FQHE in higher Chern number bands

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TCM

T. Scaffidi & GM, Phys. Rev. Lett. (2012)
GM & N.R. Cooper, Phys. Rev. Lett. (2009, 2015)
T. Jackson, GM, R. Roy, Nature Communications (2015)

Entanglement in Strongly Correlated Systems

Centro de Ciencas de Benasque Pedro Pascual

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Trinity Hall Cambridge
The Leverhulme Trust
The Royal Society
University of Cambridge

Overview

Motivation & Background

- Why magnetic fields are exciting: fractional quantum Hall effect
- Emulating the effects of magnetic fields without external fields.
- Berry curvature and the Chern number

Fate of the Fractional Quantum Hall Effect in 'Artificial Fields'

- Q1: Can FQHE states be stabilised?
- Q2: Are they the same states as in the continuum?
- Q3: How does the band geometry influence their stability?

New Features of the FQHE in General Topological Bands

Chern numbers |C|>1:

- prediction of series of states at filling $\nu = \frac{r}{r|Ck|+1}$
- numerical evidence for those states

Topological Order in the Quantum Hall Effect

- a macroscopic quantum phenomenon: magnetoresistance in 2D electron gases

Where?

- in semiconductor hetero-structures with clean two-dimensional electron gases
- at low temperatures (~0.1K) and in strong magnetic fields

$$k_B T \ll \hbar\omega_c = \hbar eB/m_e$$

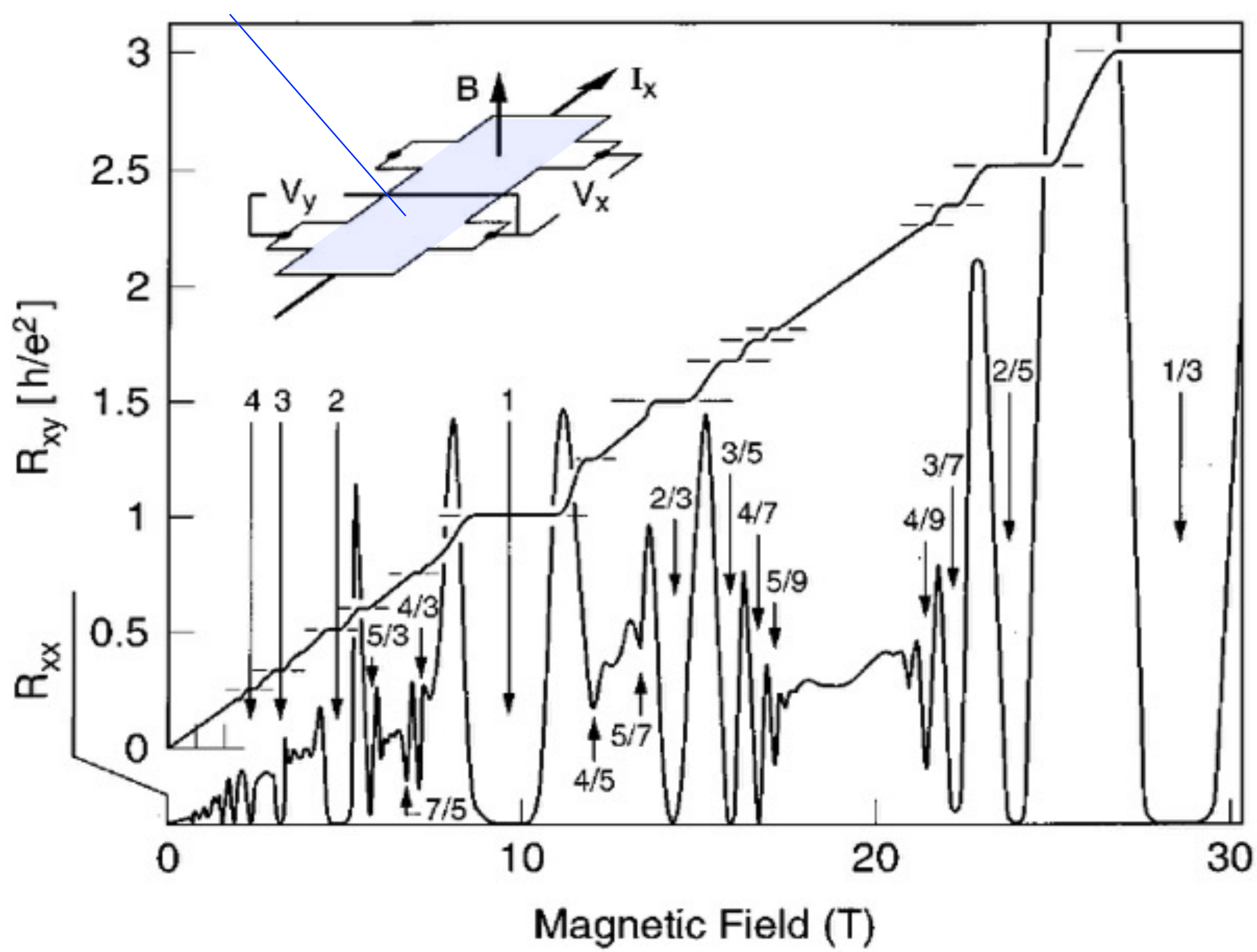
What?

- plateaus in Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

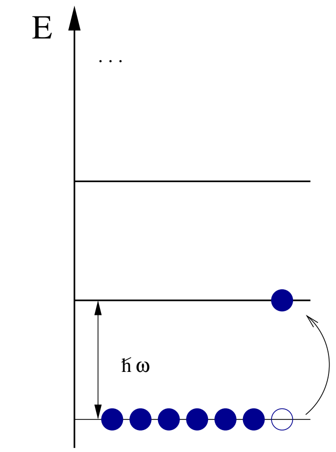
- simultaneously: (near) zero longitudinal resistance

2D electron gas



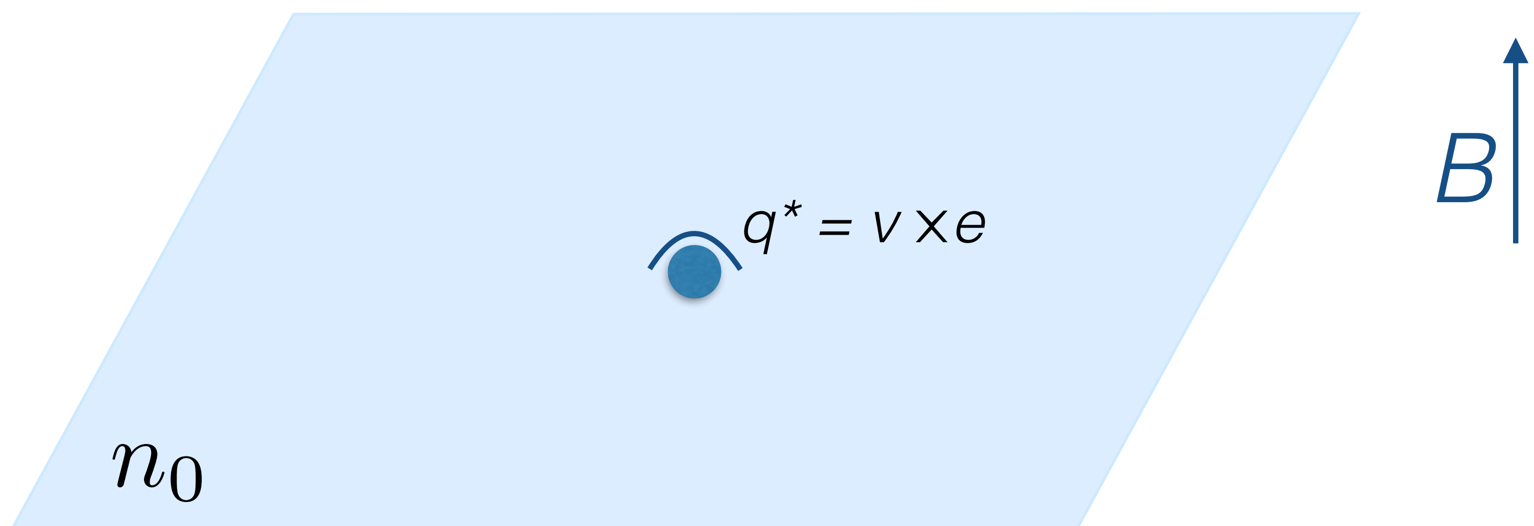
- different topologically ordered phases at each Hall plateau
- supporting fractionalized abelian and non-Abelian excitations

Properties of Quantum Hall Liquids



- quantum Hall plateaus ➜ quantum liquids:
  - no long-range order
  - preferred density: $n_0 = \nu d_{\rm LL}$ $\quad d_{LL} = eB/h$

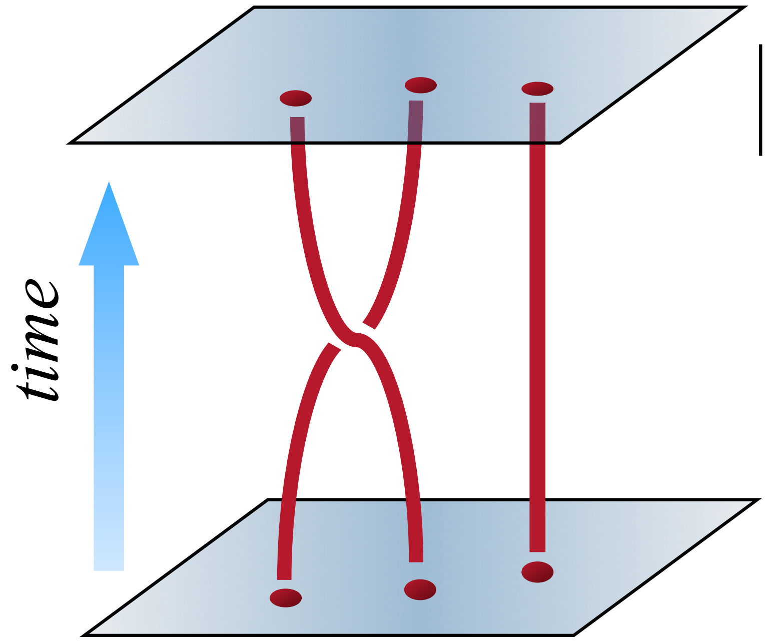
quantum number ν, observed to take integer or simple fractional values



- changes in density = defects: <u>localized</u> quasiparticles

> - quasiparticles have *<u>fractional charge</u>* & *<u>fractional statistics</u>*

Fractional statistics - Anyons and Non-Abelions



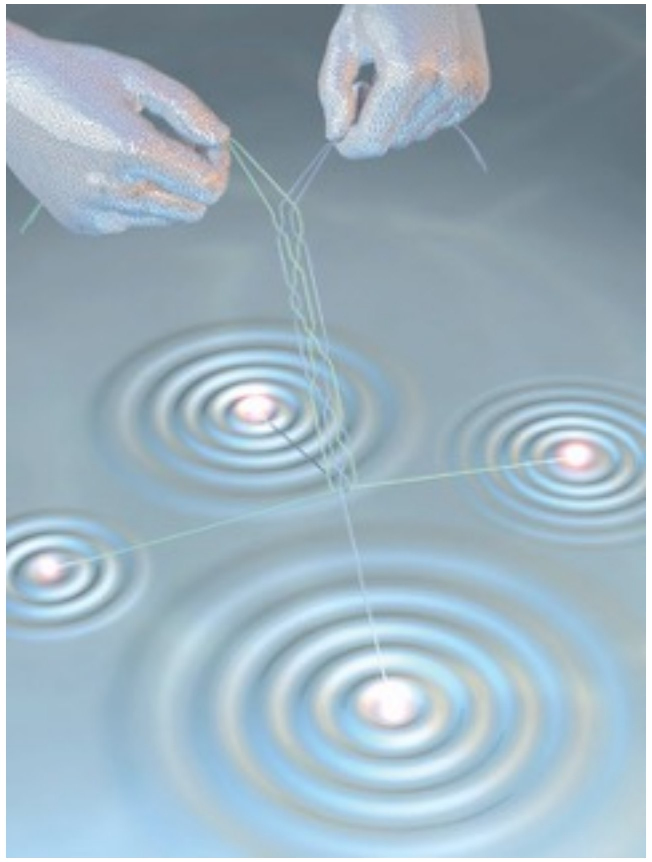
$$|\psi_f\rangle = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Non-Abelian Anyons

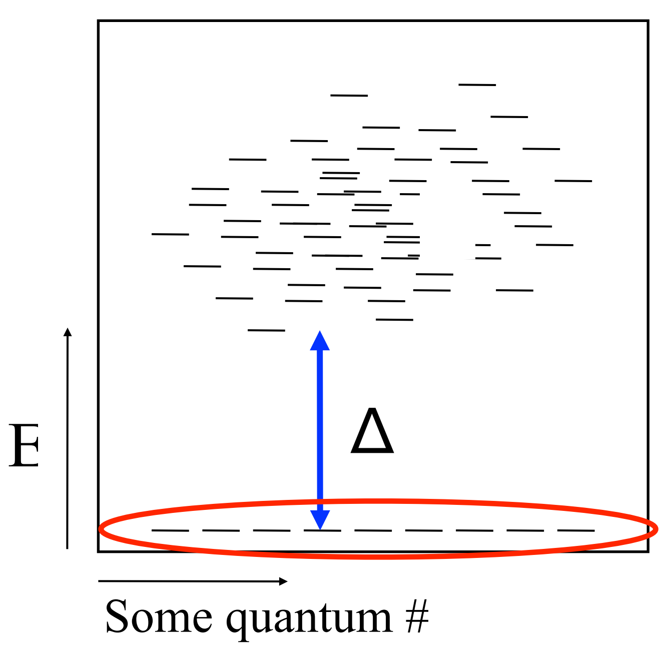
non-Abelian representation of the braid group

$$|\psi_i\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Topological Quantum Computing



- *interactions* drive *emergent realization* as quasi-particle states in quantum Hall systems!



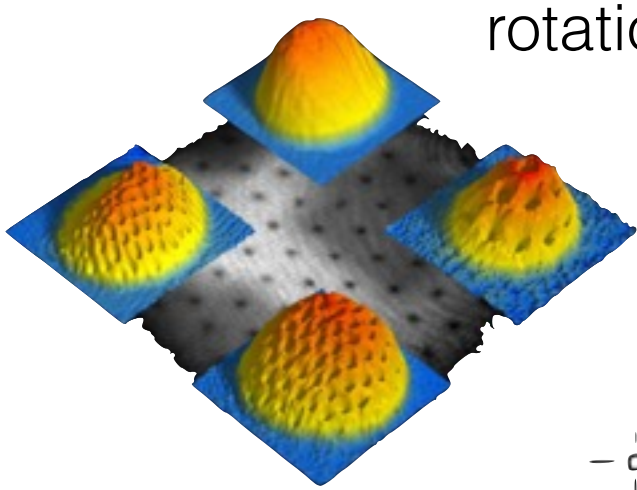
- manipulations of quasiparticles could provide the basis for a quantum computer that is protected from errors!

Quantum Hall effect without magnetic fields

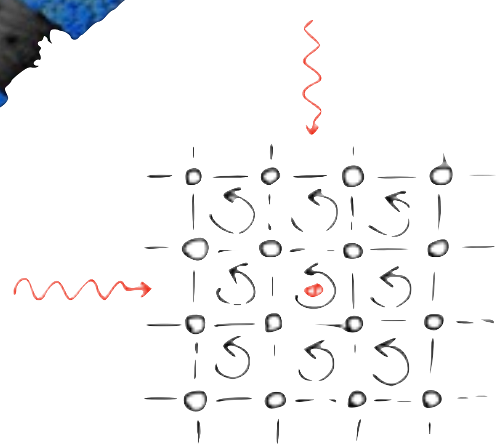
The fractional quantum Hall effect is observed under extreme conditions
- strong magnetic fields of several Tesla
- very low temperatures
- clean / high mobility semiconductor samples

Many different opportunities for emulating magnetic fields:

1. Cold Atomic Gases

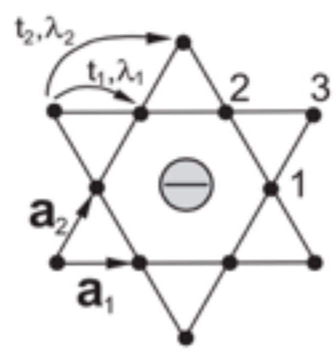


rotation



laser assisted hopping

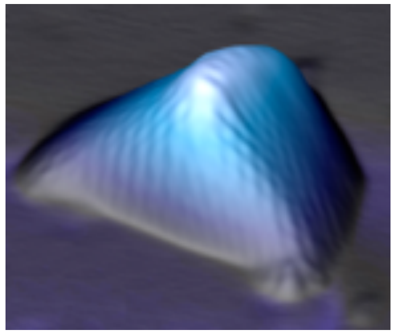
2. Solid State



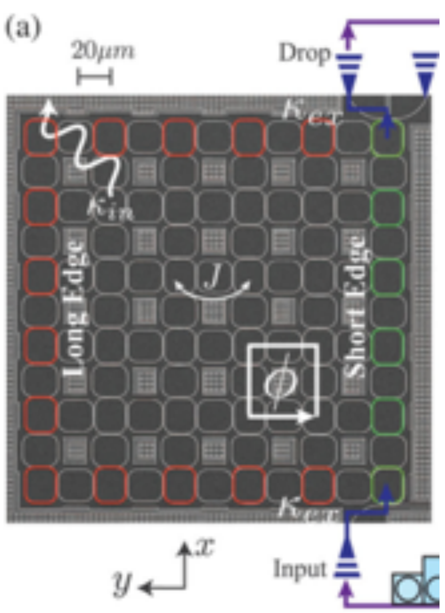
Fe_3Sn_2
(proposed)

spin orbit coupling

strain



3. Photons

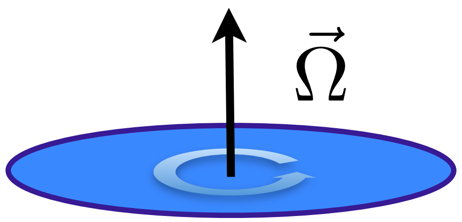


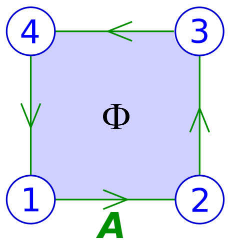
Si waveguides

Ideas behind these strategies for simulating magnetic fields

- Simulate a physical effect that a magnetic field B exerts particle of charge q

Signature		Simulated by
Lorentz Force	$F_L = q\,\vec{v} \times \vec{B}$	Coriolis Force in Rotating System
Aharonov-Bohm Effect	$\Psi \propto \exp\left\{ i\frac{q}{\hbar}\int \vec{A}\cdot d\vec{\ell}\right\}$	Complex Hopping Amplitudes A in Optical Lattices
Berry Curvature of Landau levels		Same physics seen in reciprocal space…



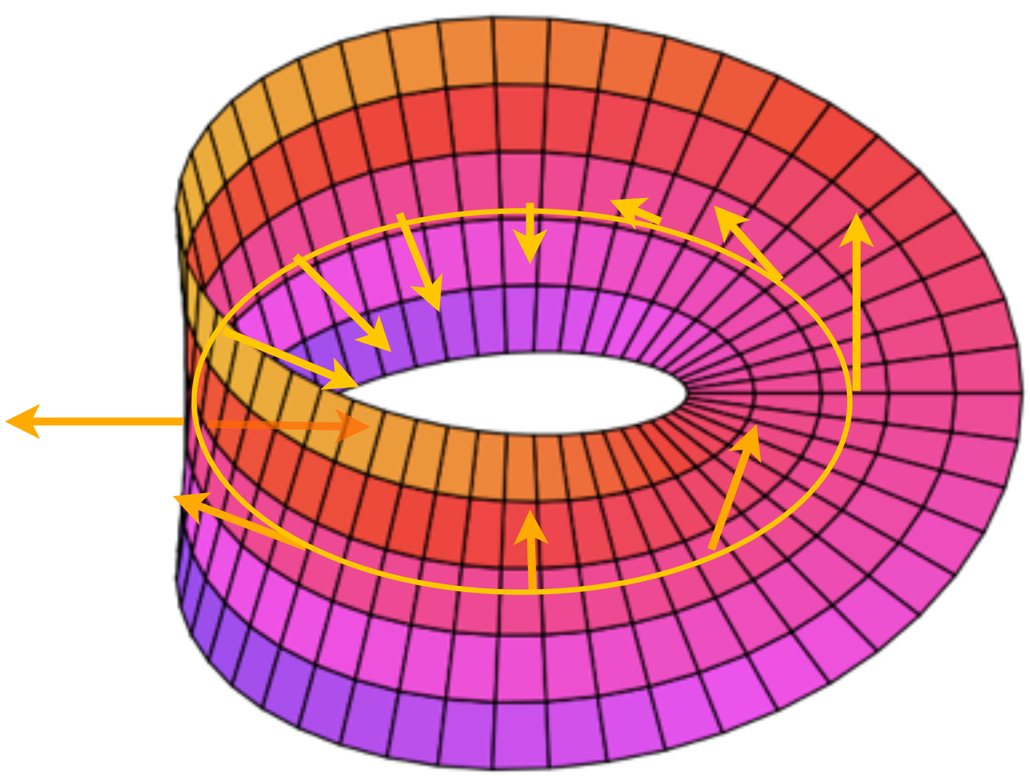


$$\sum_{\square} A_{\alpha\beta} = 2\pi n_\phi$$

Landau-levels as a topological band-structure

- Can we see in which way Landau-levels are special, just by looking at the wavefunctions?

Start with an analogy:



Recipe for calculating the twist in this Möbius band:

- choose a closed path around the surface
- construct normal vector to the surface at points along the curve
- add up the twist angle while moving along this contour

Calculating the Berry phases in reciprocal space

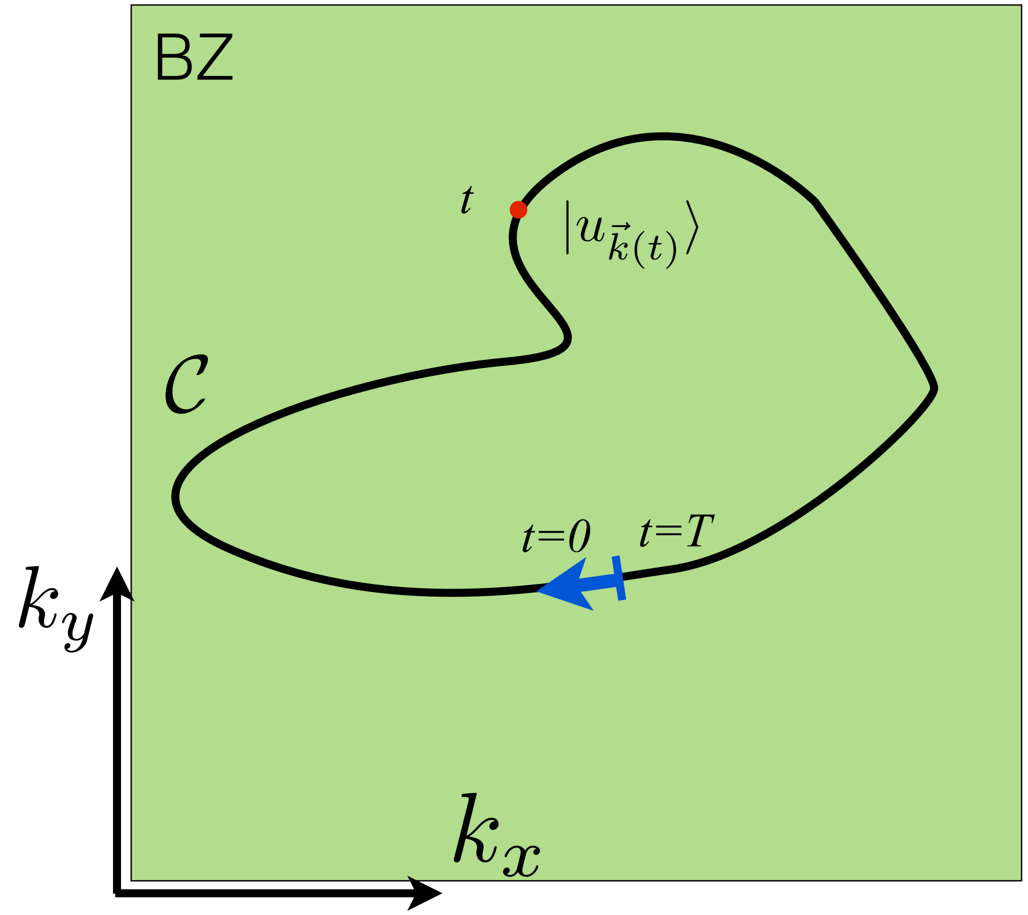
Michael Berry (1984)

Calculate how wavefunction evolves while moving adiabatically through curve C : $\mathbf{k}(t)$, $t=0 \ldots T$

Local basis $\quad \tilde{\mathcal{H}}|u_{\vec{k}}\rangle = \epsilon_{\vec{k}}|u_{\vec{k}}\rangle$

Phase evolution has two components:

