2-site spin interactions

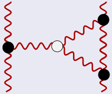
$$-S_1 \cdot S_2 =$$



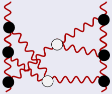
$$d_{abc} S_1^a S_1^b S_2^c =$$



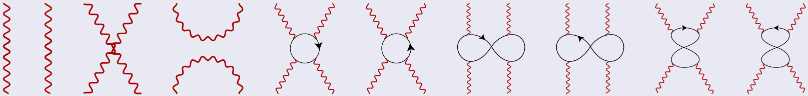
$$d_{abc} S_1^a S_2^b S_2^c = -$$



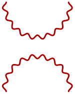
$$d_{abc} d_{efg} S_1^a S_1^f S_1^g S_2^e S_2^b S_2^c =$$



Basis of invariant operators

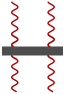
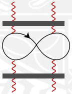


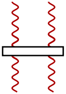
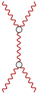
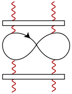
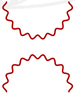
Examples of projectors

$\mathbb{P}_{\bullet} = \frac{1}{N^2-1}$ 

$\mathbb{P}_A = \frac{1}{2N}$ 

$\mathbb{P}_S = \frac{N}{2(N^2-4)}$ 

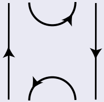
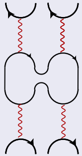
$\mathbb{P}_{A_1} = \frac{1}{2} \Bigg\{$  $+ \frac{1}{2N}$  $+$  $\Bigg\}$

$\mathbb{P}_{S_1} = \frac{1}{2} \Bigg\{$  $+ \frac{1}{2(N-2)}$  $-$  $- \frac{1}{N(N-1)}$  $\Bigg\}$

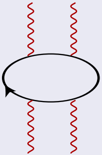
A universal AKLT Hamiltonian

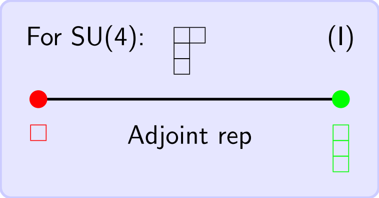
Hamiltonian on the auxiliary layer

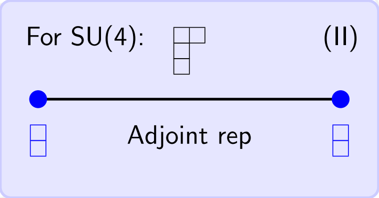
[Roy,TQ]

$$h_{\text{aux}} = \mathbb{I} - \frac{1}{\dim(\mathcal{B})}$$  $$\Rightarrow \quad \mathbb{P}_{\text{phys}} h_{\text{aux}} \mathbb{P}_{\text{phys}} = \mathbb{I}_{\text{phys}} -$$ 

Projection to the physical level

$$\tilde{H}_{\text{2-site}} = \mathbb{I} - \frac{1}{\dim(\mathcal{B})}$$  $\Rightarrow$ generally not equal weight superposition of projectors (but can be adjusted)





AKLT Hamiltonians for PSU(4)

[Roy, TQ]

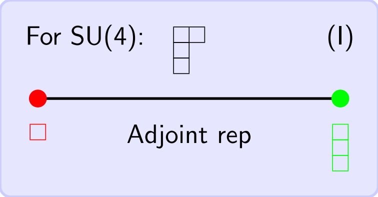
Case (I):

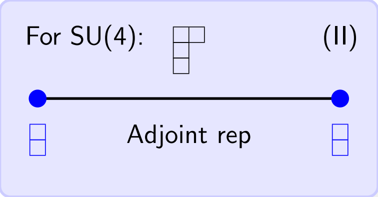
$$H_{2\text{-site}} = \mathbb{I} - \frac{1}{56}\,\mathbb{C}_A - \frac{1}{896}\,\mathbb{K} + \frac{13}{210}\,\vec{S}_1 \cdot \vec{S}_2 - \frac{17}{840}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{420}\,(\vec{S}_1 \cdot \vec{S}_2)^3 + \frac{1}{1680}\,(\vec{S}_1 \cdot \vec{S}_2)^4$$

Case (II):

$$H_{2\text{-site}} = \mathbb{I} - \frac{1}{128}\,\mathbb{K} + \frac{31}{20}\,\vec{S}_1 \cdot \vec{S}_2 - \frac{7}{40}\,(\vec{S}_1 \cdot \vec{S}_2)^2 - \frac{1}{5}\,(\vec{S}_1 \cdot \vec{S}_2)^3 - \frac{3}{160}\,(\vec{S}_1 \cdot \vec{S}_2)^4$$

Summary





Results

[Duivenvoorden,TQ] [Roy,TQ]