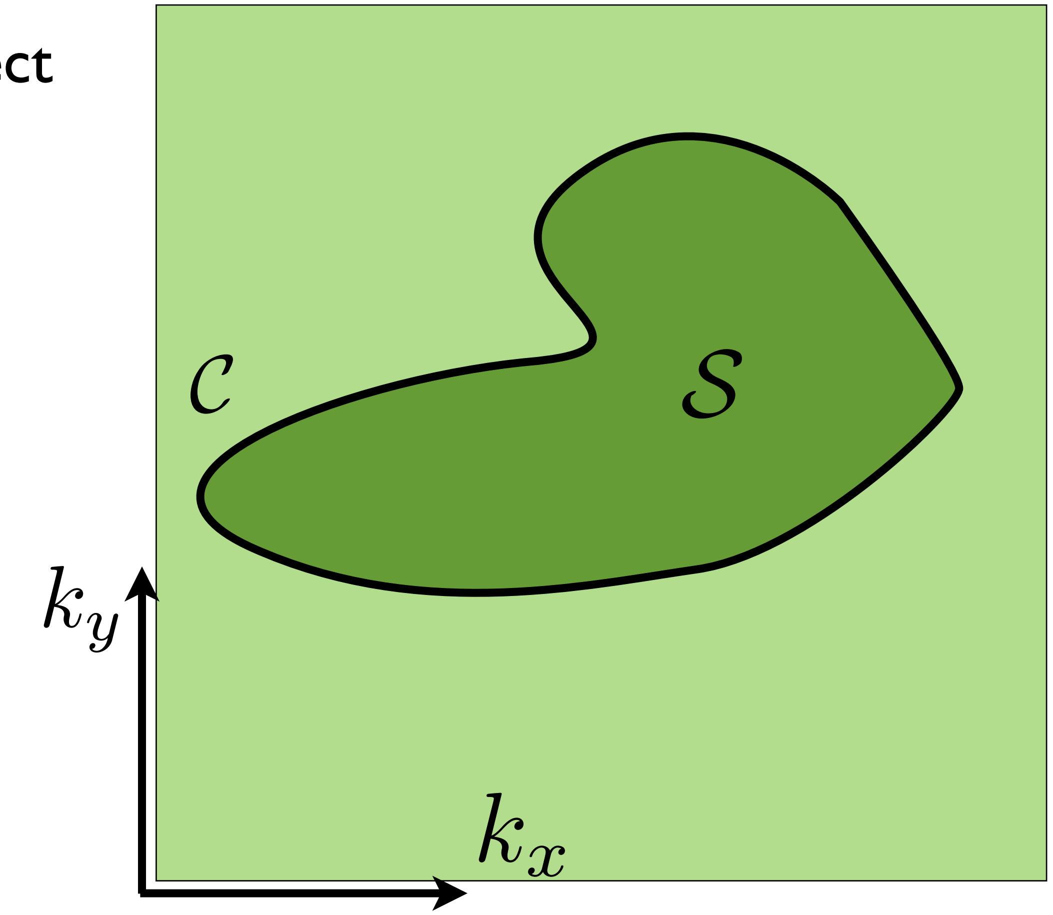
$$|U(t)\rangle = \underbrace{\exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon_{\vec{k}(t')}dt'\right\}}_{\text{dynamical time evolution}} \underbrace{\exp\{i\gamma(t)\}}_{\text{'twist'}} |u_{\vec{k}(t)}\rangle$$



Berry curvature and Chern number

Geometrical phase analogous to Aharonov-Bohm effect

$$\gamma(\mathcal{C}) = i\int_{\mathcal{C}} \langle u_{\vec{k}}|\frac{d}{d\vec{k}}|u_{\vec{k}}\rangle d\vec{k} \equiv \int_{\mathcal{C}} \vec{\mathcal{A}}(\vec{k})d\vec{k}$$



Effective 'vector potential' called *Berry connection*

$$\vec{\mathcal{A}}(\vec{k}) = i\int_{\mathrm{UC}} u_{\vec{k}}(\vec{r})^*\vec{\nabla}_k u_{\vec{k}}(\vec{r})\, d^2r$$

Using Stokes' theorem:

$$\gamma(\mathcal{C}) = \int_{\mathcal{C}} \vec{\mathcal{A}}(\vec{k})d\vec{k} = \int_{\mathcal{S}} \vec{\nabla}_k \times \vec{\mathcal{A}}(\vec{k})d\vec{\sigma}$$

$\mathcal{C} = \partial\mathcal{S}$

Berry curvature: $\vec{\mathcal{B}} = \vec{\nabla}_k \times \vec{\mathcal{A}}(\vec{k})$ is a property of the band eigenfunctions, only!

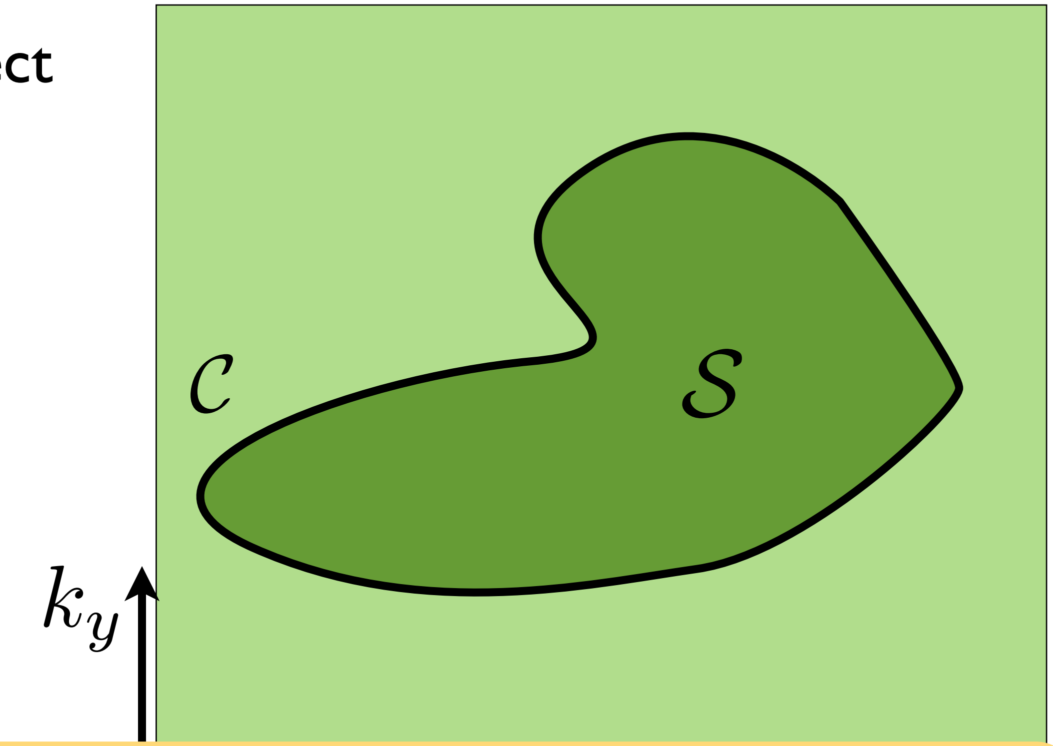
Chern number: $C = \frac{1}{2\pi}\int_{BZ} d^2\mathbf{k}\,\mathcal{B}(\mathbf{k})$ takes only integer values!

- Chern number provides classification of all possible single-particle bands (class A)

Berry curvature and Chern number

Geometrical phase analogous to Aharonov-Bohm effect

$$\gamma(\mathcal{C}) = i\int_{\mathcal{C}} \langle u_{\vec{k}}|\frac{d}{d\vec{k}}|u_{\vec{k}}\rangle d\vec{k} \equiv \int_{\mathcal{C}} \vec{\mathcal{A}}(\vec{k})d\vec{k}$$



Effective 'vector potential' called *Berry connection*

$$\vec{\mathcal{A}}(\vec{k}) = i\int_{\mathrm{UC}} u_{\vec{k}}(\vec{r})^* \vec{\nabla}_k u_{\vec{k}}(\vec{r})\, d^2r$$

integral over real-space, so it does not matter in what physical space the system 'lives'

in reciprocal space, only the change of the scalar product on that space matters

Berry curvature: $\vec{\mathcal{B}} = \vec{\nabla}_k \times \vec{\mathcal{A}}(\vec{k})$ is a property of the band eigenfunctions, only!

Chern number: $C = \frac{1}{2\pi}\int_{BZ} d^2\mathbf{k}\,\mathcal{B}(\mathbf{k})$ topological: takes only integer values!

- Chern number provides *universal* classification of all possible single-particle bands (cl. A)

Q1: Do bands resembling Landau-levels support QH states?

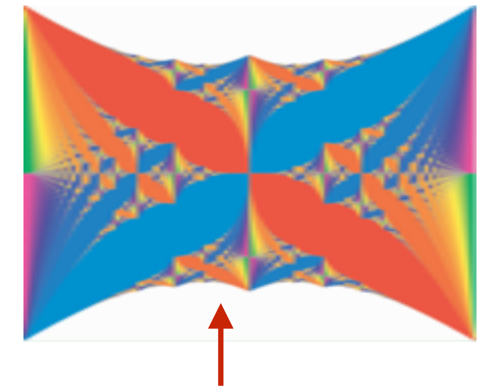
continuum

$$\mathcal{H} = \frac{1}{2m}|\mathbf{p} - e\mathbf{A}|^2 + \hat{V}$$

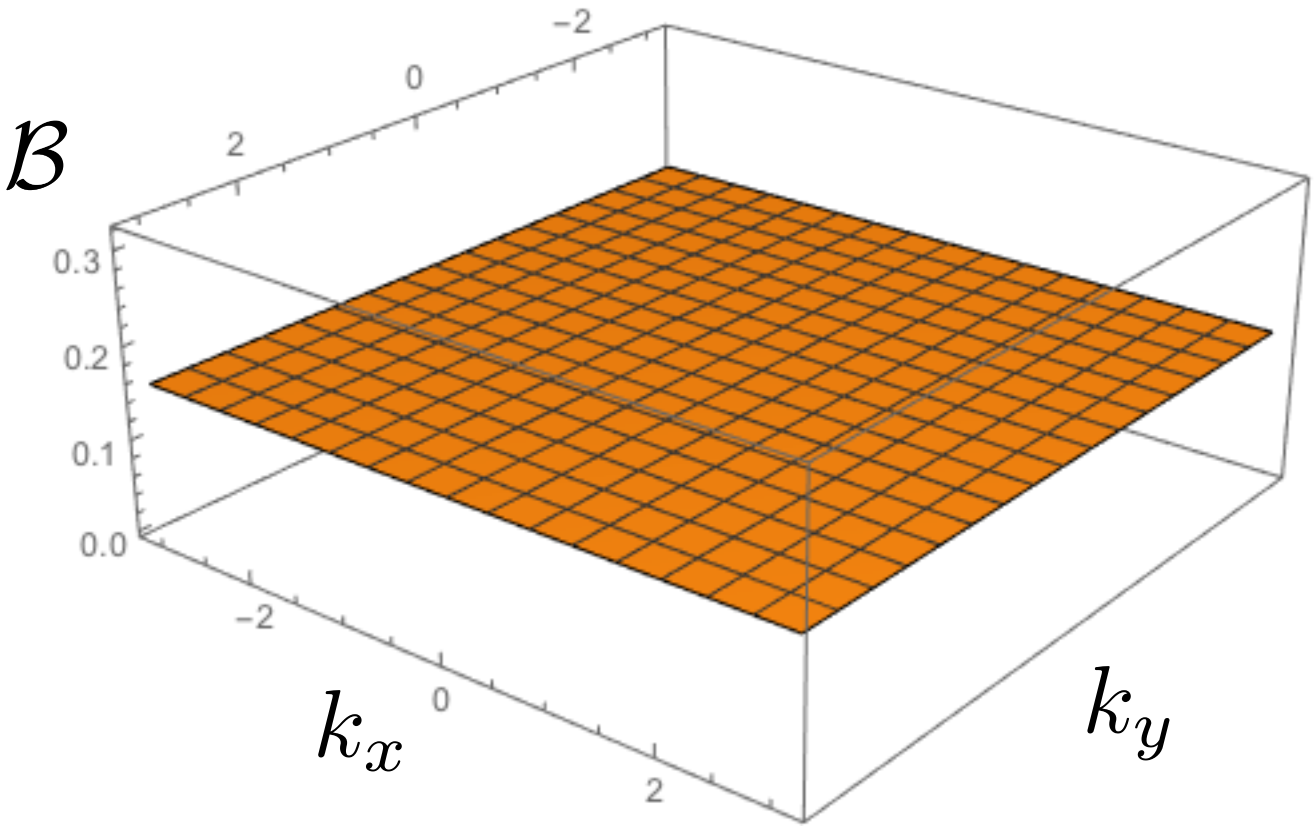
$$\mathbf{A} = B_z x \mathbf{e}_y$$

lattice

$$\mathcal{H} = -J \sum_{\langle\alpha,\beta\rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c.\right] + \sum V_{ij}\hat{n}_i\hat{n}_j$$

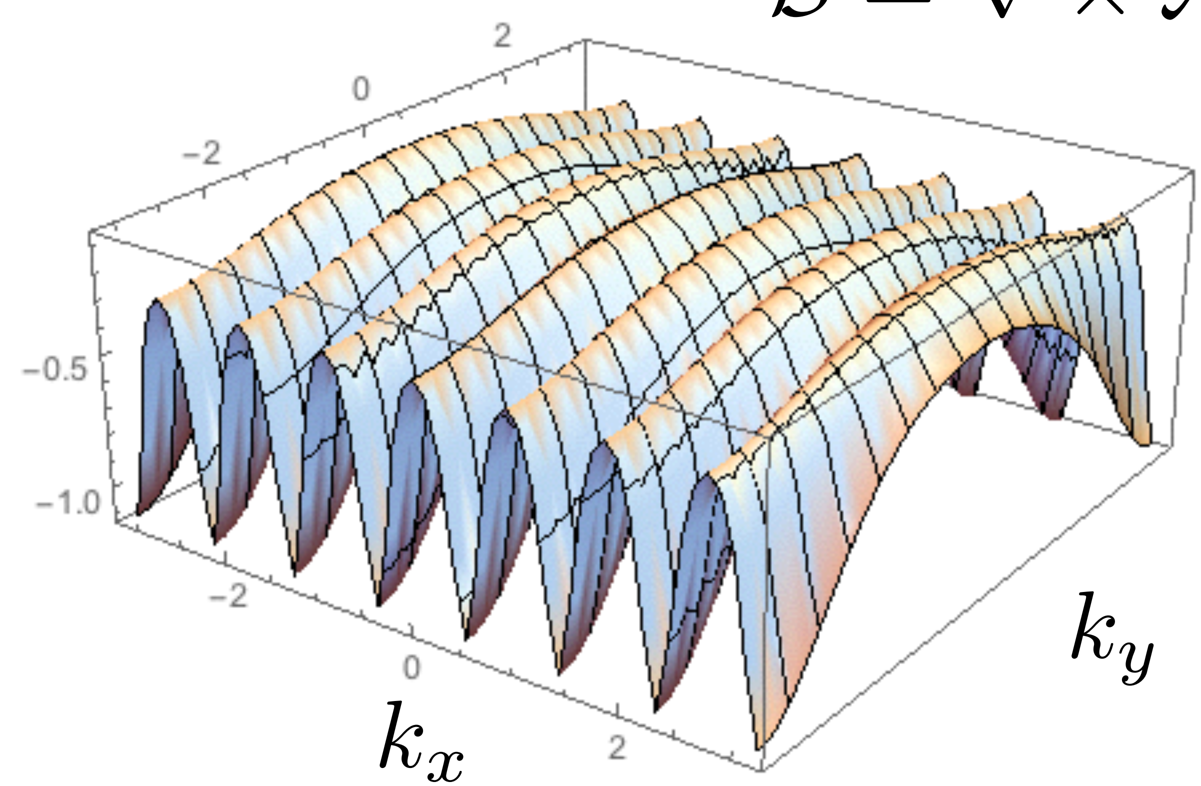


Berry curvature in a Landau-Level: flat



Berry curvature in the Hofstadter model

$$\mathcal{B} = \nabla \times \mathcal{A}$$



Fractional Quantum Hall Effect in Periodic Potentials

- quantized Hall response in *partially* filled bands?
- THEORY: Kol & Read (1993)

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \sum V_{ij} \hat{n}_i \hat{n}_j$$

PHYSICAL REVIEW B VOLUME 48, NUMBER 12 15 SEPTEMBER 1993-II

Fractional quantum Hall effect in a periodic potential

A. Kol and N. Read*

Departments of Physics and Applied Physics, P. O. Box 2157, Yale University, New Haven, Connecticut 06520

(Received 28 May 1993)

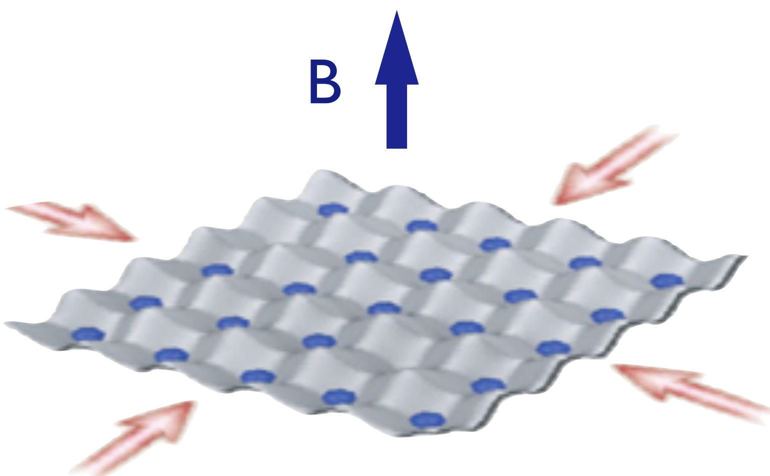
The fractional quantum Hall effect in a periodic potential or modulation of the magnetic field is studied by symmetry, topological, and Chern-Simons field-theoretic methods. With periodic boundary conditions, the Hall conductance in a finite system is known to be a fraction whose denominator is the degeneracy of the ground state. We show that in a finite system, translational symmetry predicts a degeneracy that varies periodically with system size and equals 1 for certain commensurate cases which we argue are physically representative. However, this analysis may overlook gaps due to finite-size effects that vanish in the thermodynamic limit. This possibility is addressed using a fermionic Chern-Simons field theory in the mean-field approximation. In addition to solutions describing the usual Laughlin or Jain states whose properties are only weakly modified by the periodic background, we also find solutions whose existence depends on the presence of the background. In these incompressible states, the Hall conductance is a fraction not equal to the filling factor, and its denominator is the same as that of the fractional charge and statistics of the elementary quasiparticle excitations.

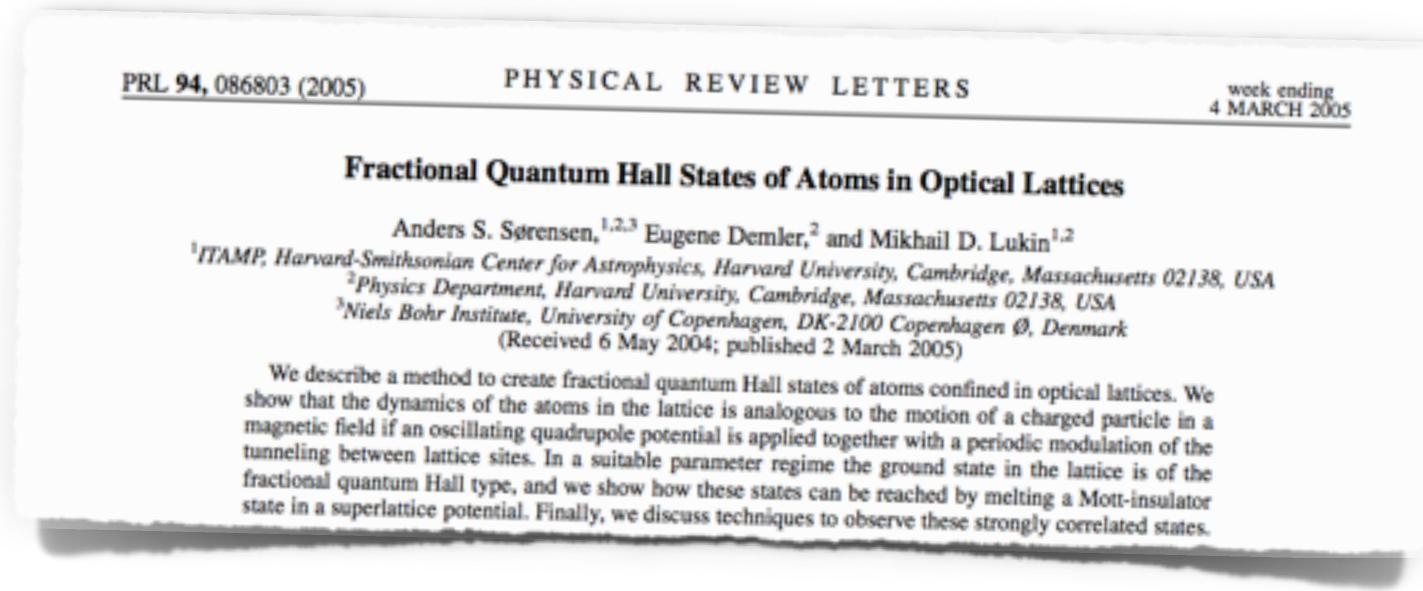
- Confirmations for such states?

Fractional Quantum Hall on lattices: Numerical Evidence

- interest in cold atom community 2000’s:

- realisations of tight-binding models with complex hopping from light-matter coupling:



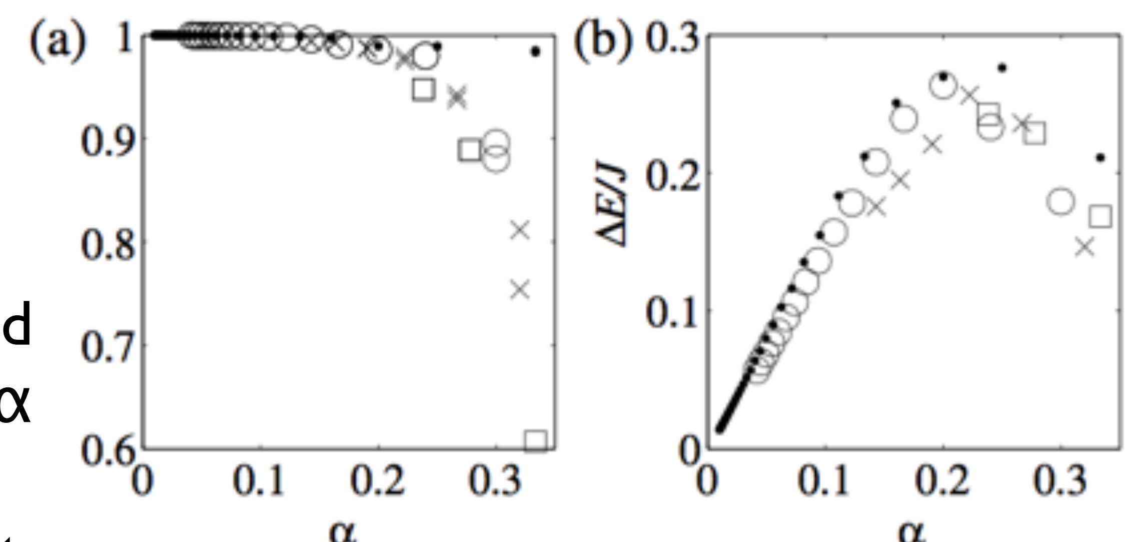

PRL 94, 086803 (2005) PHYSICAL REVIEW LETTERS week ending 4 MARCH 2005

Fractional Quantum Hall States of Atoms in Optical Lattices

Anders S. Sørensen,[1,2,3] Eugene Demler,[2] and Mikhail D. Lukin[1,2]
[1]ITAMP, Harvard-Smithsonian Center for Astrophysics, Harvard University, Cambridge, Massachusetts 02138, USA
[2]Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA
[3]Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark
(Received 6 May 2004; published 2 March 2005)

We describe a method to create fractional quantum Hall states of atoms confined in optical lattices. We show that the dynamics of the atoms in the lattice is analogous to the motion of a charged particle in a magnetic field if an oscillating quadrupole potential is applied together with a periodic modulation of the tunneling between lattice sites. In a suitable parameter regime the ground state in the lattice is of the fractional quantum Hall type, and we show how these states can be reached by melting a Mott-insulator state in a superlattice potential. Finally, we discuss techniques to observe these strongly correlated states.

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha (\hat{n}_\alpha - 1)$$



flux density

- bosons with onsite U: many-body gap in the half-filled “synthetic Landau-level” persists to large flux density α

- correct GS degeneracy + good overlap with trial state

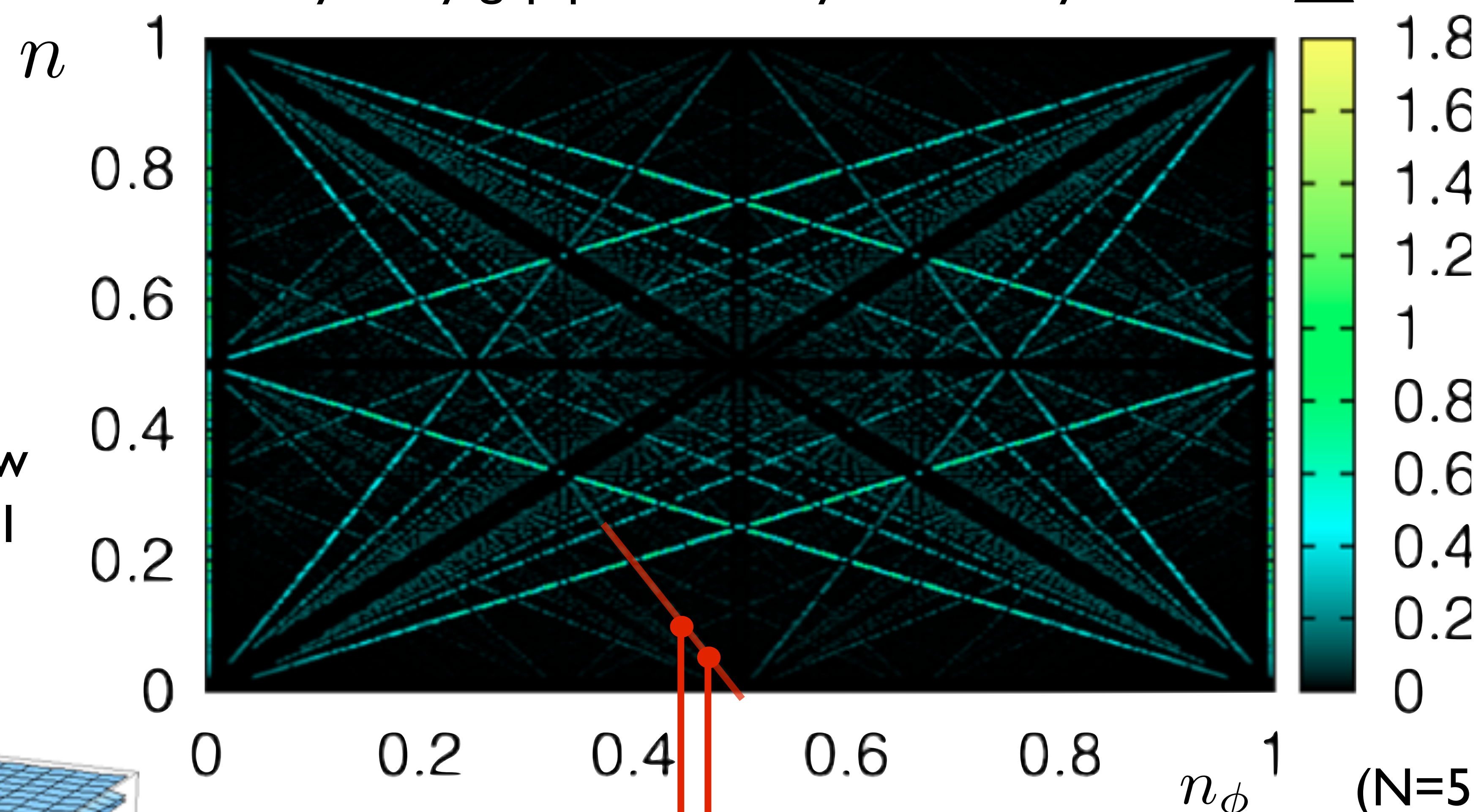
Fractional Quantum Hall on lattices with higher Chern-# bands

- bands of the Hofstadter model go *beyond* the continuum limit and support *new classes* of quantum Hall states

theory:
bosonic Hall states on the lattice

numerical verification
for what we would now call FCI states with ν=1
- C=2 band
- hardcore bosons

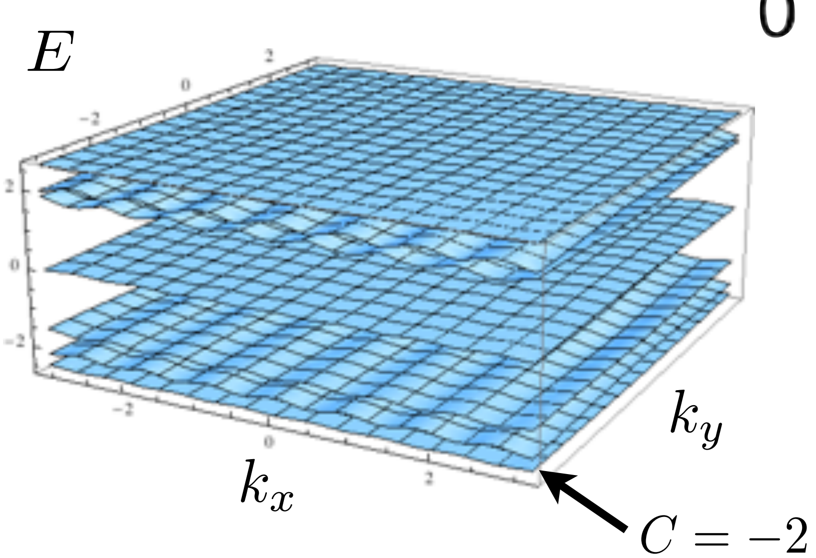
many-body gap predicted by CF theory



(N=5 particles)

$n = 1/9$: $\mathcal{O} = |\langle \Psi_{\mathrm{CF}} | \mathrm{GS} \rangle|^2 \simeq 0.46$

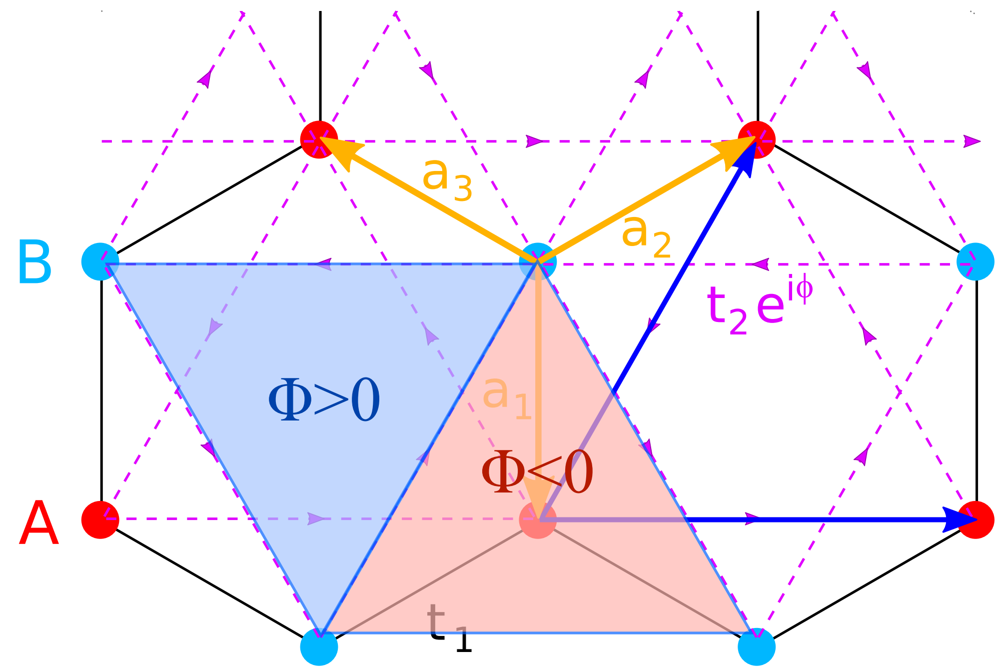
$n = 1/7$: $\mathcal{O} = |\langle \Psi_{\mathrm{CF}} | \mathrm{GS} \rangle|^2 \simeq 0.56$



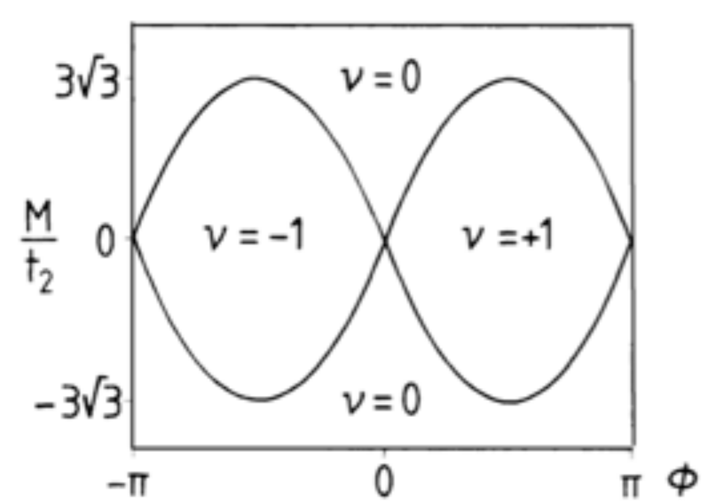
GM & NR Cooper, PRL 2009

Chern bands in more general tight binding models

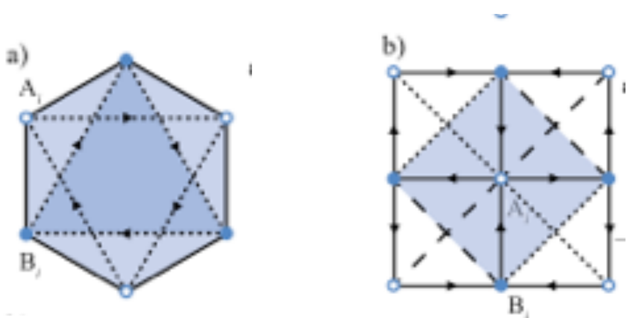
- Original proposal for IQHE without magnetic fields: Haldane (1988)



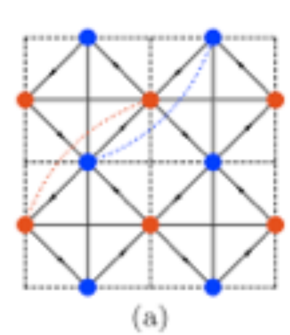
Chern numbers



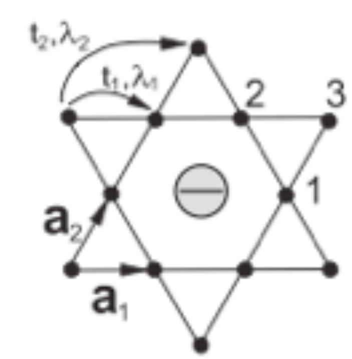
- 2011: FQHE expected in models with spin-orbit coupling + interactions



T. Neupert et al.



K. Sun et al.

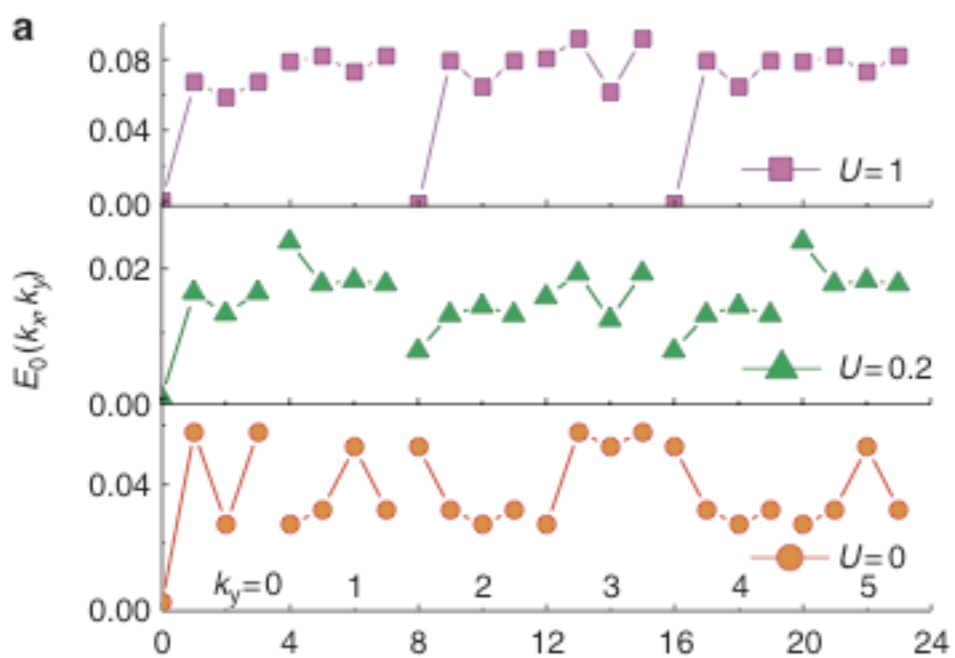


E. Tang et al.

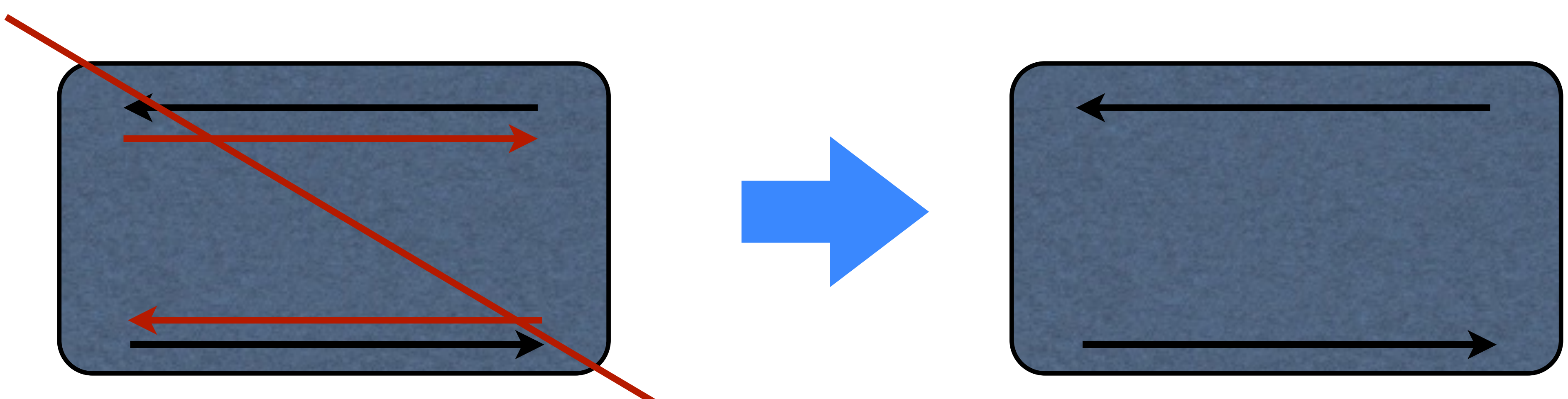
Numerical evidence: GS degeneracy, gap, PES,… D. Sheng; C. Chamon; N. Regnault & A. Bernevig, …

Numerical evidence for "Fractional Chern Insulators"

- existence of a gap & groundstate degeneracy [checkerboard lattice]
- Chern number of groundstate manifold



[D. Sheng]



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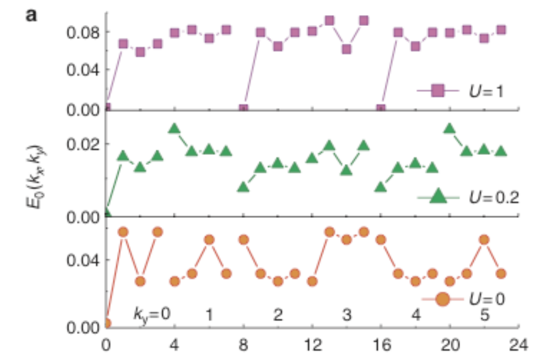
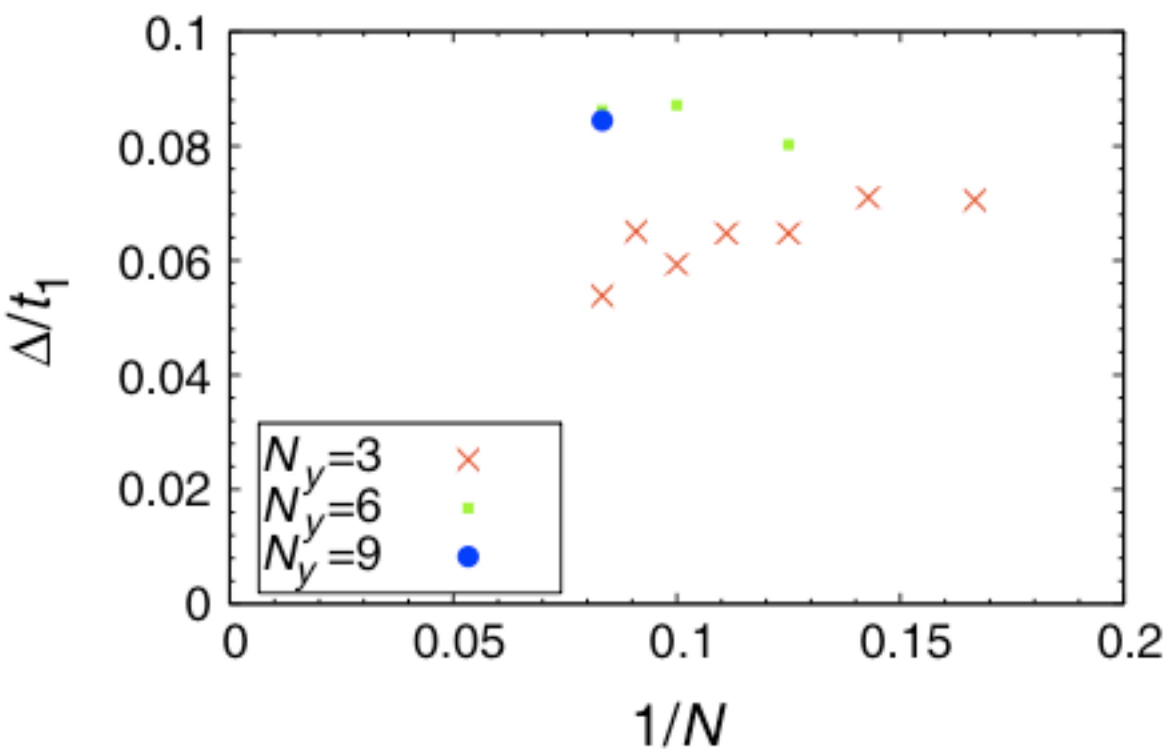
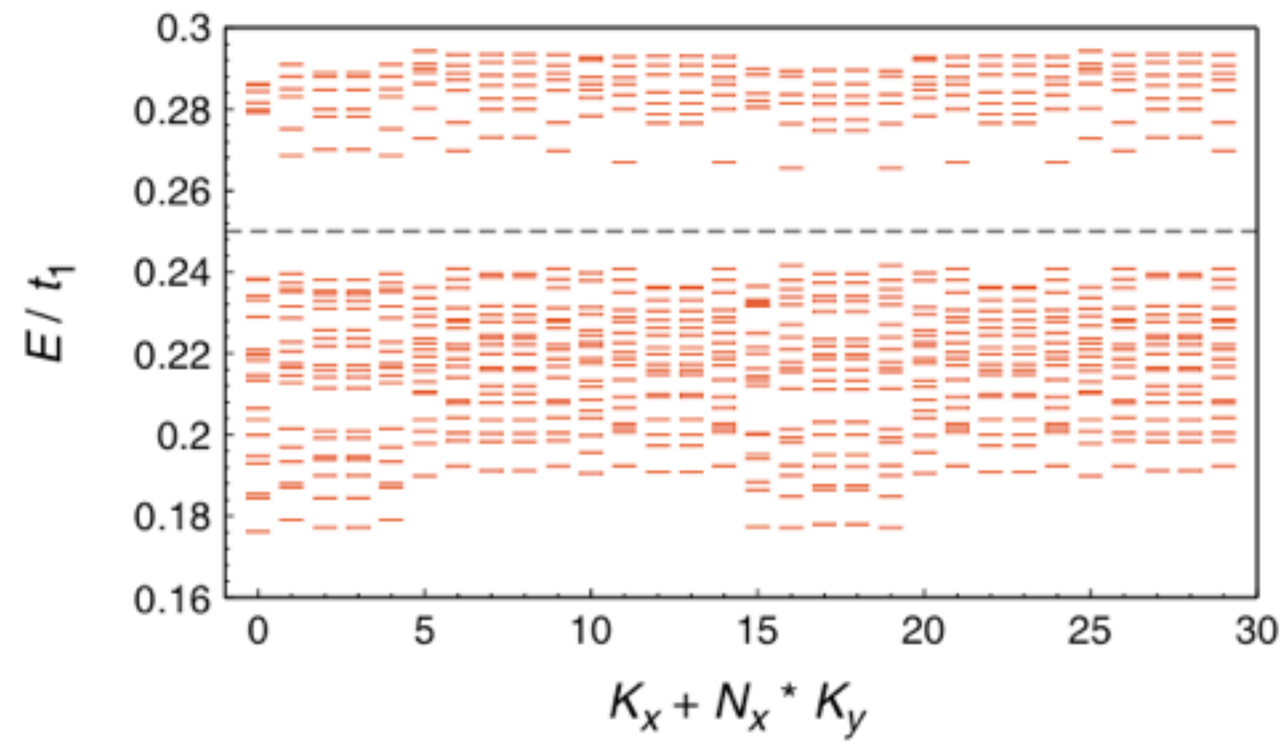


- Finite size scaling of gap



- Particle Entanglement Spectra : count of excitations matches FQHE (here - Laughlin state)

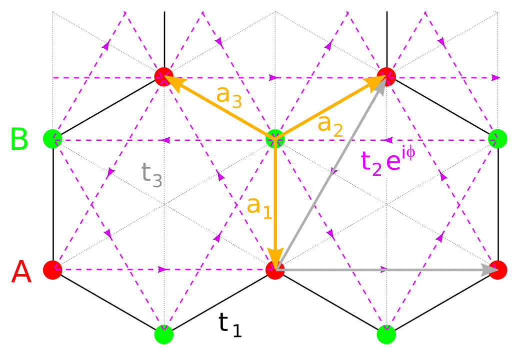


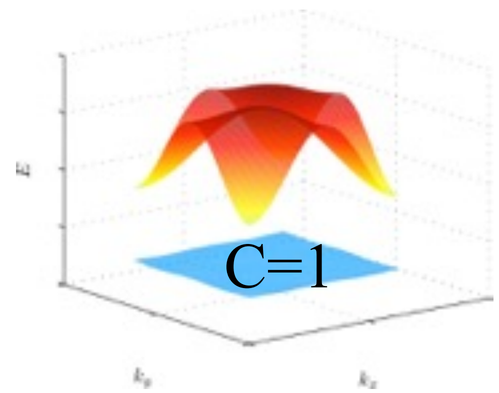
"Fractional Chern Insulators (FCI)" [PRX '11, PRB '12, N. Regnault & A. Bernevig]

- Strong numerical evidence for QHE like quantum liquids

Q2: Do Chern bands support the same phases as LLs?

- Are the quantum Hall states in general lattice models the same as in LL?





\+ Interactions = FQHE ?

- Analogy for topological order of many-body states:

- Topological order is invariant under continuous / adiabatic deformations!

- Approach: Continuously deform a fractional quantum Hall state into a fractional Chern insulator *without closing the gap*.

Adiabatic Continuation of QH liquids in different systems

- use Hilbert spaces with the same overall structure (based on Wannier states) to study the low-lying spectrum numerically (exact diagonalization)
- adiabatically deform many-body Hamiltonian of FQHE to a fractionally filled Chern band:

$$\mathcal{H}(x) = \frac{\Delta_{\mathrm{FCI}}}{\Delta_{\mathrm{FQHE}}}(1-x)\mathcal{H}^{\mathrm{FQHE}} + x\mathcal{H}^{\mathrm{FCI}}$$

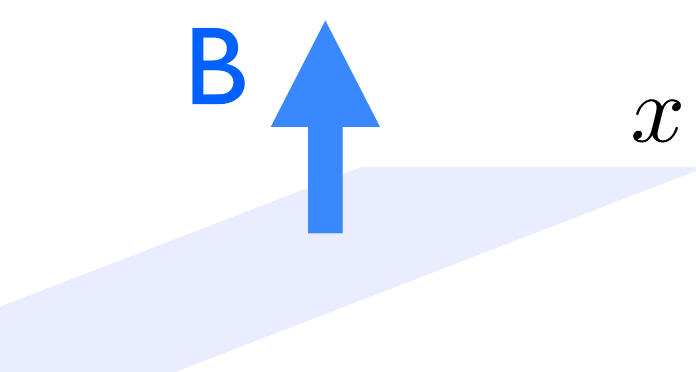
- E.g.: half-filled band for bosons & *contact repulsion*

FQHE of Bosons at

$\nu = 1/2$

continuum Landau-level

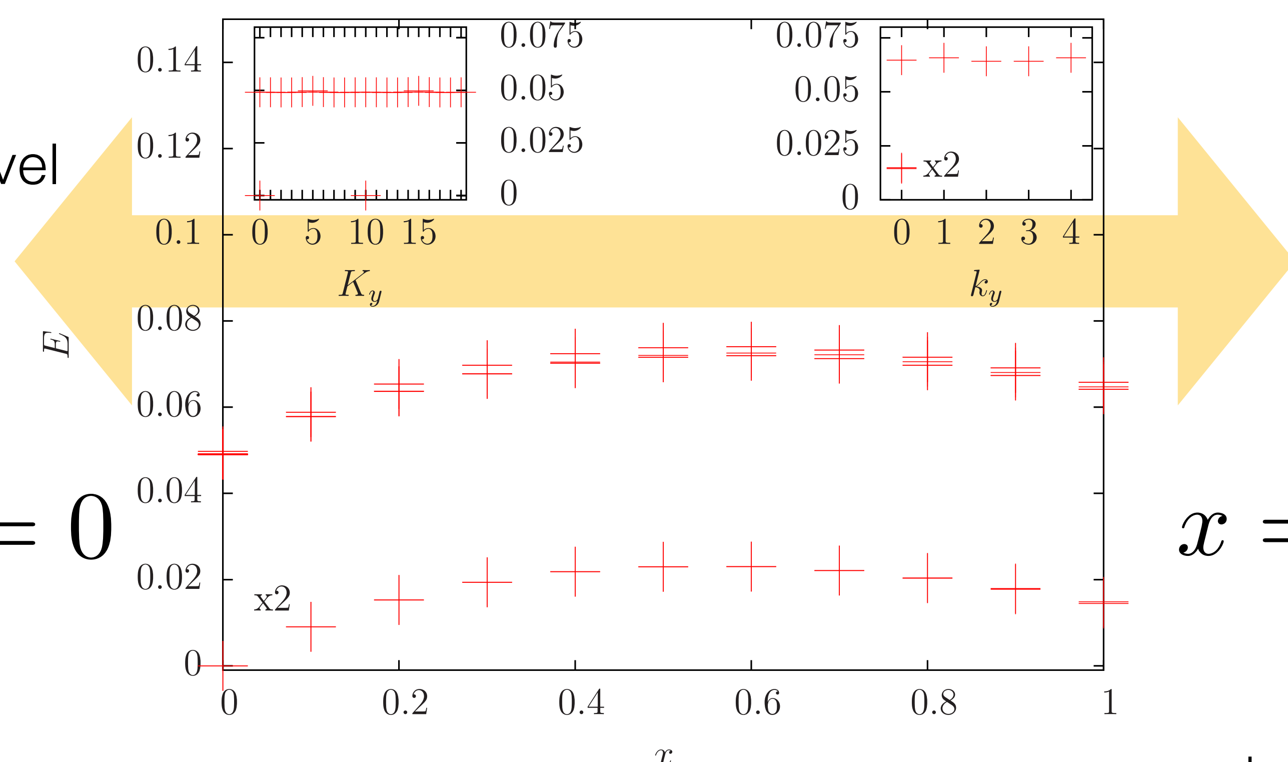
Laughlin state



B

$x = 0$

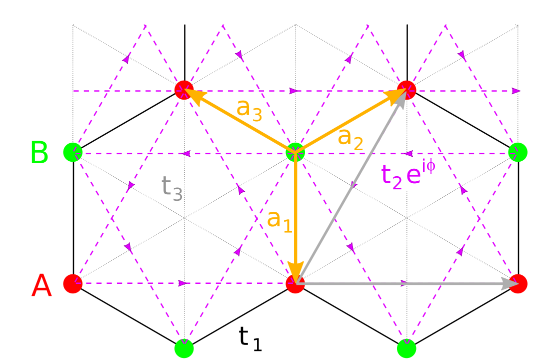
(N=10)



$x = 1$

Half filled band of the (flattened) Haldane-model

Same topological phase!