The AKLT Hamiltonians feature higher-order Casimir operators, e.g. terms like

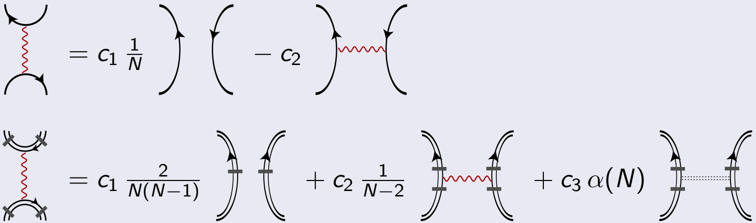
$$\mathbb{C}_A = d_{abc}\big(S_1^a S_1^b S_2^c - S_1^a S_2^b S_2^c\big) \qquad \text{or} \qquad \mathbb{K} = d_{abc} d_{def} S_1^a S_1^b S_1^d S_2^c S_2^e S_2^f$$

where d_{abc} is the completely symmetric rank-3 tensor

Transfer matrix and correlation lengths

Step 1: Write the transfer matrix as a sum over projectors

[Orus,Tu] [Roy,TQ]



Step 2: The largest two eigenvalues determine the correlation length

[Orus,Tu] [Roy,TQ]

$$\xi_{\square} = 1/\ln(N^2 - 1) \qquad \xi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = 1/\ln\left[\frac{N^2-2N-4}{2(N+1)(N-2)}\right]$$

Some open issues and conjectures (adjoint representation)

Open issue

Which phase is realized in the SU(N) Heisenberg model?

Conjecture for the case SU(4) [Roy,TQ]

The Heisenberg model realizes the class [2] Haldane-SPT phase

Expectation for SU(odd) [Roy,TQ]

The Heisenberg model realizes a superposition of two Haldane-SPT phases

Conjecture on topological phase transitions [Roy,TQ]

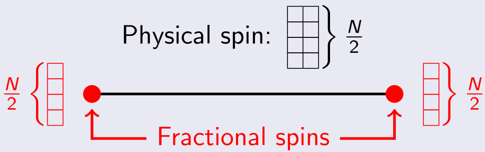
2nd order topological phase transitions are generically described by $SU(N)_1$ for odd N and by $SU(N)_2$ for even N (absence of a $\mathbb{Z}_N$-anomaly). Fine-tuned transitions may lead to larger values of the level

Other SU(N) setups

Realization of non-trivial topological phases (even N)

Potential symmetry fractionalization

[Nonne,Moliner,Capponi,Lecheminant,Totsuka]



Topological class $\left[\frac{N}{2}\right]$

Conjecture (so far only verified for $N = 4$)

These non-trivial topological phases are realized in the Heisenberg model

[Nonne,Moliner,Capponi,Lecheminant,Totsuka] [Bois,Capponi,Lecheminant,Moliner,Totsuka] [Tanimoto,Totsuka] [Weichselbaum,TQ] to appear

Symmetry group	SU(2)	SU(4)	SU(6)	SU(8)	SU(10)	SU(N)
Dim of physical spins	3	20	175	1 764	19 404	$\frac{n!(n+1)!}{[(n/2)!(n/2+1)!]^2}$
Dim of boundary spins	2	6	20	70	252	$\frac{n!}{(n/2)!^2}$

Corresponding AKLT Hamiltonians

SU(2) "AKLT" Hamiltonian

[Affleck,Kennedy,Lieb,Tasaki]

$$H_{\text{2-site}} = \frac{2}{3} + \vec{S}_1 \cdot \vec{S}_2 + \frac{1}{3}(\vec{S}_1 \cdot \vec{S}_2)^2$$

SU(4) "AKLT" Hamiltonian

[Nonne,Moliner,Capponi,Lecheminant,Totsuka]

$$H_{\text{2-site}} = \frac{8}{3} + \vec{S}_1 \cdot \vec{S}_2 + \frac{13}{108}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{216}(\vec{S}_1 \cdot \vec{S}_2)^3$$

SU(6) "AKLT" Hamiltonian

[Tanimoto,Totsuka] [Weichselbaum,TQ] to appear

$$H_{\text{2-site}} = \frac{504}{127} + \vec{S}_1 \cdot \vec{S}_2 + \frac{47}{508}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{17}{4572}(\vec{S}_1 \cdot \vec{S}_2)^3 + \frac{1}{18288}(\vec{S}_1 \cdot \vec{S}_2)^4$$

SU(8) and above

[Weichselbaum,TQ] to appear

Hamiltonians involve operators beyond powers of $\vec{S}_1 \cdot \vec{S}_2$

Conclusions

Conclusions

Summary

[Roy,TQ]

SU(N) spin chains exhibit various types of Haldane-SPT phases, most of them chiral. The construction of parent Hamiltonians for the adjoint representation is by no means straightforward but can be achieved for general N using birdtracks

Features

- They can exhibit different types of protected gapless edge modes
- For SU(4) we have a complete realization of Haldane-SPT phases

Related topics not covered in this talk

- Non-local string order, hidden symmetry breaking, etc.
- Multifractality of topological phase transitions
- Realization in alkaline-earth Fermi gases ($\rightarrow$ Talk by Lecheminant)