Th. Scaffidi & GM, Phys. Rev. Lett. (2012)

detail

Q3: Relation of band geometry and stability FQHE like states?

How to decide which lattice models have stable fractional Chern Insulators?

- single-particle dispersion - want flat bands

  many groups

  finite size matters a lot - success by iDMRG A. Grushin et al.

- shape of interactions - clear hierarchy of two-body energies desirable "Pseudopotentials"

  Läuchli, Liu, Bergholtz, Moessner + other proposals

- band geometry - ideally want even Berry curvature

  Regnault, Bernevig; Dobardzic, Milovanovic, …

  systematic study of geometric measures beyond Berry curvature

This Talk!

- Full story: all three aspects contribute
- Focus here: band geometry, while keeping flat dispersion + identical interactions

Which Berry Curvature?

Gauge invariance of the Bloch functions: one arbitrary U(1) phase for each **k**-point

$$|u_{\mathbf{k}}^{\alpha}\rangle \to e^{i\phi_\alpha(\mathbf{k})}|u_{\mathbf{k}}^{\alpha}\rangle$$

The above manifestly leaves H invariant:

$$H_{bc}(\mathbf{k}) = \sum_{\alpha=1}^{\mathcal{N}} E_\alpha(\mathbf{k}) u_b^{\alpha *}(\mathbf{k}) u_c^{\alpha}(\mathbf{k})$$

However, sublattice dependent phases are *not gauges*:

$$u_a^{\alpha}(\mathbf{k}) \to \widetilde{u}_b^{\alpha}(\mathbf{k}) = e^{i\mathbf{r}_b\cdot\mathbf{k}} u_b^{\alpha}(\mathbf{k})$$

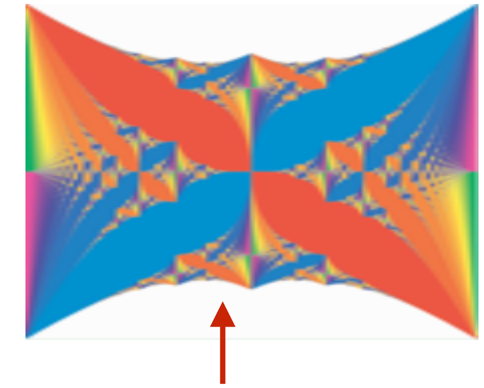
as this substitution yields a modified Berry curvature:

$$\widetilde{B}_\alpha(\mathbf{k}) - B_\alpha(\mathbf{k}) = \sum_{b=1}^{\mathcal{N}} r_{b,y} \frac{\partial}{\partial k_x} |u_b^{\alpha}(\mathbf{k})|^2 - r_{b,x} \frac{\partial}{\partial k_y} |u_b^{\alpha}(\mathbf{k})|^2$$

There is a unique choice such that the polarisation reduces to the correct semi-classical expression and canonical position operator $\hat{R}_\mu \to -i\dfrac{\partial}{\partial k_\mu}$

see, e.g. Zak PRL (1989)

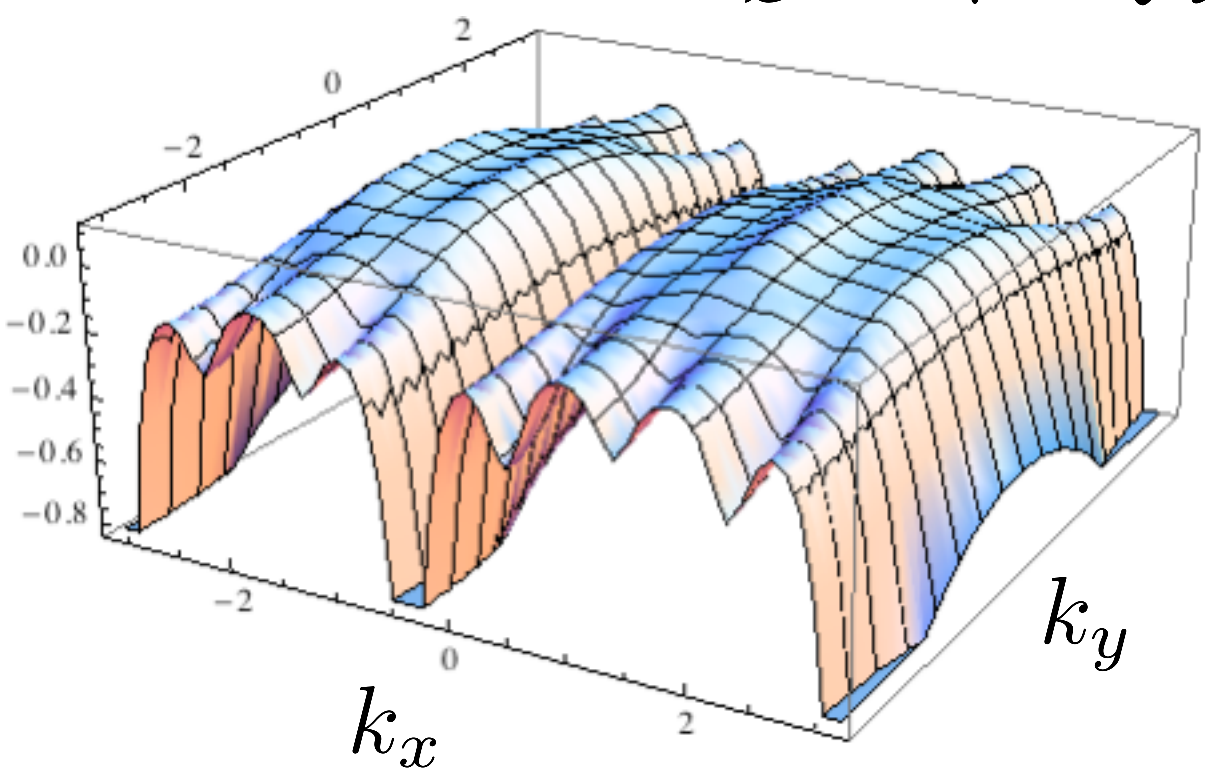
Conventions for Berry Curvature: in pictures



an example: Hofstadter spectrum in magnetic unit cell of 7x1, $n_\phi = 3/7$

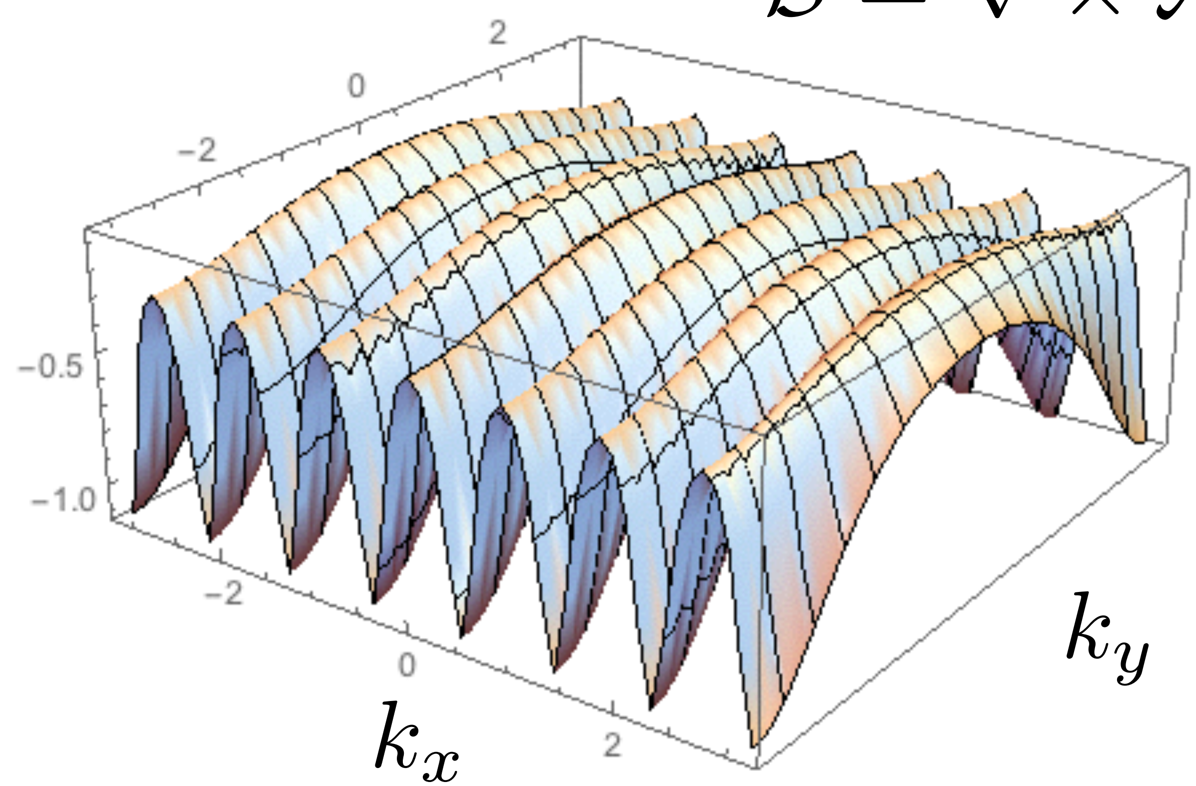
Curvature for Fourier transform with respect to unit cell position

$$\tilde{\mathcal{B}} = \nabla \times \tilde{\mathcal{A}}$$



Curvature for canonical Fourier transform

$$\mathcal{B} = \nabla \times \mathcal{A}$$



Magnetic unit cell

$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$-\frac{6\Phi}{7}$

net flux defined only mod Φ_0

GMP Algebra: Generating low-lying excitations

Girvin, MacDonald and Platzman, PRB **33**, 2481 (1986).

- single mode approximation captures low-lying neutral excitations in quantum Hall systems:

$$|\Psi_{\mathbf{k}}^{\mathrm{SMA}}\rangle = \hat{\rho}_{\mathbf{k}}|\Psi_0\rangle$$

with single particle density operators

$$\hat{\rho}_{\mathbf{k}} = \sum_q \hat{\gamma}^{\dagger}_{\mathbf{k}+\mathbf{q}}\hat{\gamma}_{\mathbf{q}}$$

these satisfy: GMP algebra (w/LLL form factor):

$$[\rho_{\mathrm{LLL}}(\mathbf{q}),\rho_{\mathrm{LLL}}(\mathbf{q}')] = 2i\sin\left(\tfrac{1}{2}\mathbf{q}\wedge\mathbf{q}'\ell_B^2\right)\exp\left(\tfrac{1}{2}\mathbf{q}\cdot\mathbf{q}'\ell_B^2\right)\rho_{\mathrm{LLL}}(\mathbf{q}+\mathbf{q}')$$

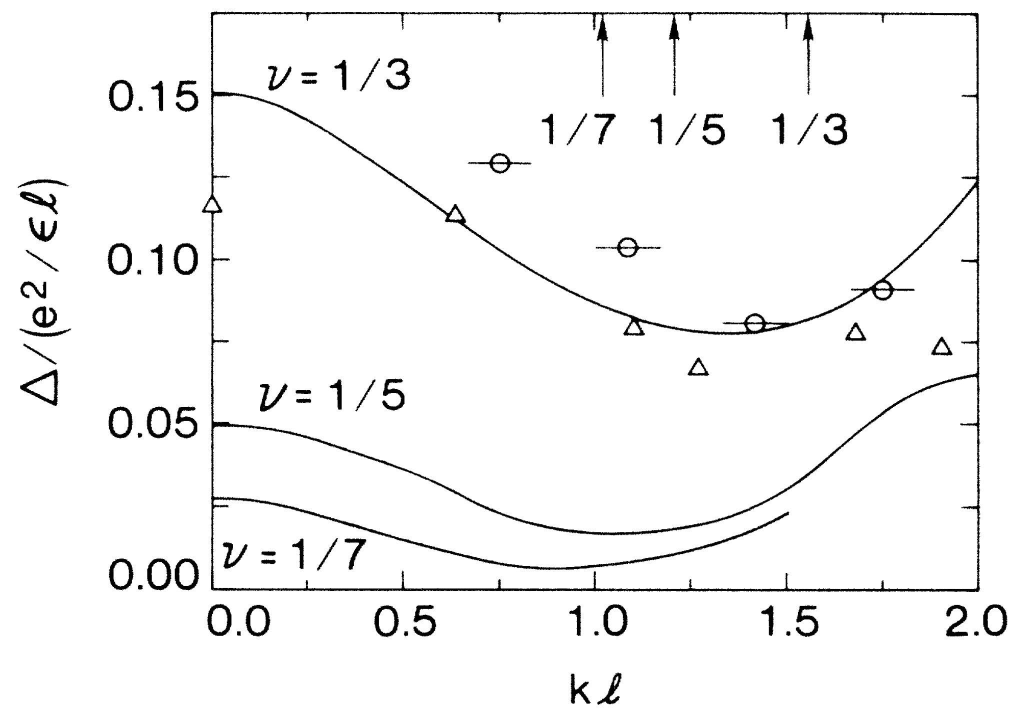


FIG. 4. Comparison of SMA prediction of collective mode energy for $\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ with numerical results of Haldane and Rezayi (Ref. 20) for $\nu = \frac{1}{3}$. Circles are from a seven-particle spherical system. Horizontal error bars indicate the uncertainty

SMA carries over to Chern bands: Repellin, Neupert, Papić, Regnault, Phys. Rev. B 90 (2014)

Chern bands: generalised GMP algebra

- consider band-projected density operators for general Chern bands:

$$\widetilde{\rho}_{\mathbf{q}} \equiv P_\alpha e^{i\mathbf{q}\cdot\widehat{\mathbf{r}}} P_\alpha = \sum_{\mathbf{k}} \sum_{b=1}^{\mathcal{N}} u_b^{\alpha *}(\mathbf{k}+\mathbf{q}/2) u_b^{\alpha}(\mathbf{k}-\mathbf{q}/2) \gamma^{\alpha\dagger}_{\mathbf{k}+\mathbf{q}/2} \gamma^{\alpha}_{\mathbf{k}-\mathbf{q}/2}$$

- in general, the algebra of density operators does not close, i.e.

$$[\widetilde{\rho}_{\mathbf{q}}, \widetilde{\rho}_{\mathbf{k}}] \neq F(\mathbf{k},\mathbf{q}) \widetilde{\rho}_{\mathbf{k}+\mathbf{q}}$$

- intuitive consequences for FQH states:
  - no finite, closed set of low-energy excitations corresponding to the GMP single mode states
  - $\widetilde{\rho}_{\mathbf{q}}$ can generate many distinct eigenstates
  - strong violation of the algebra should signal an unstable, gapless phase

Conditions for closure of the generalised GMP algebra I

• conditions for closure of the algebra can be derived in a long-wavelength expansion

i) $\mathcal{O}(k^2)$:

$$\sigma_c \equiv \sqrt{\frac{A_{BZ}^2}{4\pi^2}\langle B^2\rangle - c_1^2}$$

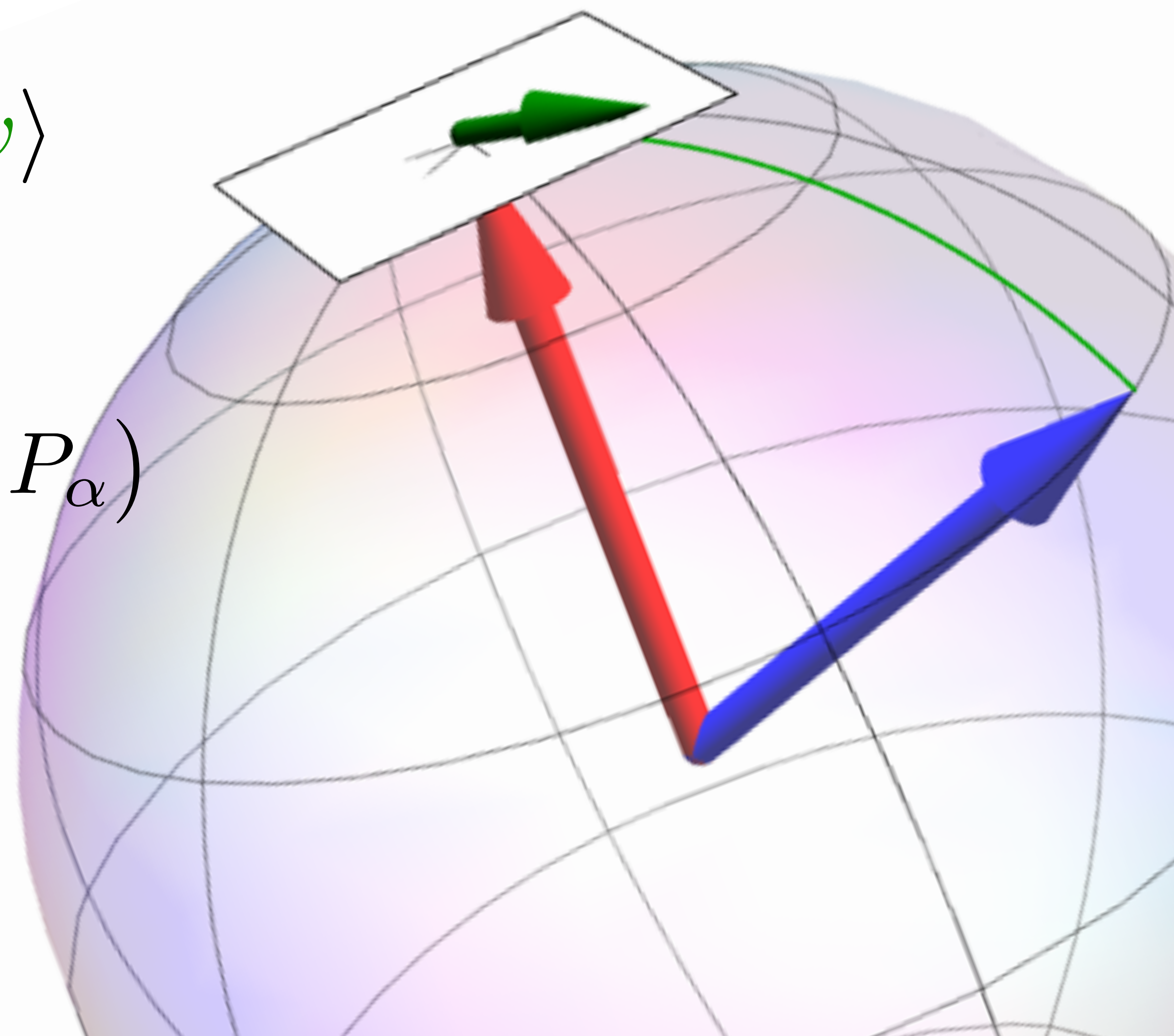
flatness of Berry curvature

ii) $\mathcal{O}(k^3)$: Pullback of Hilbert space metric constant over BZ

$$ds^2 = \langle\delta\psi|\delta\psi\rangle - \langle\delta\psi|\psi\rangle\langle\psi|\delta\psi\rangle$$

$$g_{\mu\nu} + \frac{i}{2}F_{\mu\nu}$$

$$= \sum_{\alpha\in\mathrm{occ}} \mathrm{tr}\left(\frac{\partial}{\partial k_\mu}P_\alpha\right)(1-P_\alpha)\left(\frac{\partial}{\partial k_\nu}P_\alpha\right)$$



deviations $\sigma_g \equiv \sqrt{\frac{1}{2}\sum_{\mu,\nu}\langle g_{\mu\nu}g_{\nu\mu}\rangle - \langle g_{\mu\nu}\rangle\langle g_{\nu\mu}\rangle}$

Conditions for closure of the generalised GMP algebra II

- single condition in terms of metric g:

$$T(\mathbf{k}) \equiv \mathrm{tr}\ g^{\alpha}(\mathbf{k}) - |B_{\alpha}(\mathbf{k})| = 0$$

- algebra of projected density operators reduces exactly to the GMP algebra

- Next step: test how violations of the closure constraints correlate with gap

R. Roy, arxiv:1208.2055 (PRB 2014); Parameswaran, Roy, Sondhi C. R. Physique (2013)

Target models to examine

- Hamiltonian: bosonic states with on-site interactions — defined independent of specific lattice

2-body contact

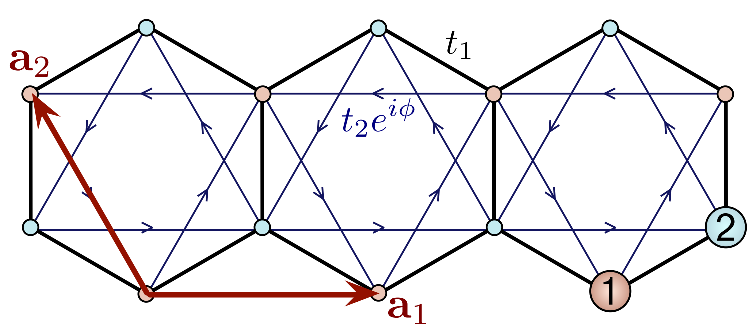
Laughlin $\nu = \frac{1}{2}$

3-body contact

Moore-Read $\nu = 1$

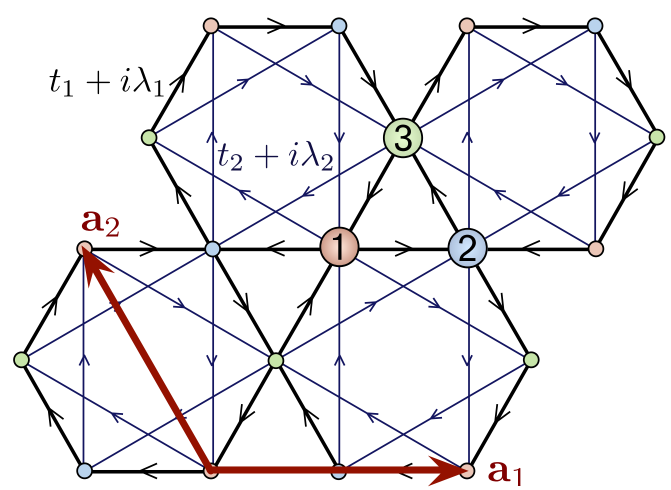
- Lattice Geometry:

Haldane model



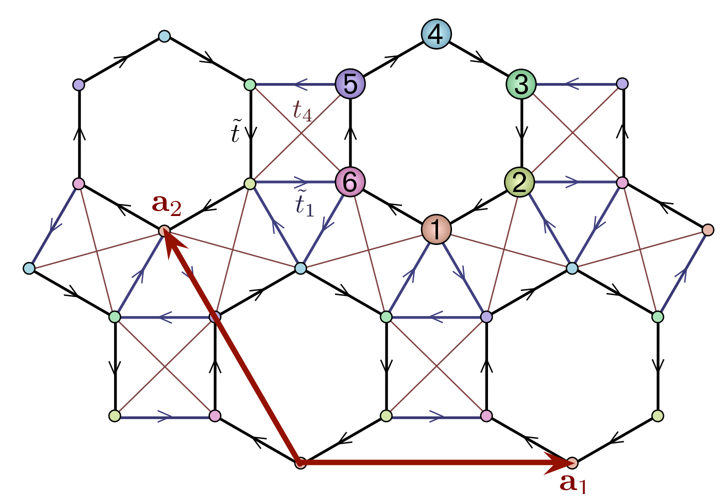
$\mathcal{N} = 2$

Kagomé model



$\mathcal{N} = 3$

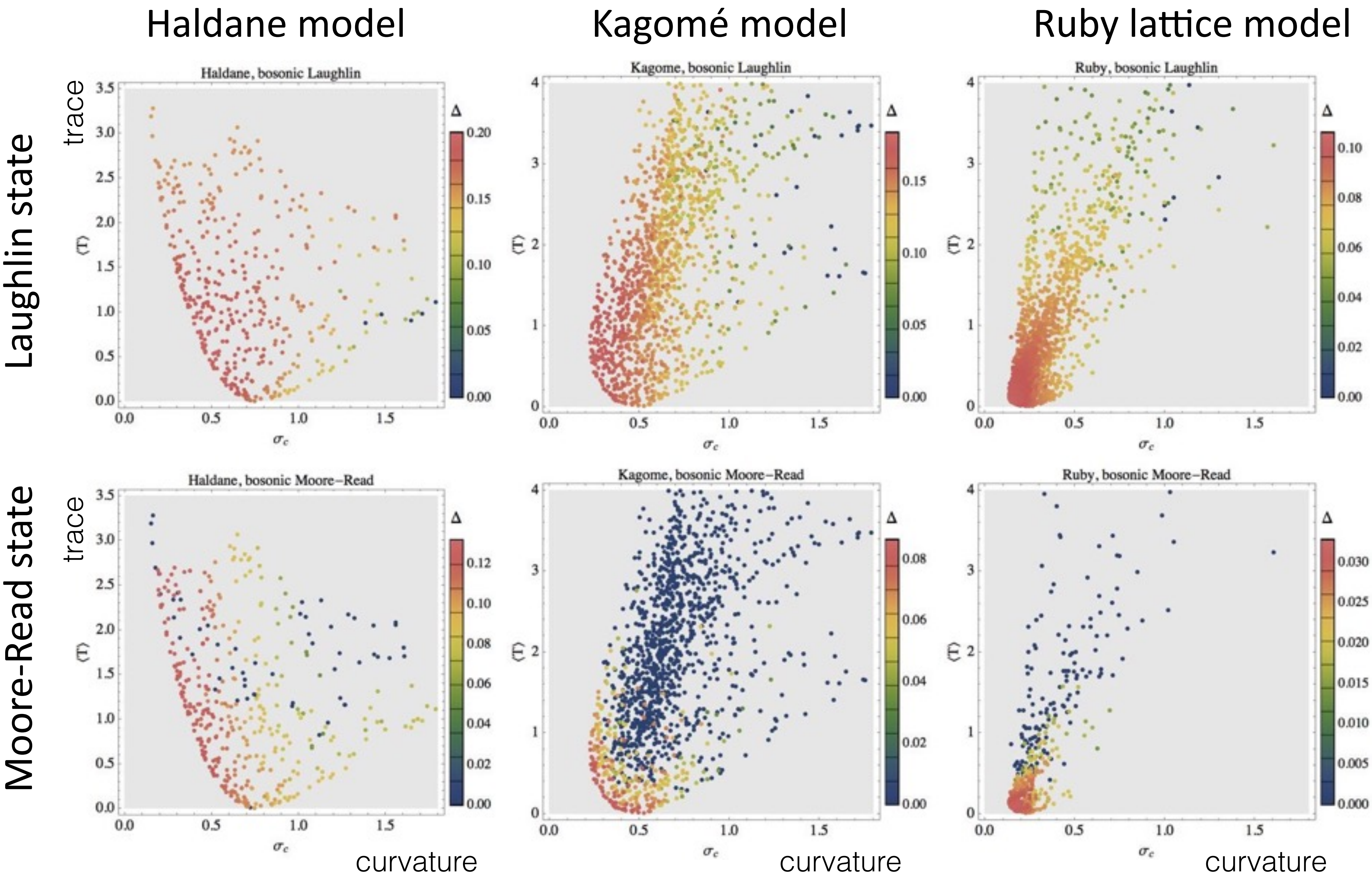
Ruby lattice model



$\mathcal{N} = 6$

T. Jackson, GM, R. Roy, Nature Comm. (2015)

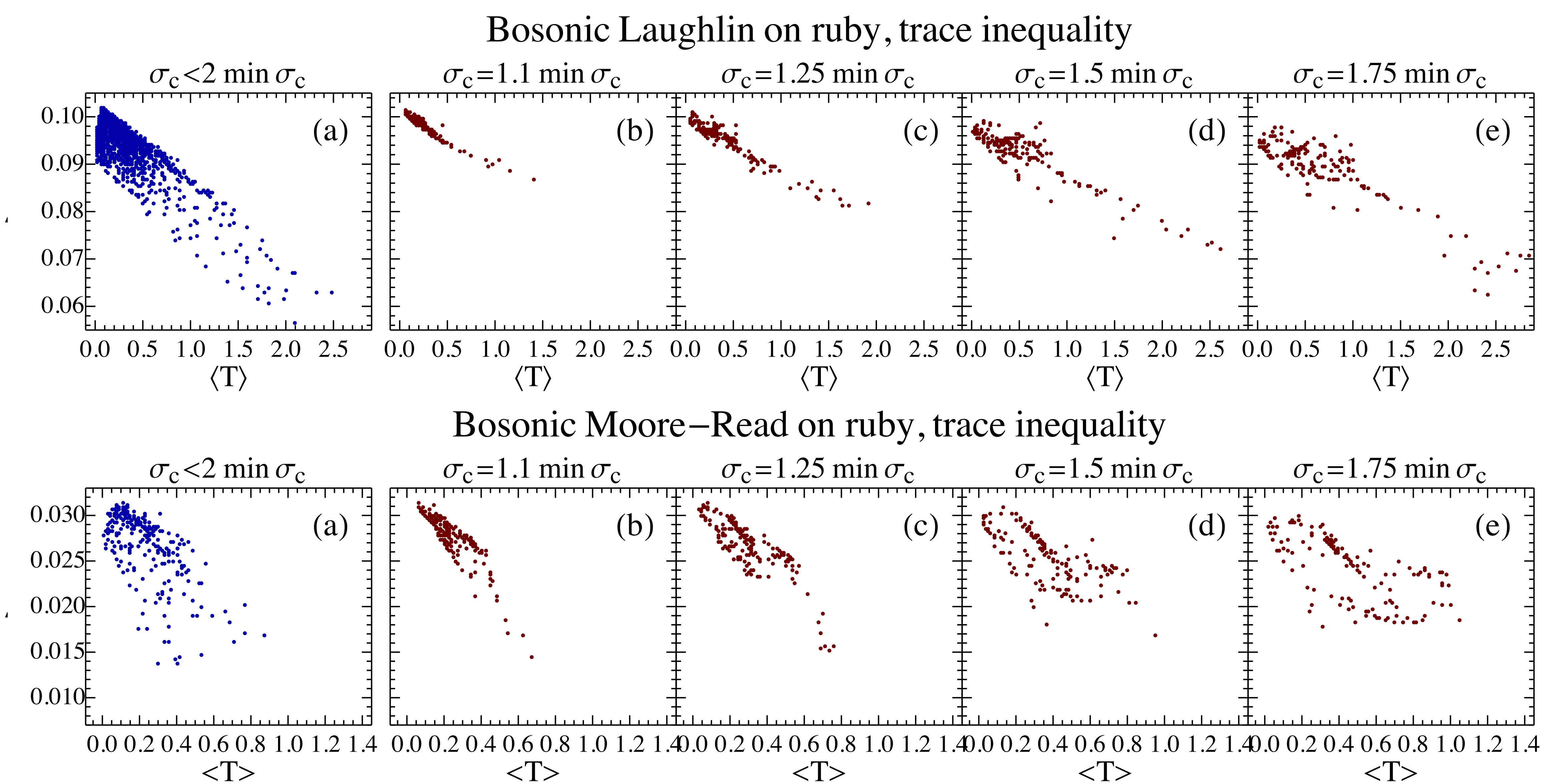
Model Comparison: Gaps vs. RMS B and trace inequality



- 1 point ⇔1 set of parameters: max gap tends to be found in lower-left corner
- Demonstrates relevance of both band-geometric quantities

Role of band geometry: trace of the metric tensor

Influence of metric tensor g via "trace": $T(\mathbf{k}) \equiv \mathrm{tr}\ g^{\alpha}(\mathbf{k}) - |B_{\alpha}(\mathbf{k})|$



- Systematic correlation of many-body gap and trace of metric tensor

T. Jackson, GM, R. Roy, Nature Comm. (2015)

Intermediate conclusions

Q1: Do bands resembling Landau-levels support QH states?

evidence: ‣ ground state degeneracy, gap, entanglement spectrum match ✓

Q2: Do Chern bands support the same phases as LLs?

evidence: ‣ adiabatic continuity shown for multiple phases

‣ no new types of quantum Hall liquids in C=1 bands
(but potentially coexistence of QHE and Landau order) ✓

Q3: Relation of band geometry and stability FQHE like states?

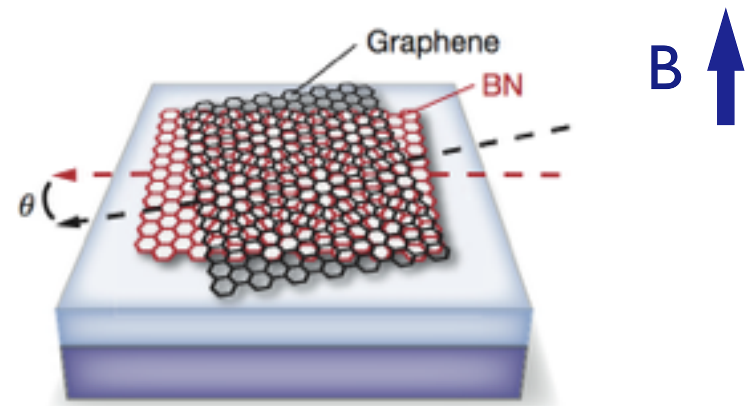
evidence: ‣ bands resembling Landau-levels closely produce
the most robust QH liquids ✓

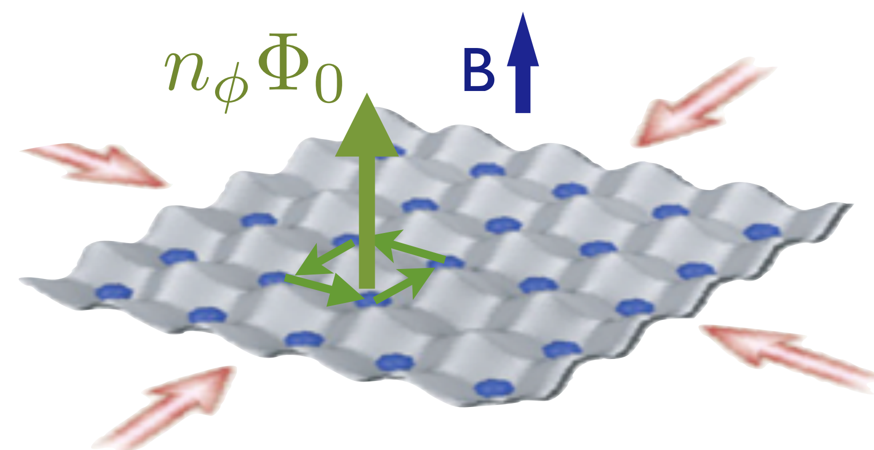
Artificial magnetic fields yield faithful realisations of the FQHE

Q4: Is there new physics in fractional Chern insulators

One novelty: Higher Chern numbers |C|>1, e.g. in *the Hofstadter Problem*

- Harper / Hofstadter: systems with Magnetic Field and periodic potentials





- twisted graphene bilayers: Kim *et al. (2013)*
- optical flux lattices: MIT / Munich (2014)

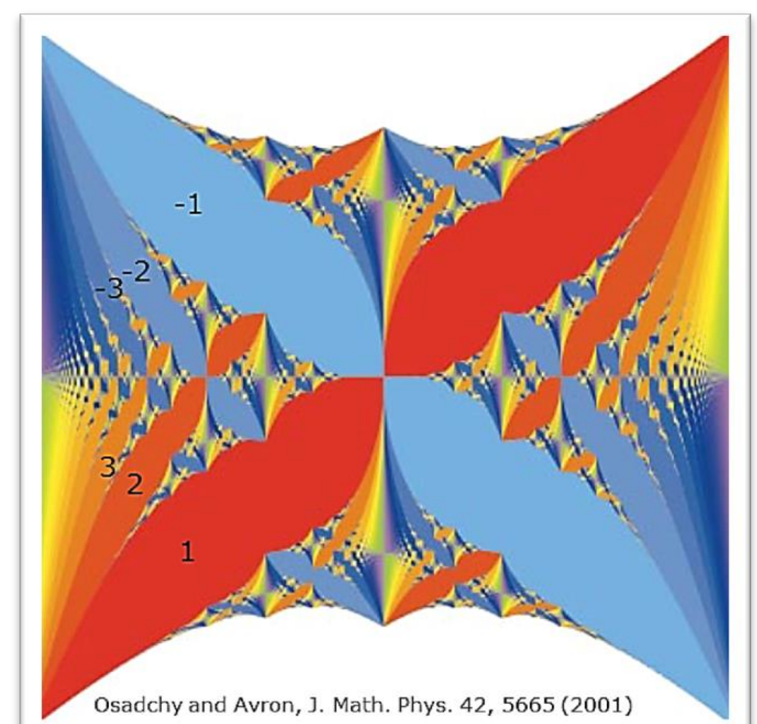
VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1982

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195

(Received 30 April 1982)

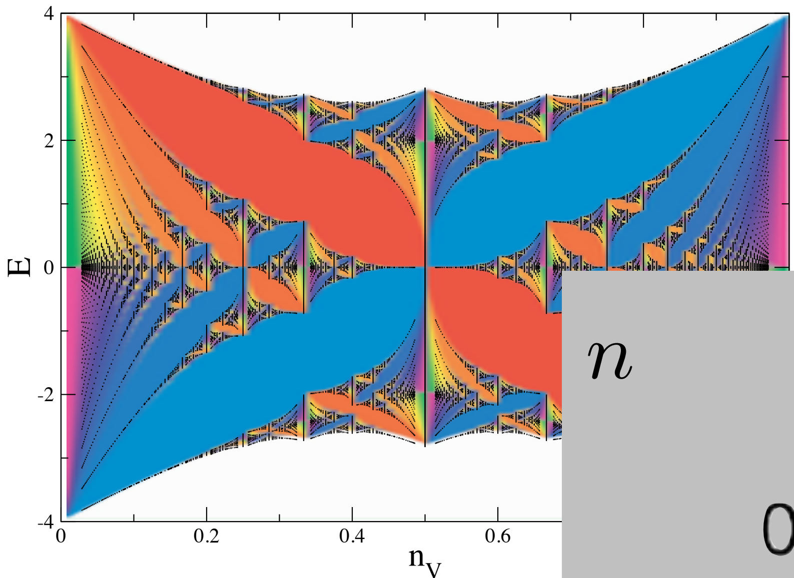
$$\sigma_H = \frac{ie^2}{2\pi h}\sum \int d^2k \int d^2r \left(\frac{\partial u^*}{\partial k_1}\frac{\partial u}{\partial k_2} - \frac{\partial u^*}{\partial k_2}\frac{\partial u}{\partial k_1}\right)$$
$$= \frac{ie^2}{4\pi h}\sum \oint dk_j \int d^2r \left(u^* \frac{\partial u}{\partial k_j} - \frac{\partial u^*}{\partial k_j} u\right), = \frac{e^2}{h}\sum_n C_n$$



Osadchy and Avron, J. Math. Phys. 42, 5665 (2001)

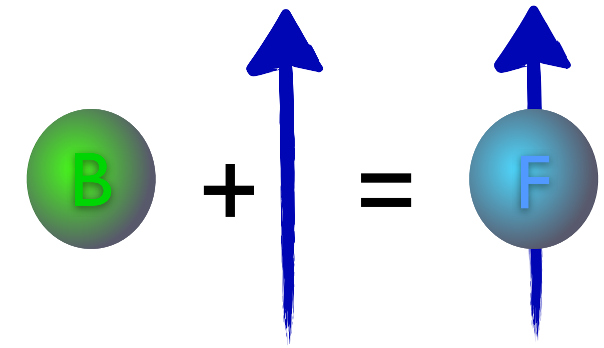
New Universality classes of FQH states

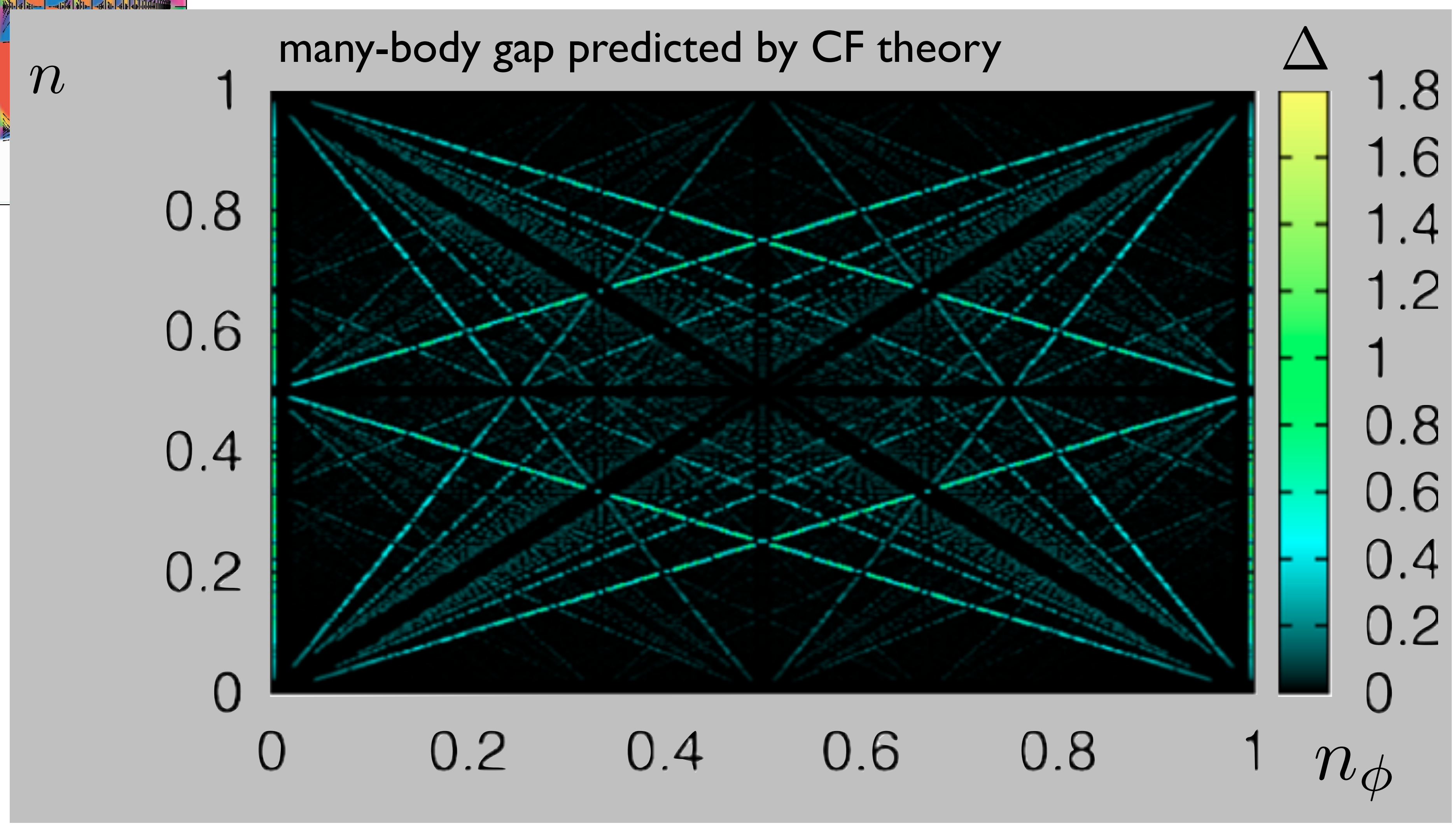
single-particle spectrum



Simple Heuristics: Composite Fermions

$$n_\phi = \pm n + n_\phi^*$$

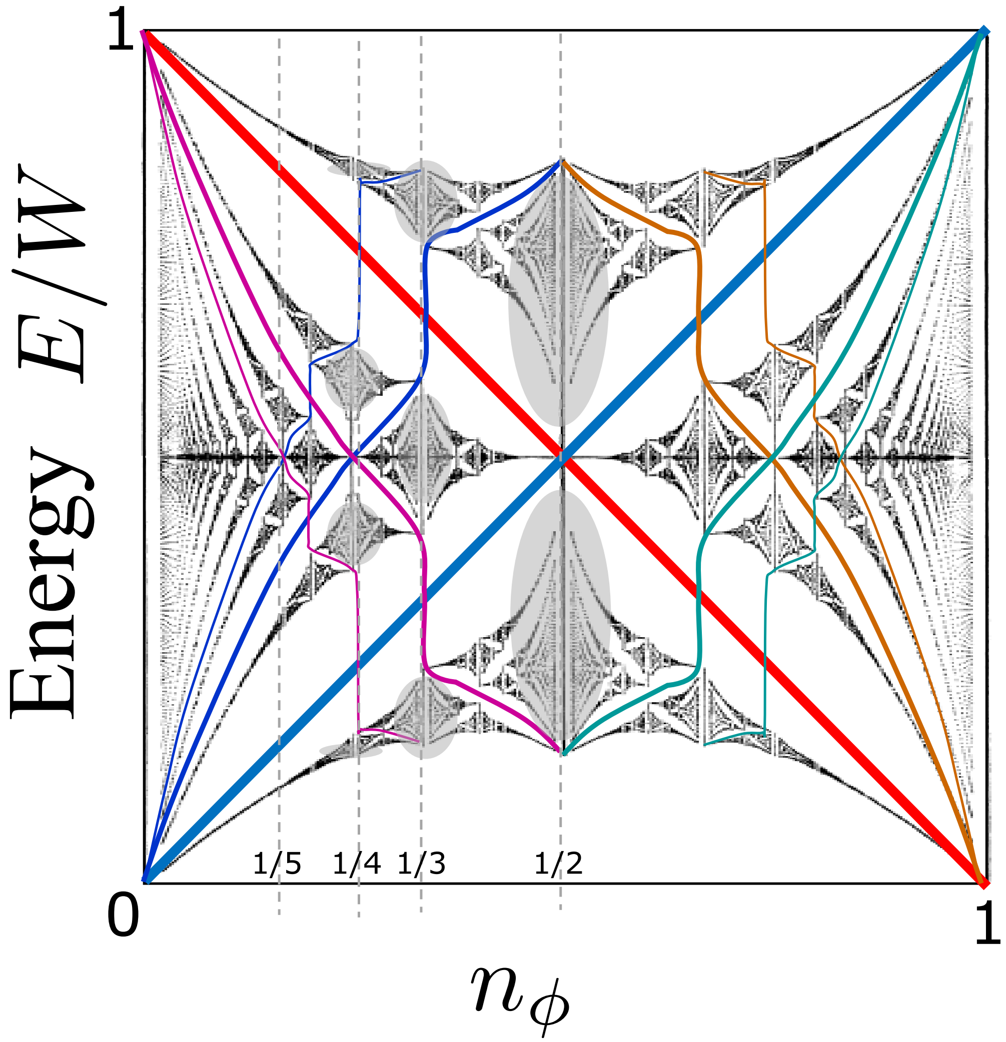


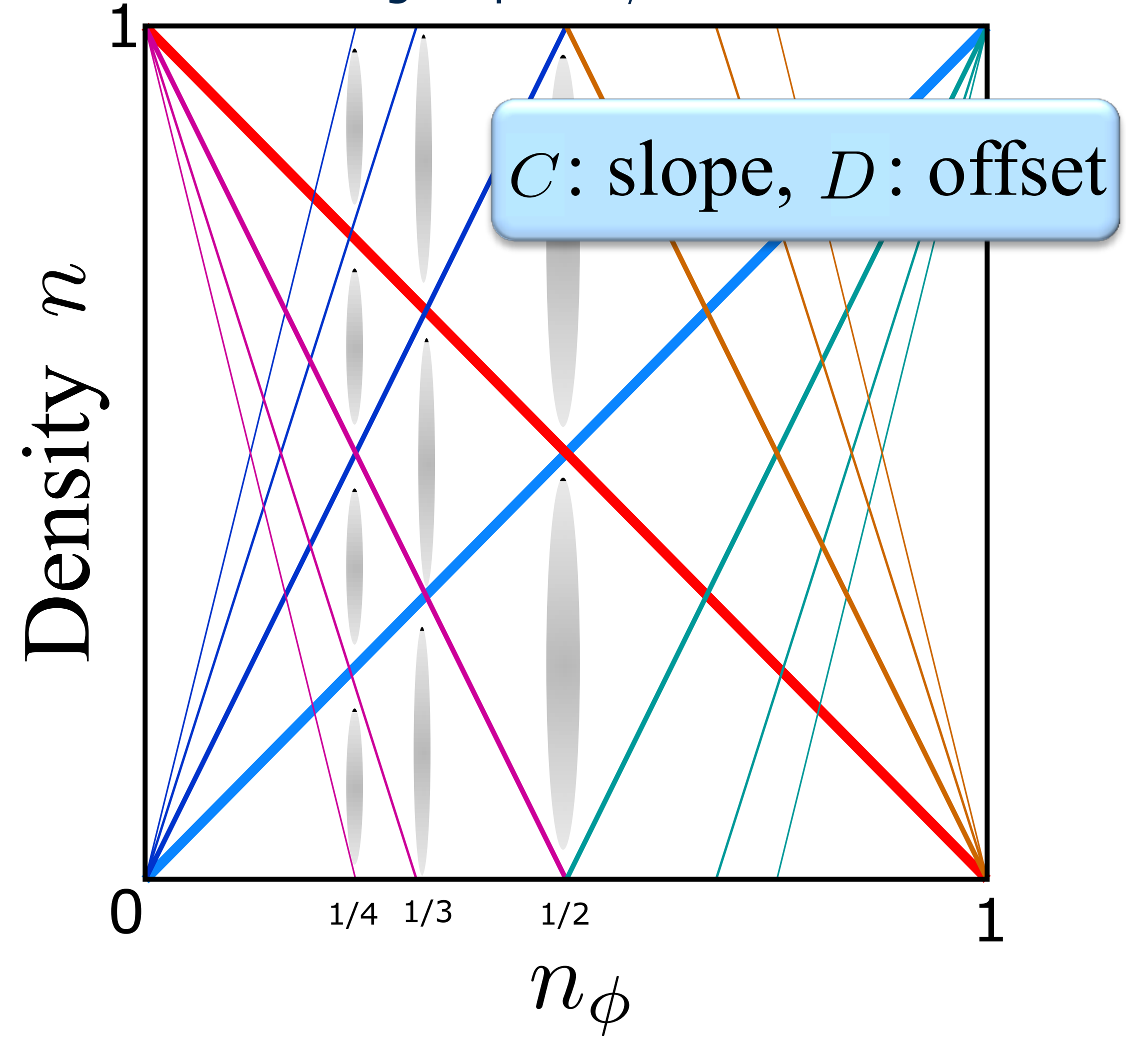


- higher Chern number bands yield new series of Abelian quantum Hall states!

first numerical evidence: GM & N.Cooper Phys. Rev. Lett. **103**, 105303 (2009)

Density of filled bands in the Butterfly: Wannier Diagram





Density of filled bands n

Flux density n_ϕ

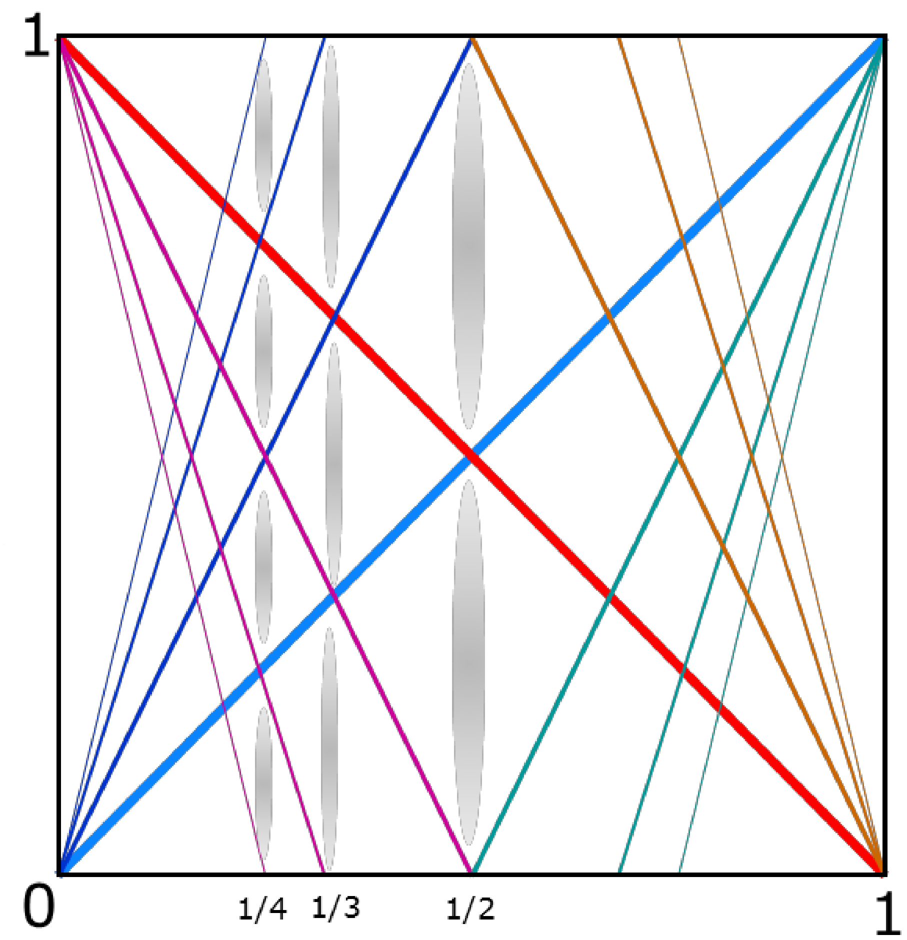
Diophantine equation for gaps

$$n = Cn_\phi + D, \quad C, D \in \mathbb{Z}$$

Streda & Thouless: Quantization of Hall conductivity

Wannier:

$$n = Cn_\phi + D, \quad C, D \in \mathbb{Z}$$



Streda:

$$\sigma_{xy} = \frac{e}{\Phi_0}\frac{\partial n}{\partial n_\phi} = C\frac{e^2}{h}.$$

Thouless:

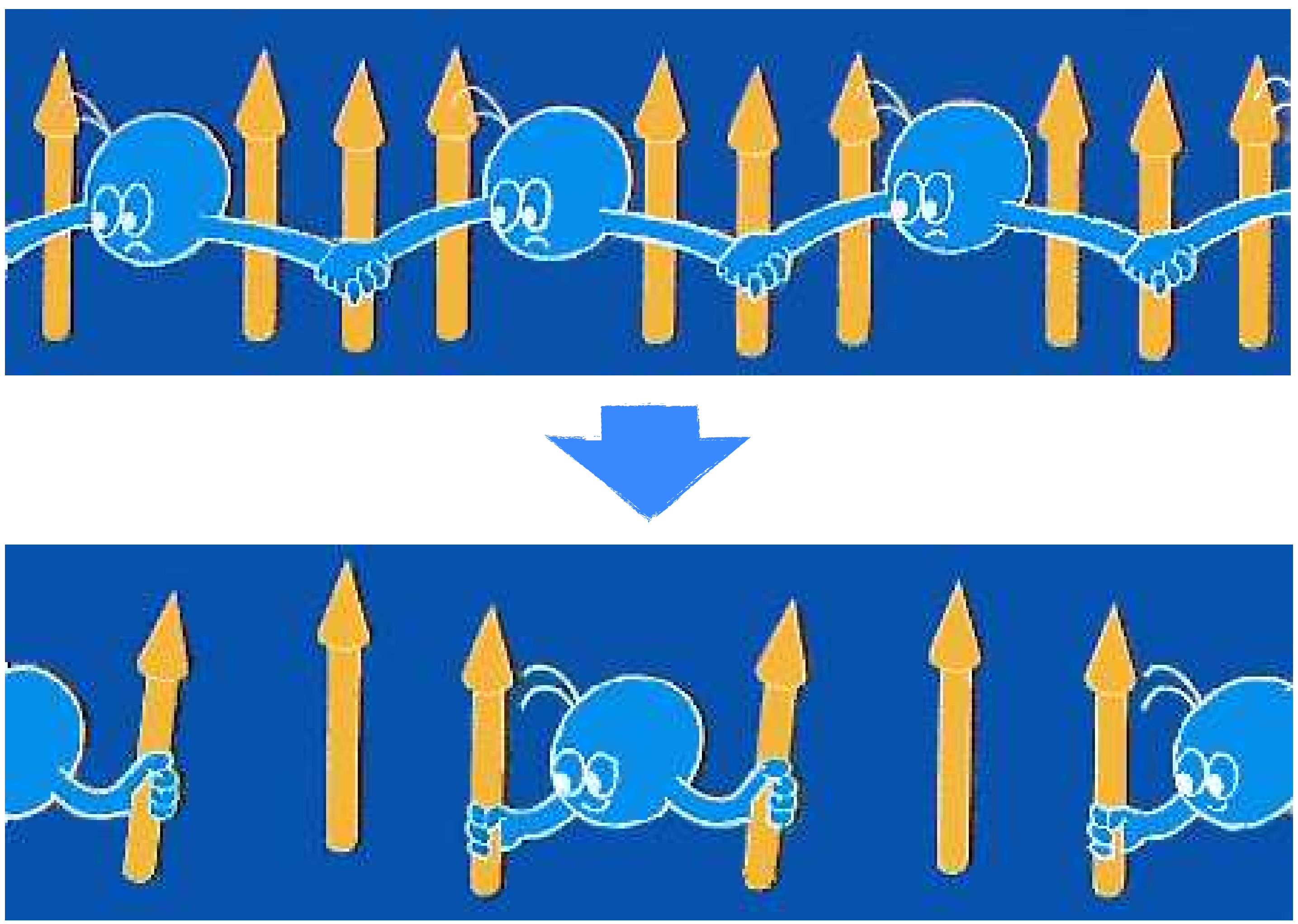
$$\sigma_{xy} = \frac{e^2}{2\pi h}\sum_{\text{filled bands } n}\int d^2k\mathcal{F}_{12}(k)$$

$$\Rightarrow \quad C = \sum_{\text{filled bands}} C_n$$

The Composite Fermion Approach

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}_\alpha^\dagger \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha (\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$

Account for repulsive interactions $U > 0$ by "flux-attachment" (Fradkin 1988, Jain 1989)



drawings: K. Park

Continuum Landau-level for fermions at filling 1/3: three flux per particle

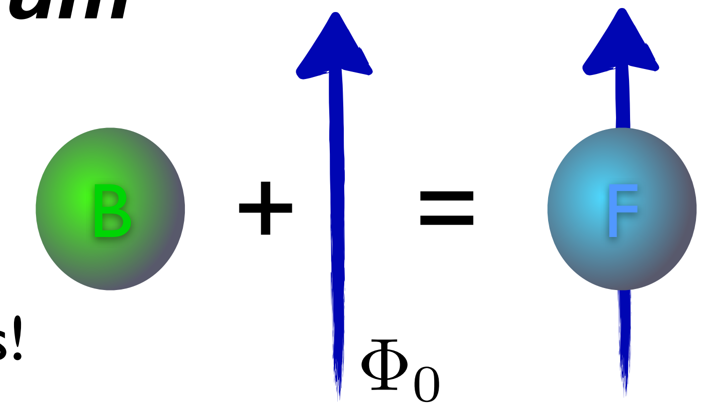
Composite fermions = electron + 2 flux quanta

$$\Psi \propto \prod_{i<j} (z_i - z_j)^2 \Psi_{\mathrm{CF}}$$

Bosons:
1 flux per composite particle

Composite Fermions in the Hofstadter Spectrum

1. Flux attachment for bosonic atoms: $n_\phi^* = n_\phi \mp n$



$$\Psi_B \propto \prod_{i<j}(z_i - z_j)\Psi_{\rm CF} \quad \Rightarrow \text{transformation of statistics!}$$

2. Effective spectrum at flux n_ϕ^* is again a Hofstadter problem

⇒ diophantine relation of flux and filling under a gap

$$n = Cn_\phi + D, \quad C, D \in \mathbb{Z}$$

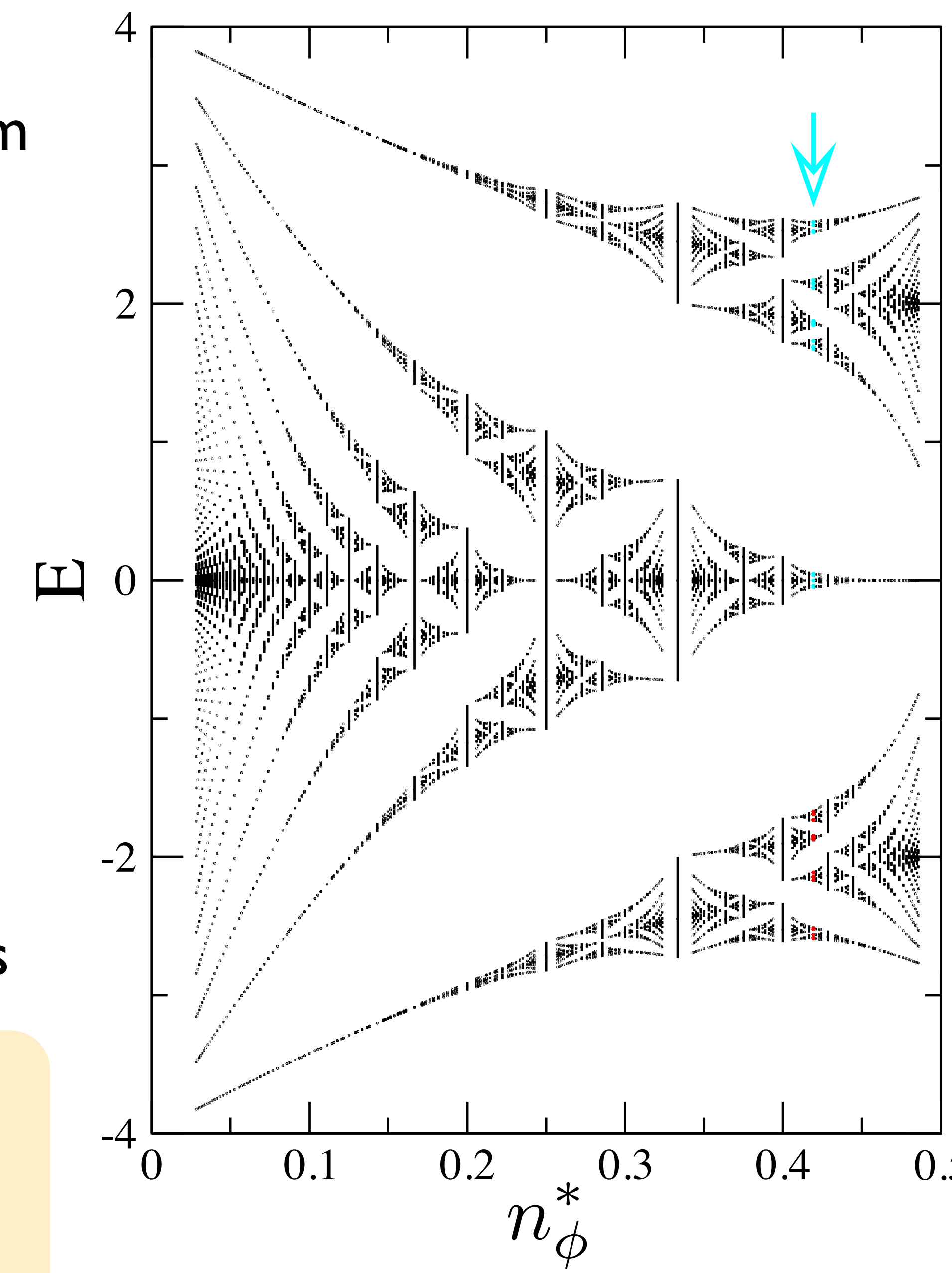
3. Parton / Composite Fermion construction

continuum: $\Psi_{\rm B}(\{\mathbf{r}_i\}) \propto \mathcal{P}_{LLL} \underbrace{\prod_{i<j}(z_i - z_j)}_{} \ \psi_{\rm CF}(\{\mathbf{r}_i\})$

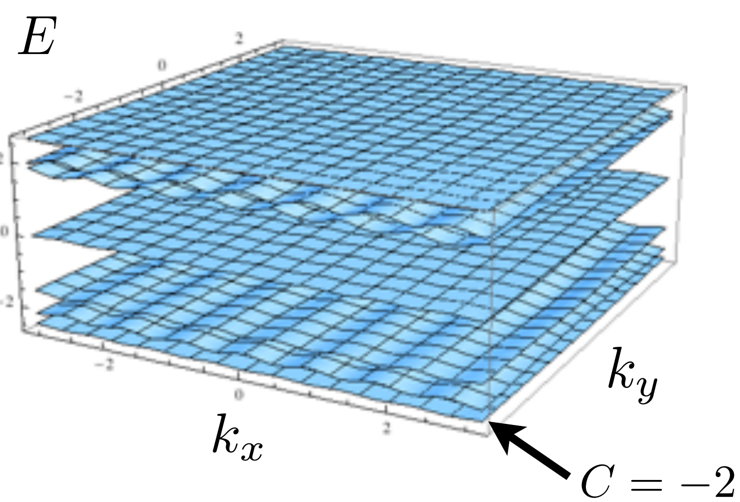
Vandermonde / Slater determinant of LLL states

lattice: $\Psi_{\rm B}(\{\mathbf{r}_i\}) \propto \underbrace{\psi_{\rm J}^{(\phi_x,\phi_y)}(\{\mathbf{r}_i\})}_{} \ \psi_{\rm CF}^{(-\phi_x,-\phi_y)}(\{\mathbf{r}_i\})$

Slater determinant of Hofstadter orbitals at flux density $n_\phi^0 = n$



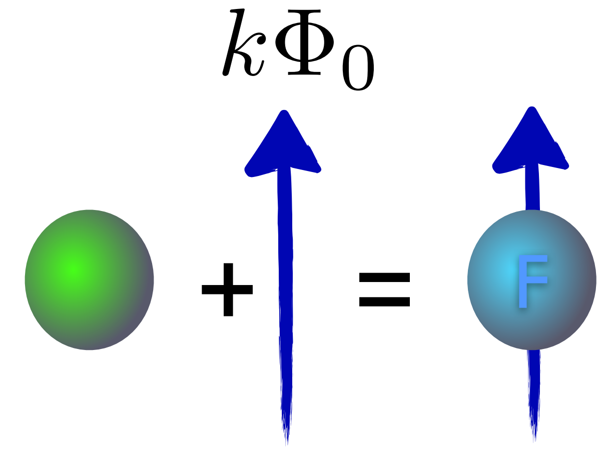
GM & N. R. Cooper, PRL 2009

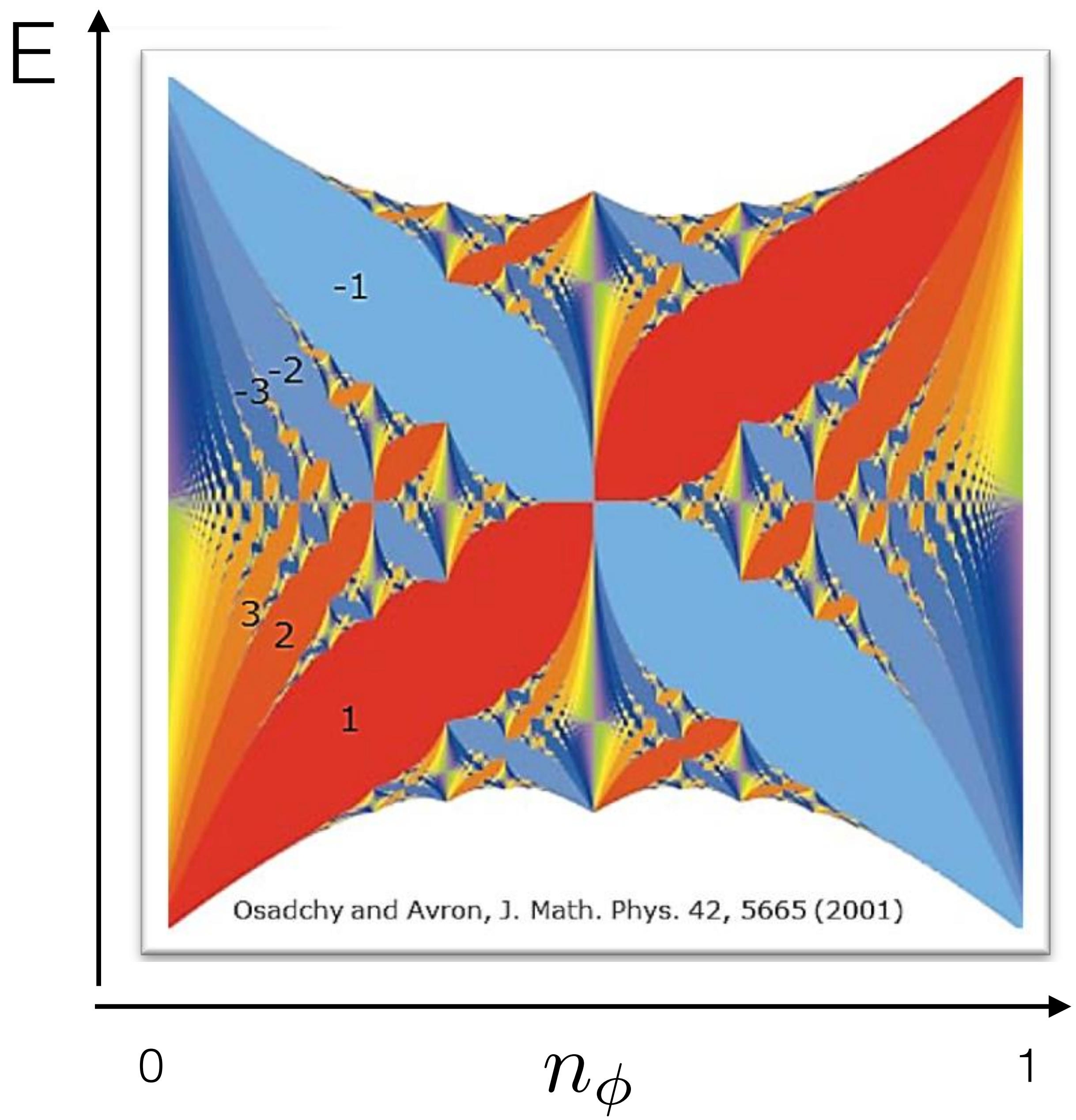


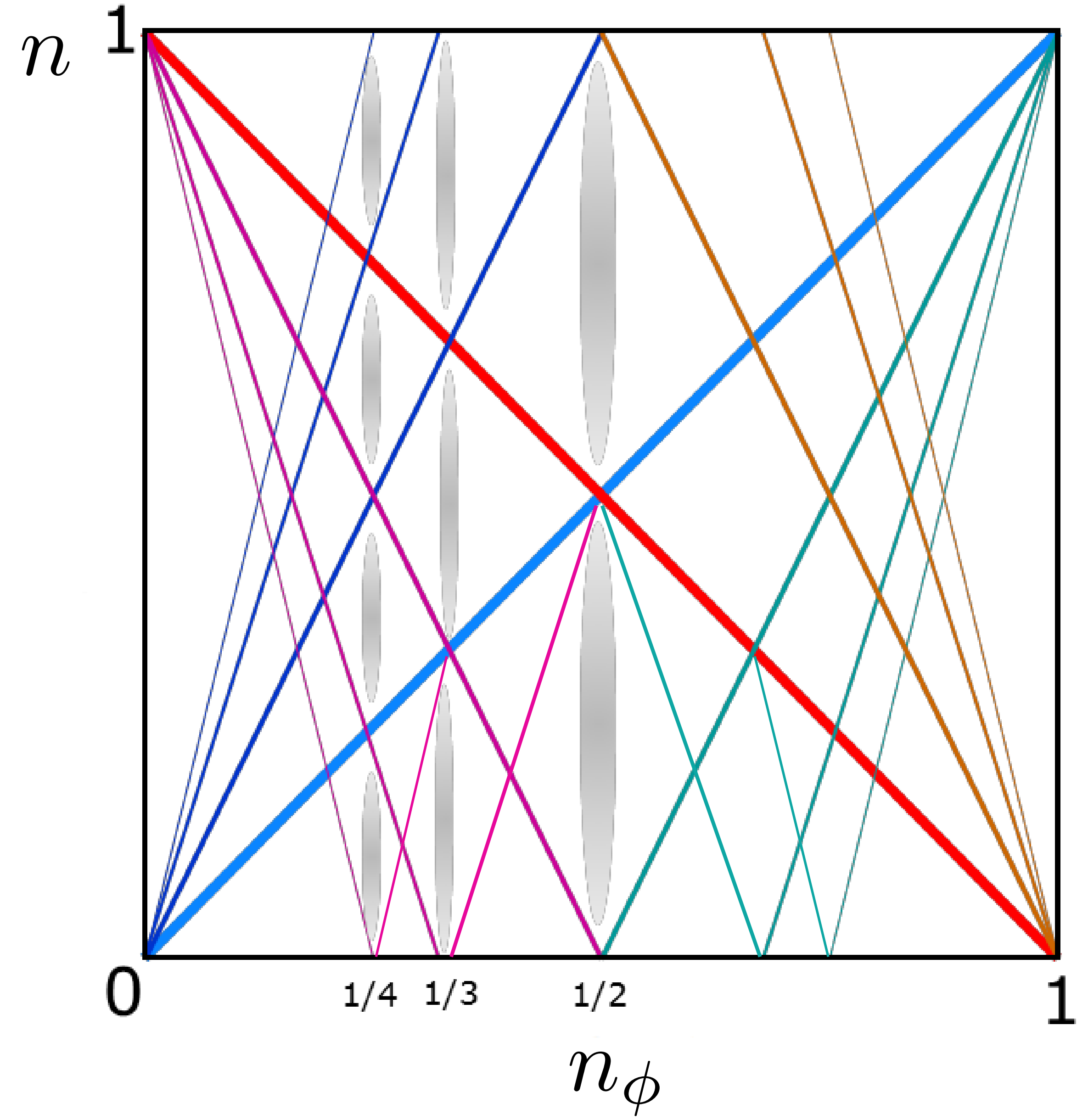
Simple Heuristics: Composite Fermions

$$n_\phi = kn + n_\phi^*$$

[k odd (even) for bosons (fermions)]







on the blackboard…

$$n_\phi = kn + n_\phi^*$$

useful to replace flux density by number of states in relevant low-energy manifold

$$n_s = C n_\phi + D$$

Composite fermions filling integer # bands, so can use the Diophantine equation for the CF gap:

$$n = n_s^* = C^* n_\phi^* + D^*$$

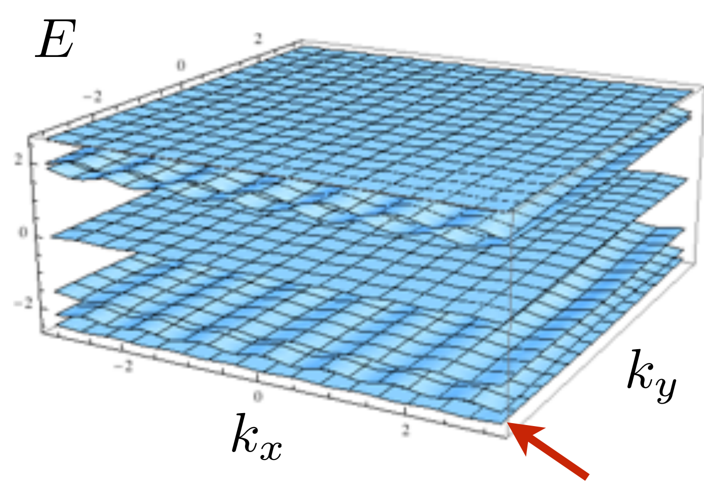
$$\Rightarrow \quad \frac{n_s}{C} - \frac{D}{C} = n\left(\frac{kC^*+1}{C^*}\right) - \frac{D^*}{C^*}$$

Hence, a constant filling factor is defined only if $\frac{D}{C} = \frac{D^*}{C^*}$ – but that is indeed a representative case: as n small, the CF band structure looks similar to the original one, but CF may fill r bands. Then, we have $C^* = rC$ and the filling factors are

$$\nu = \frac{r}{kCr+1}, \quad r \in \mathbb{Z}$$

Exact Diagonalization vs Theory Predictions:

Bosons with contact interactions, in lowest band



$$\mathcal{H} = \mathcal{P}_{\mathrm{LB}} \sum_i \hat{n}_i(\hat{n}_i - 1)\mathcal{P}_{\mathrm{LB}}$$

Check predictions for incompressible states:

filling: $$\nu = \frac{r}{r|Ck| + 1}$$

GS degeneracy: $$d_{\mathrm{GS}} = |rCk| + \mathrm{sgn}(r)$$

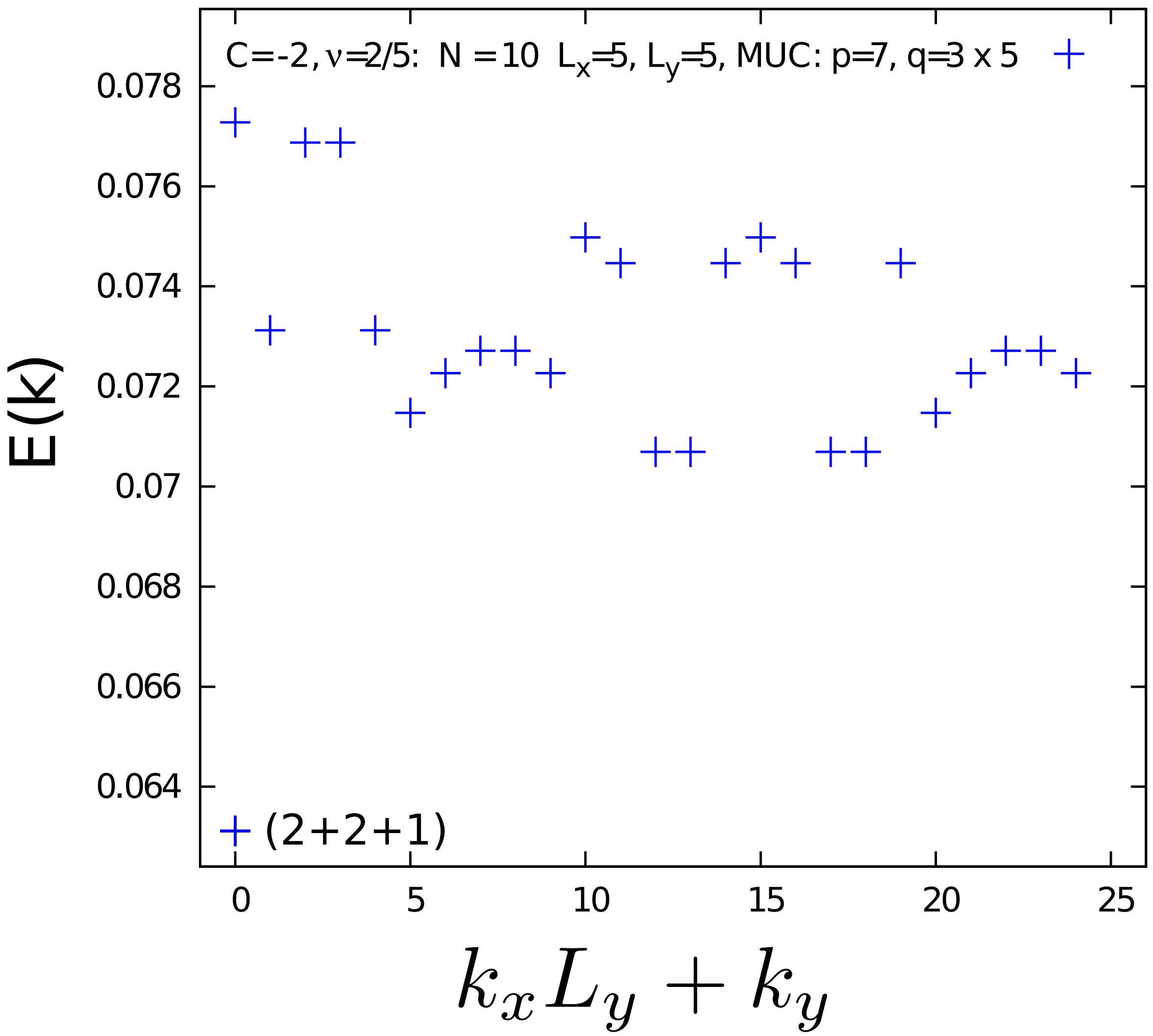
Chern number of GS's: $$C_{\mathrm{MB}} = C^* = rC$$

GM & NR Cooper, PRL (2015), arXiv:1504.06623

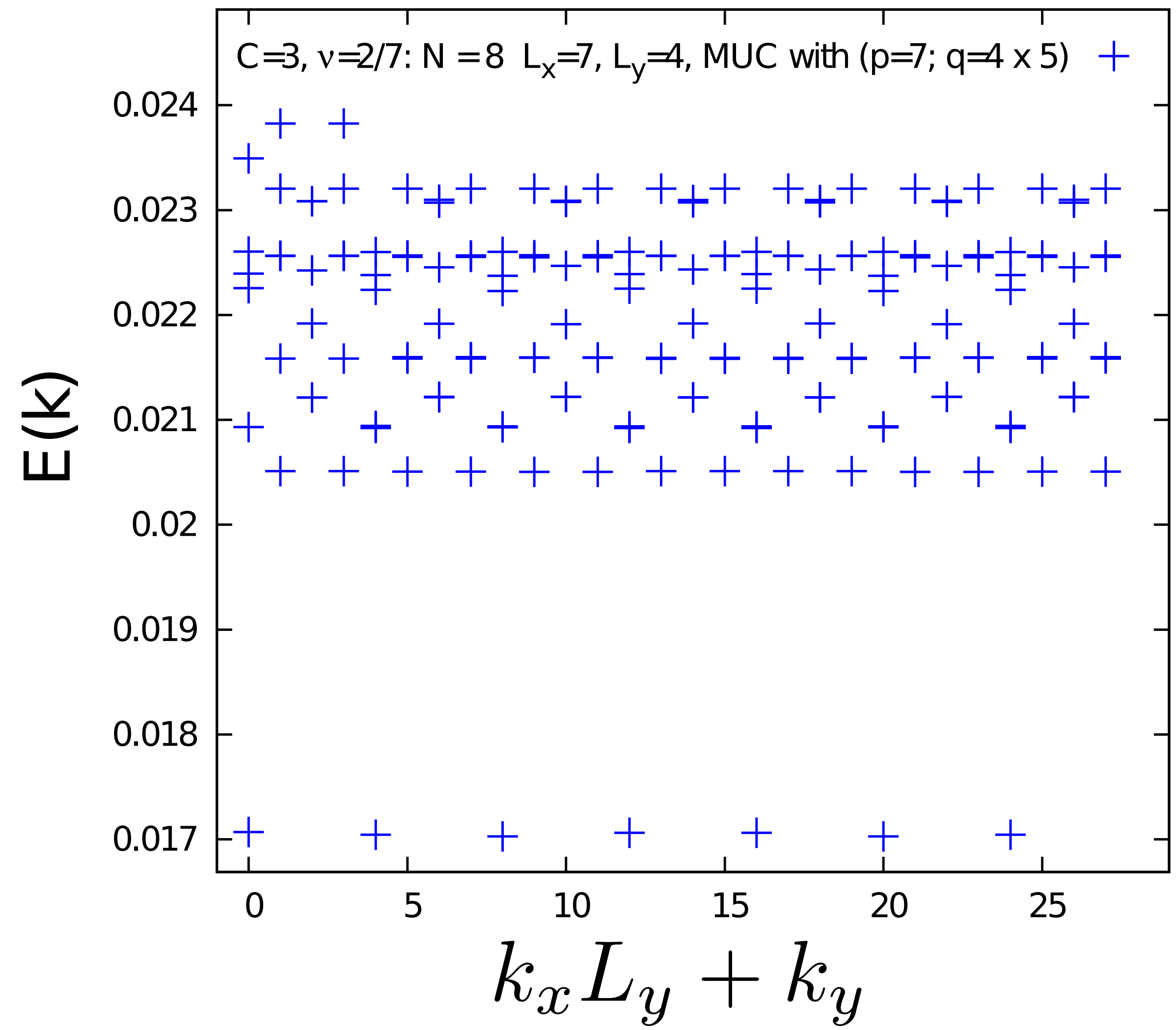
Exact Diagonalization: Spectra for new candidate

Example spectra for states with ‘positive flux attachment’: bosons, r=2

$C = 2$



$C = 3$



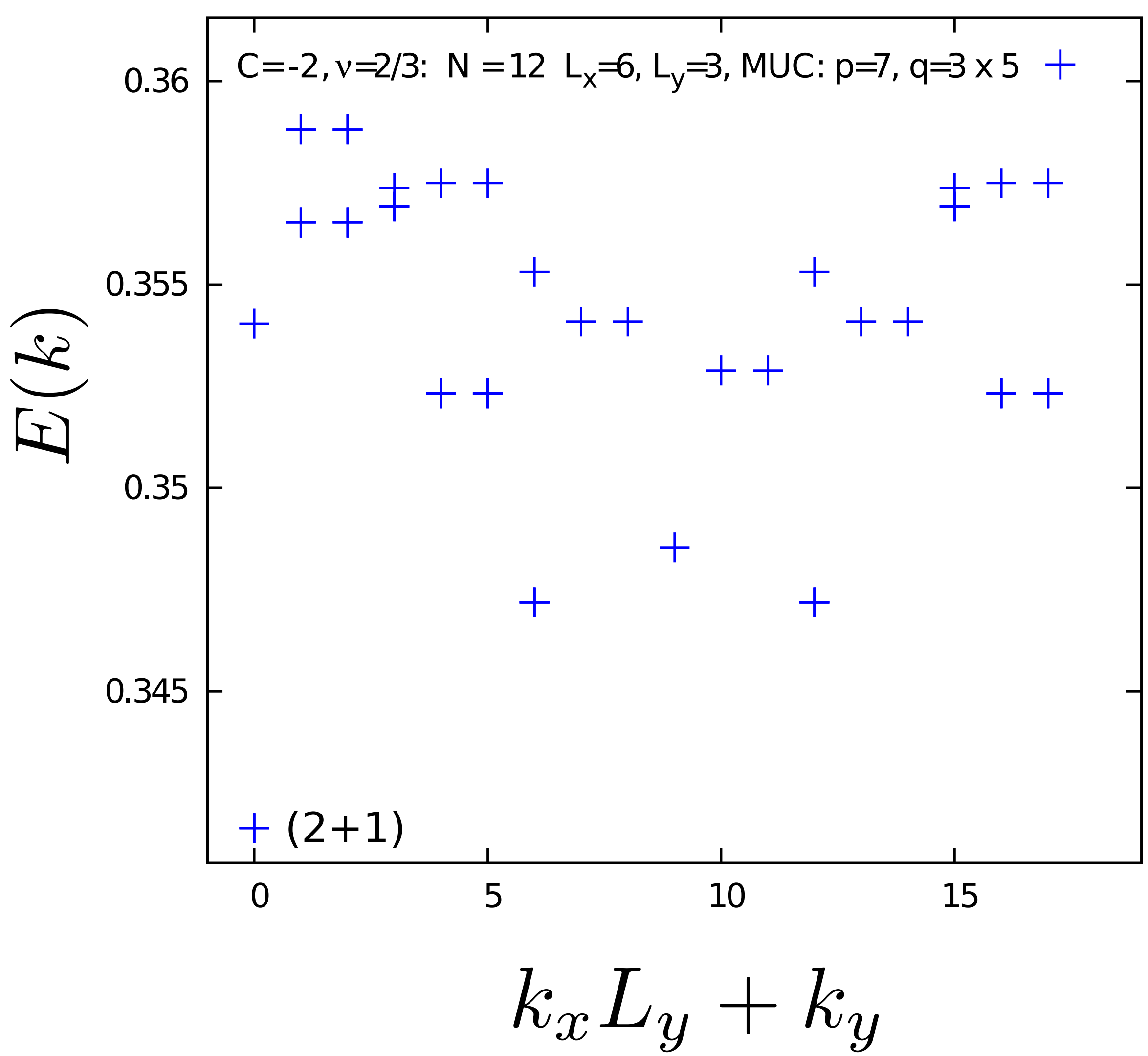
$$\nu = \frac{r}{r|Ck| + 1}$$

GM & NR Cooper, PRL (2015)

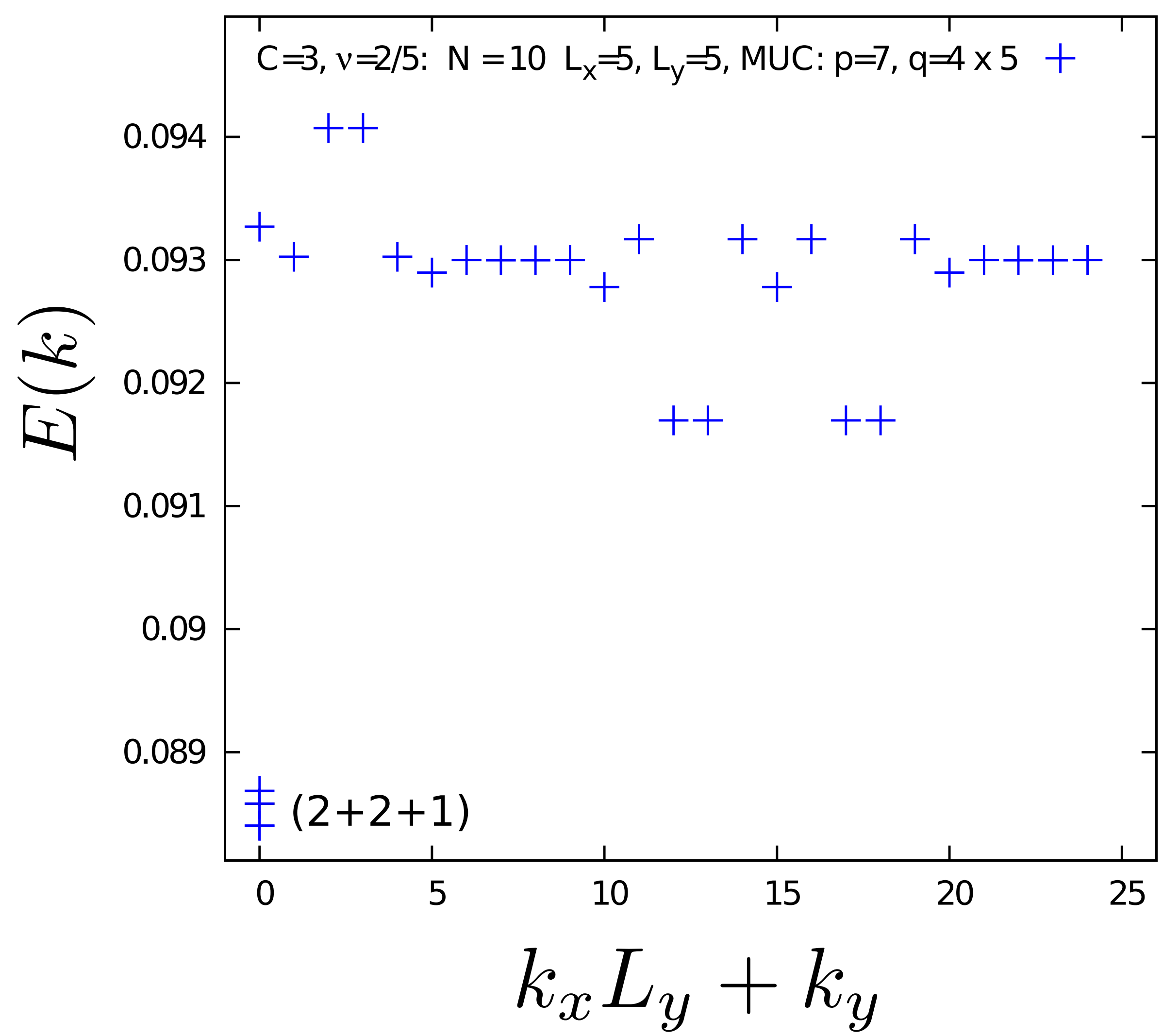
Exact Diagonalization: Spectra for new candidates

Example spectra for states with 'negative flux attachment': bosons, r=-2

$C = 2$



$C = 3$

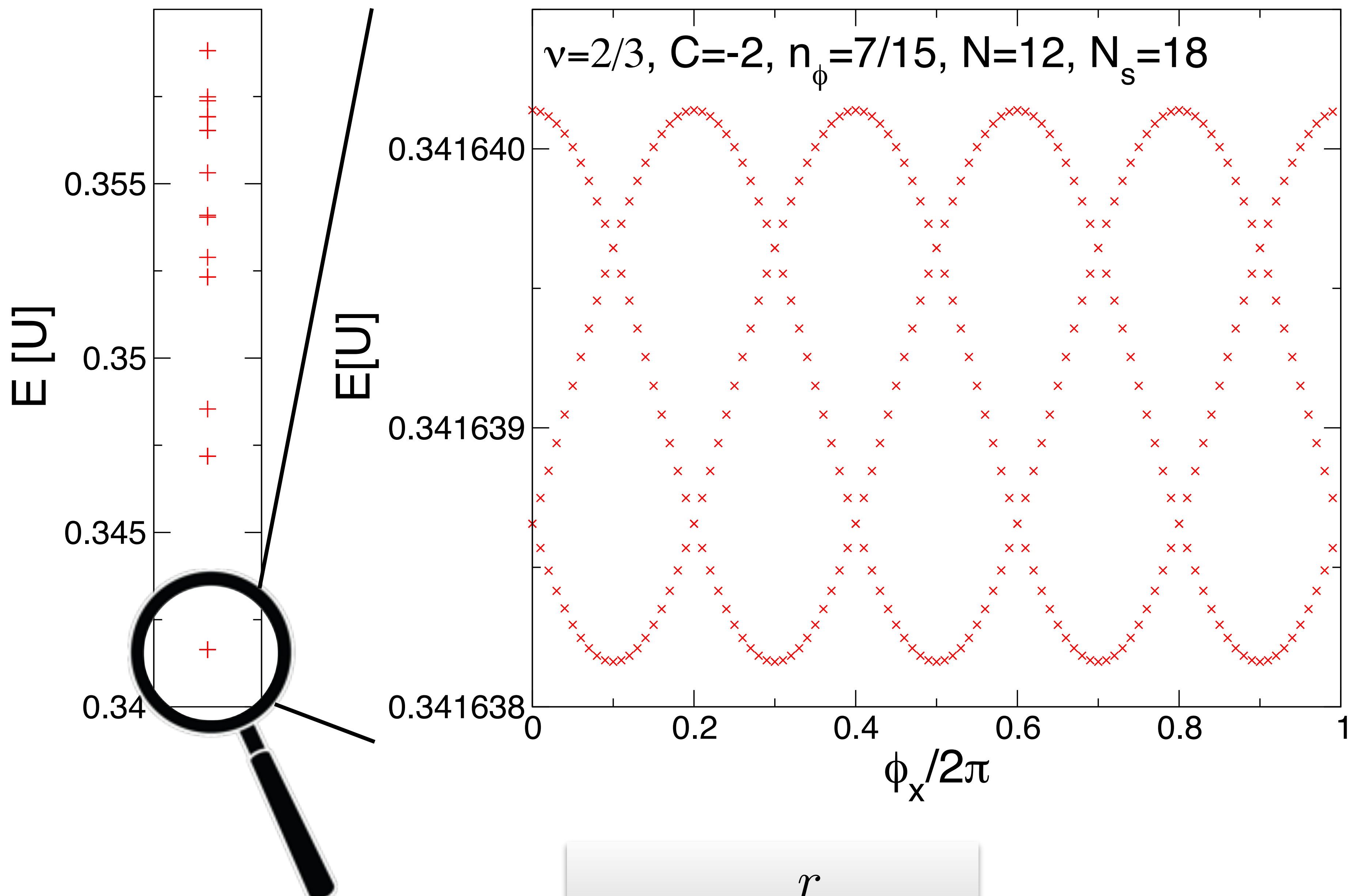


$$\nu = \frac{r}{r|Ck| + 1}$$

GM & NR Cooper, PRL (2015)

Exact Diagonalization: Spectral flow

Evolution of the ground states under "threading flux"



$$\nu = \frac{r}{r|Ck| + 1}$$

GM & NR Cooper, PRL (2015)

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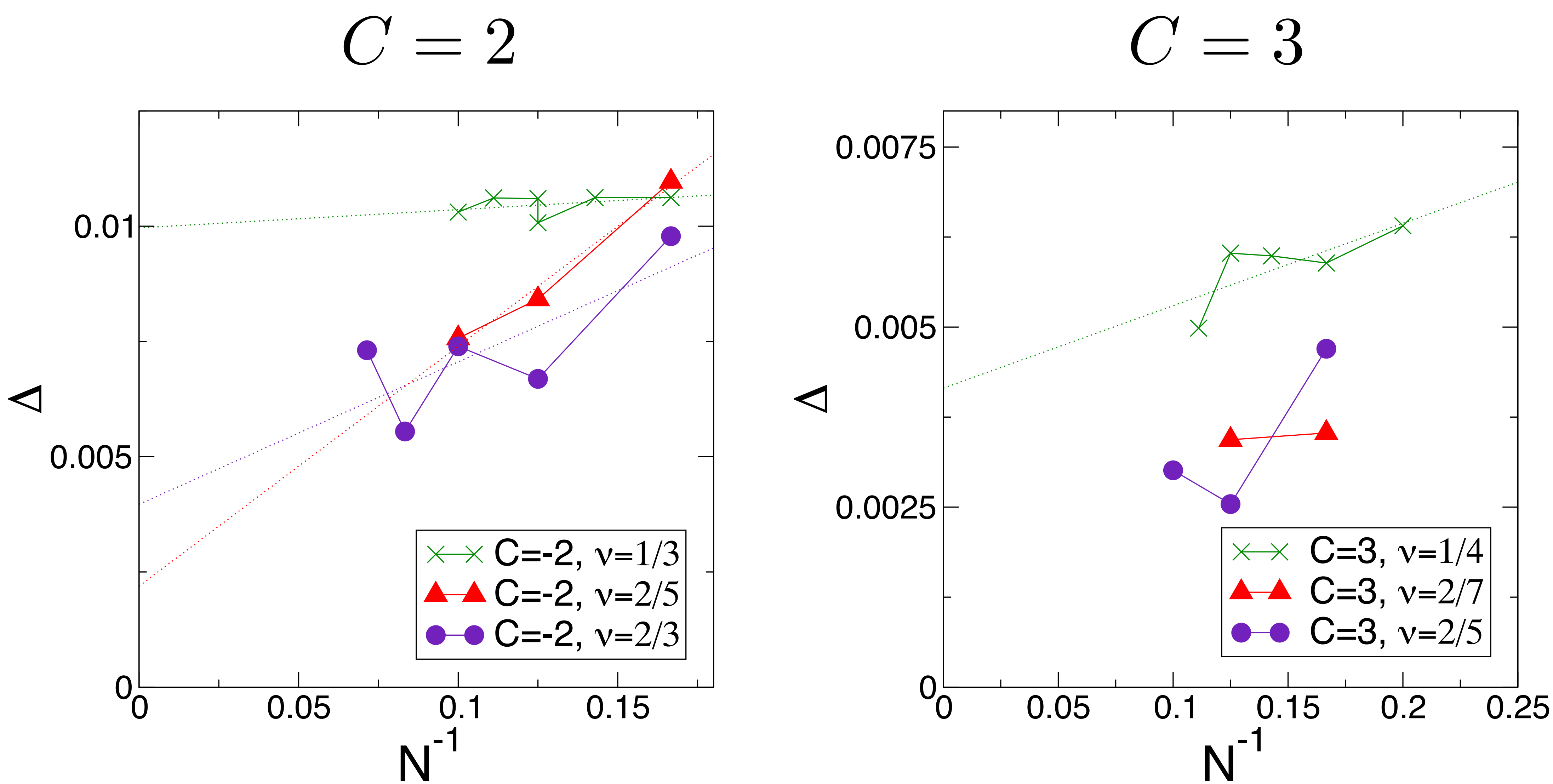
Exact Diagonalization: Finite Size Scaling of Gaps

Ascertain that GS degeneracy with finite gap is found consistently for different N_s



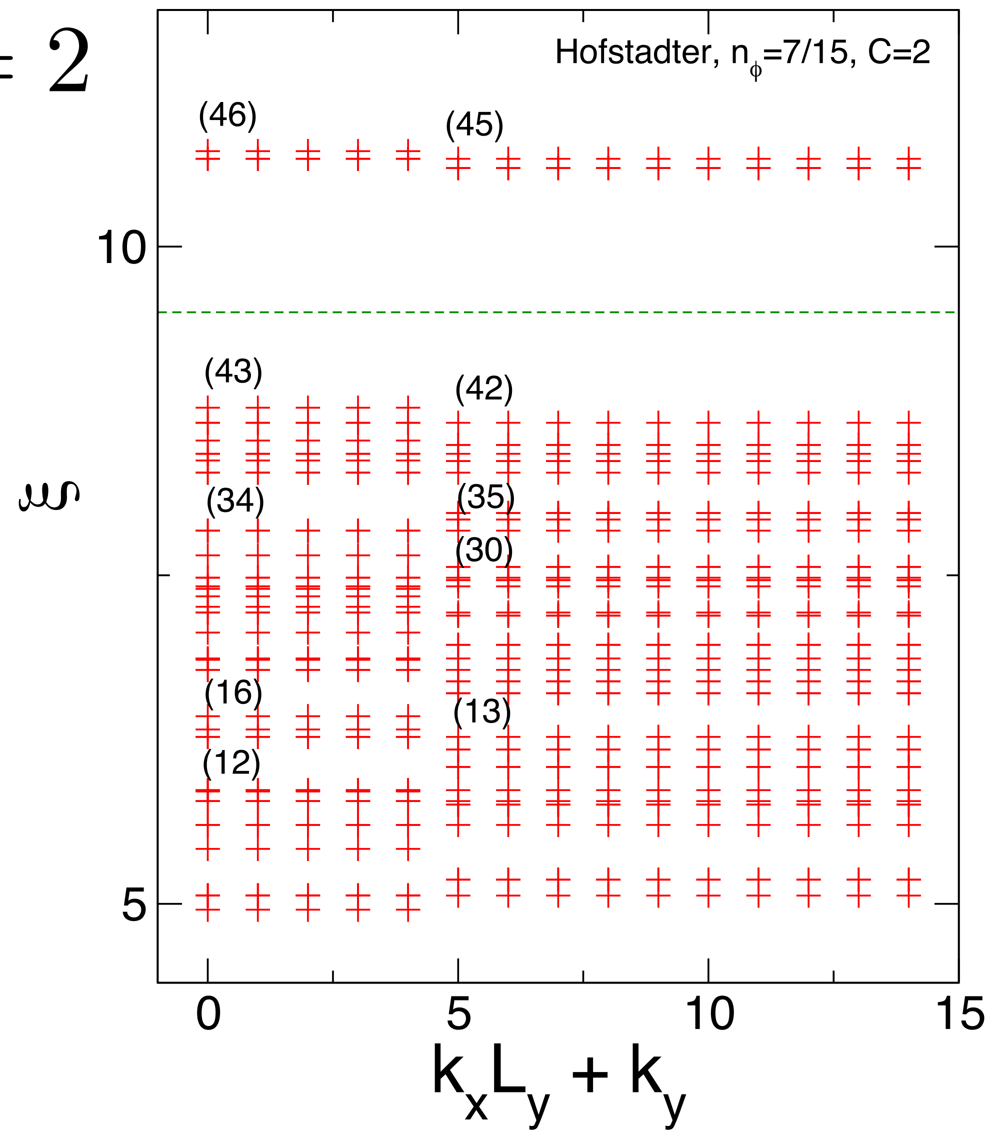
⇒ data suggests the composite fermion states are incompressible in the thermodynamic limit

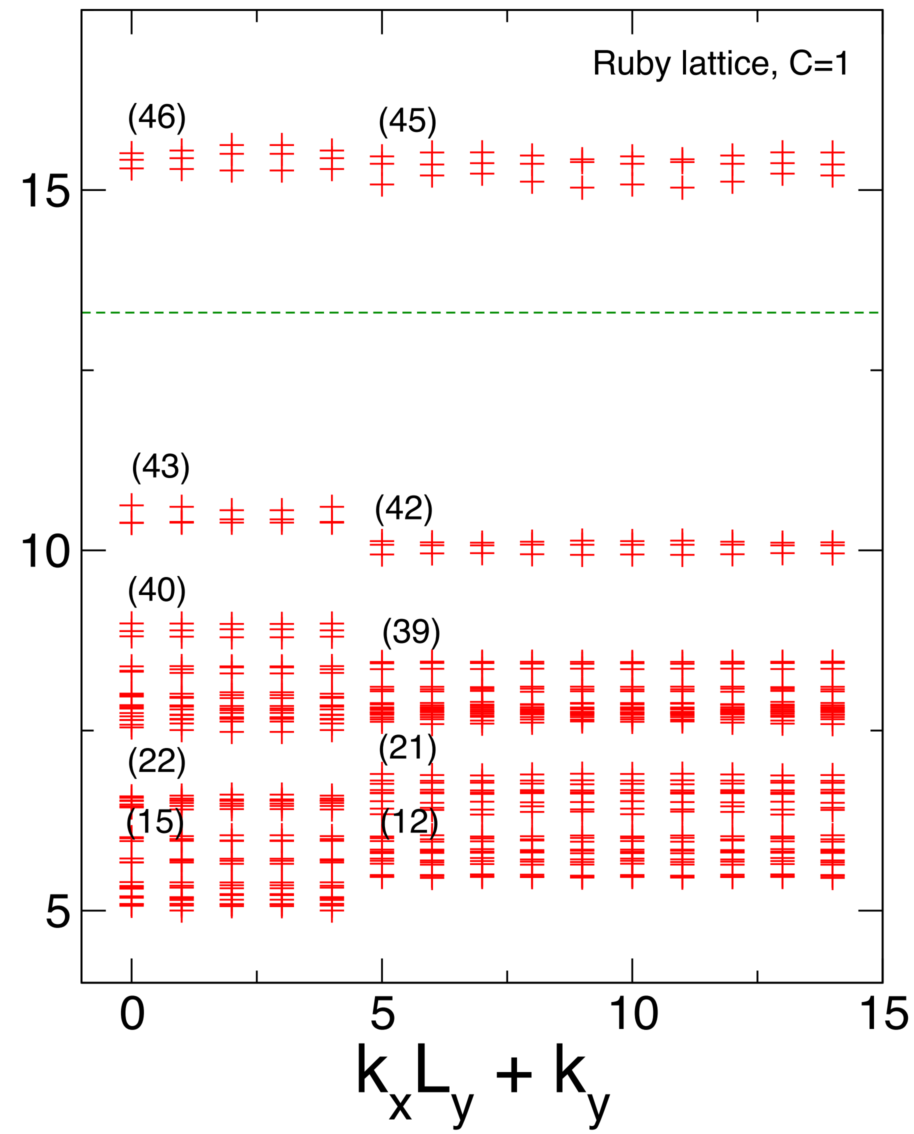
GM & NR Cooper, PRL **115**, 126401 (2015), arXiv:1504.06623

Exact Diagonalization: Particle Entanglement Spectra

Compare PES of a C=2 system to a known C=1 spectrum: ν=2/3

$C = 2$





$C = 1$

⇒ smaller entanglement energies; differences in detail

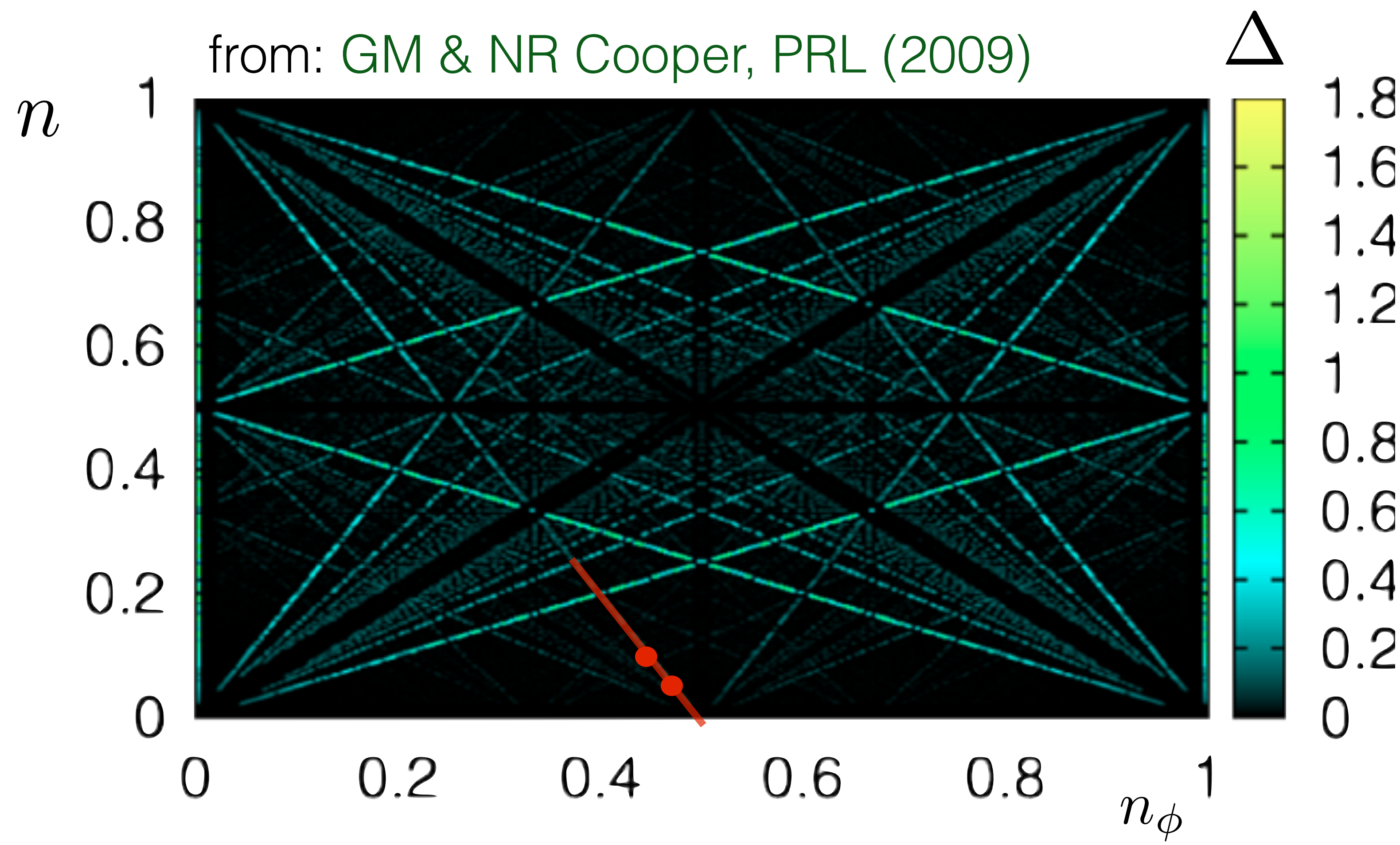
⇒ overall features similar

GM & NR Cooper, PRL **115**, 126401 (2015), arXiv:1504.06623

A Special case - Bosonic IQHE in C=2 bands

Bosons in a C=2 band with negative flux attachment (r=-1) ⇒ ν=1

from: GM & NR Cooper, PRL (2009)



alternative realisations:

- Quantum Hall Bilayers [Regnault & Senthil 2013]
- Honeycomb with correlated hopping [He et al. 2015]
- Optical Flux Lattices [Sterdyniak et al 2015]

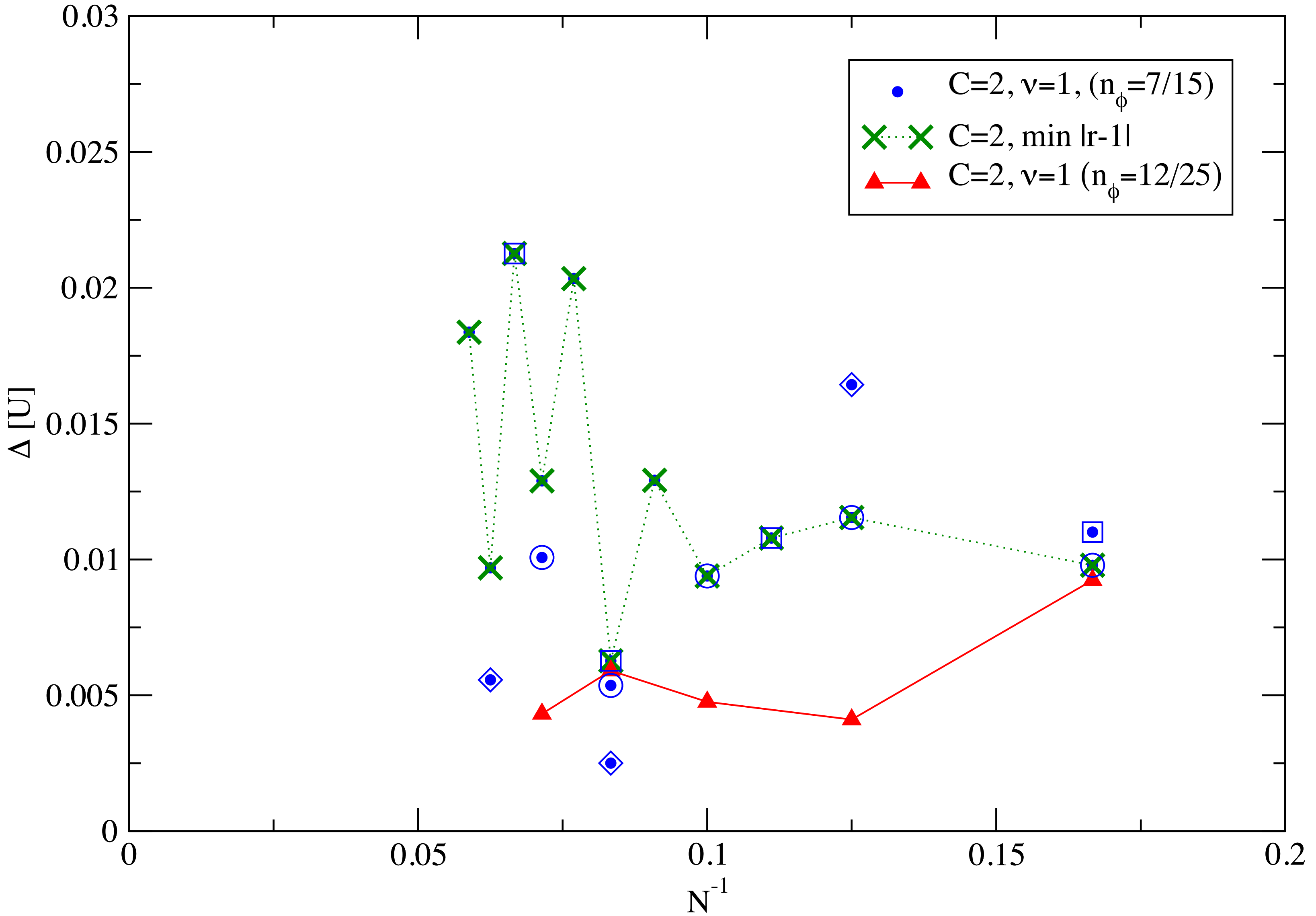
- first evidence in Hofstadter model GM & NR Cooper, PRL (2009)
- quasiparticles are fermions - not fractionalized

⇒ only symmetry protected topological phase [Senthil & Levin, PRL (2013)]

GM & NR Cooper, PRL (2009) & PRL (2015), arXiv:1504.06623; Hormozi et al. PRL 2012

A Special case - Bosonic IQHE in C=2 bands

Many-body gap: finite-size scaling at fixed flux density



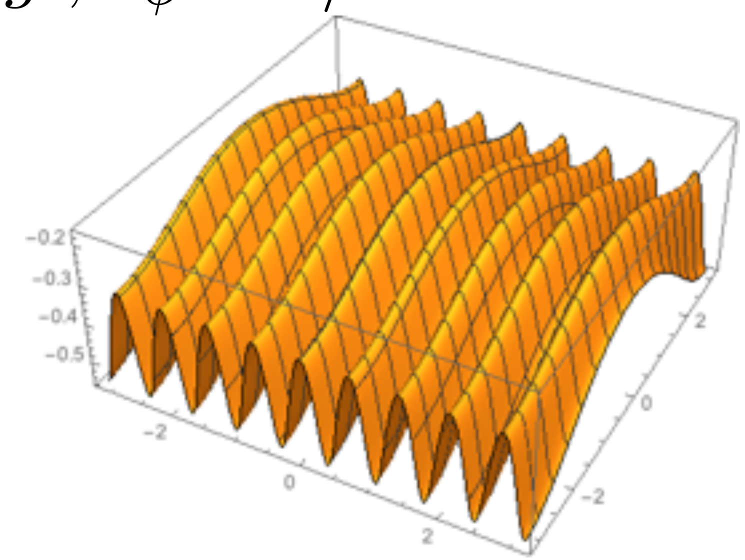
- significant geometry-dependency - but less so for flatter bands.

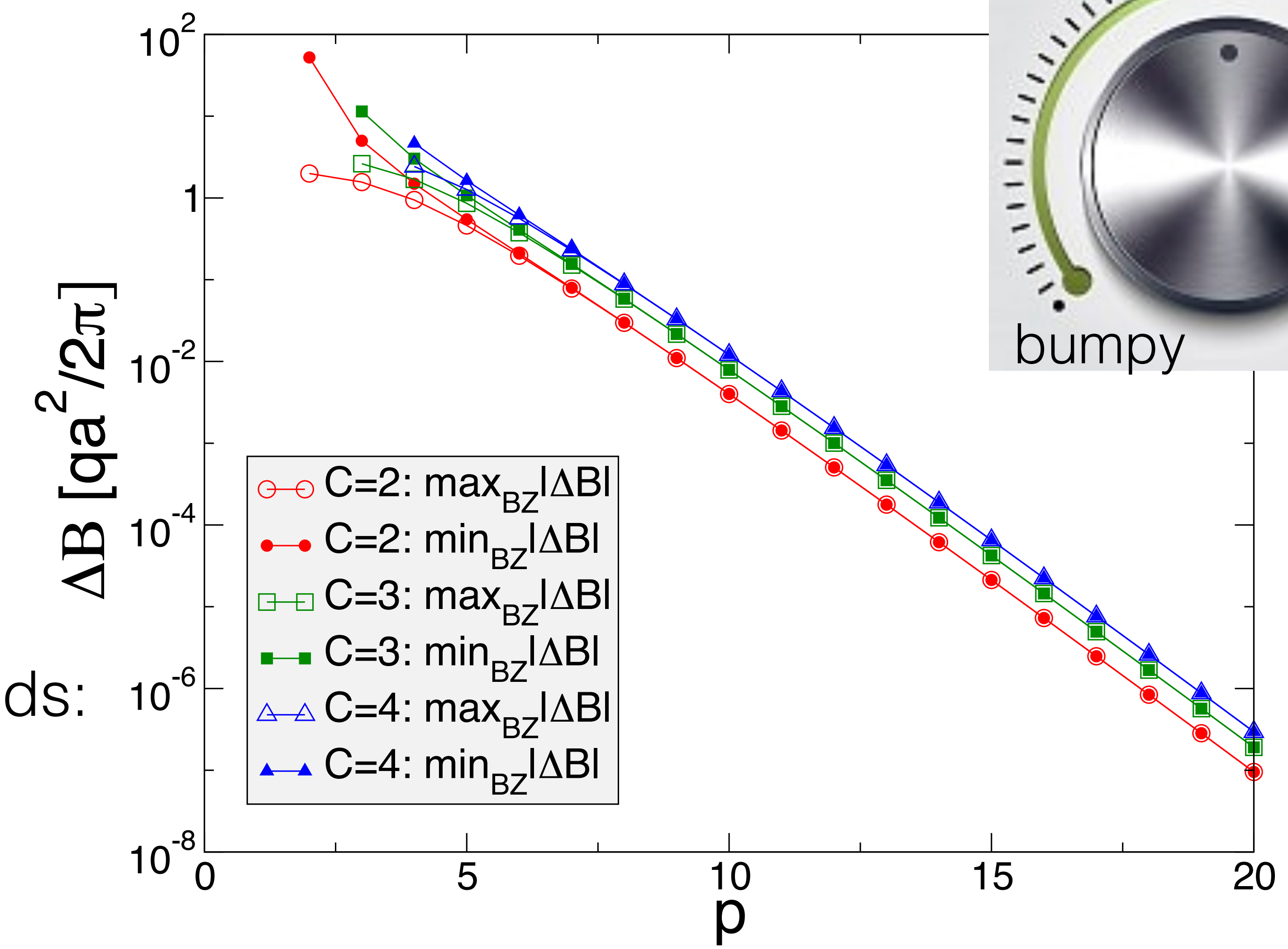
GM & NR Cooper, PRL (2009) & arXiv:1504.06623; Hormozi et al. PRL 2012

Tuning band flatness in the Hofstadter spectrum

Berry curvature exponentially flat in proximity to $n_\phi = 1/|C|$

$e.g., n_\phi = 4/9$





general case for single bands:

$$n_\phi = \frac{p}{|C|p - \mathrm{sgn}(C)}, \quad p \in \mathbb{N}$$

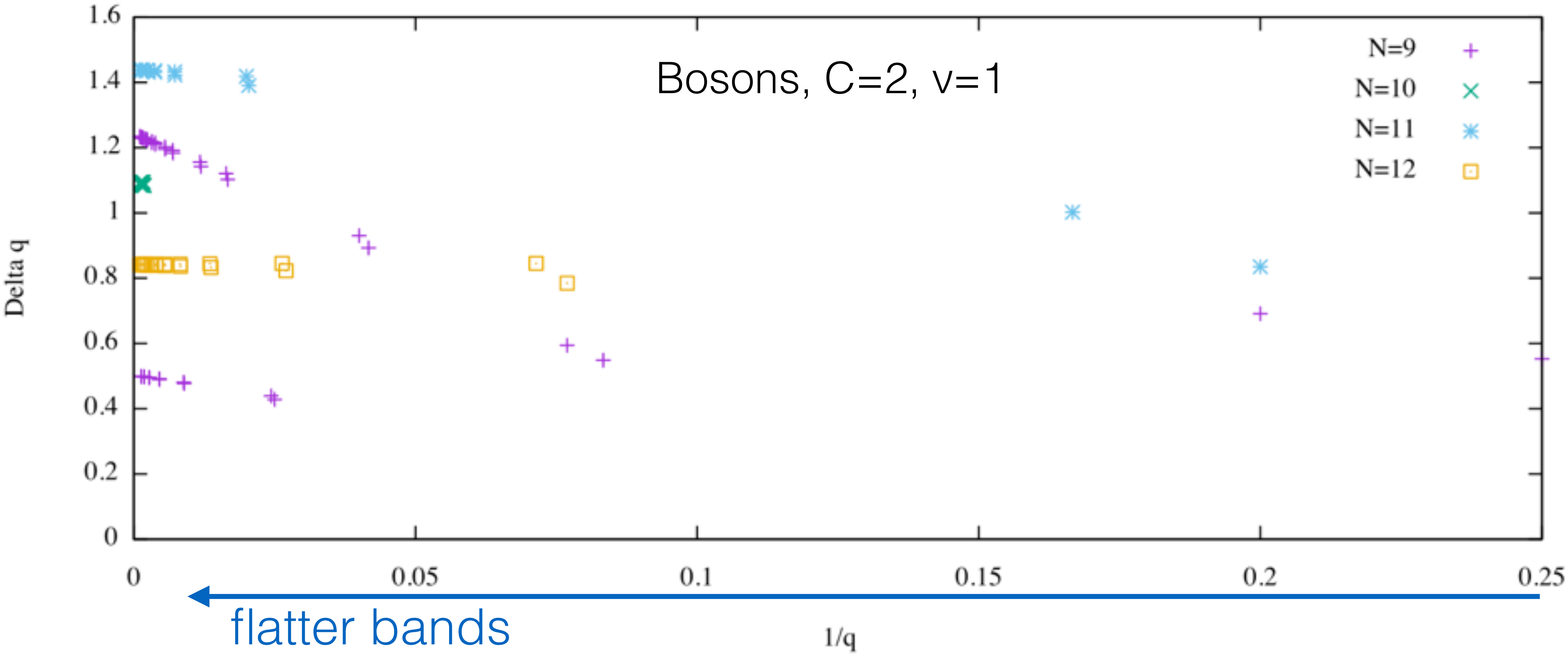
- can tune flatness of band geometry while keeping same physics

GM & NR Cooper, arXiv:1504.06623; Bauer, Jackson & Roy arxiv:1504.07185

Flat-band limit of Hofstadter bands: BIQHE

Bart

- Enlarge the magnetic unit cell (q sublattices) at fixed particle number
- Find some configurations with (near) square aspect ratio
- Compensate for natural scaling $\Delta \propto q^{-1}$ (bosons)



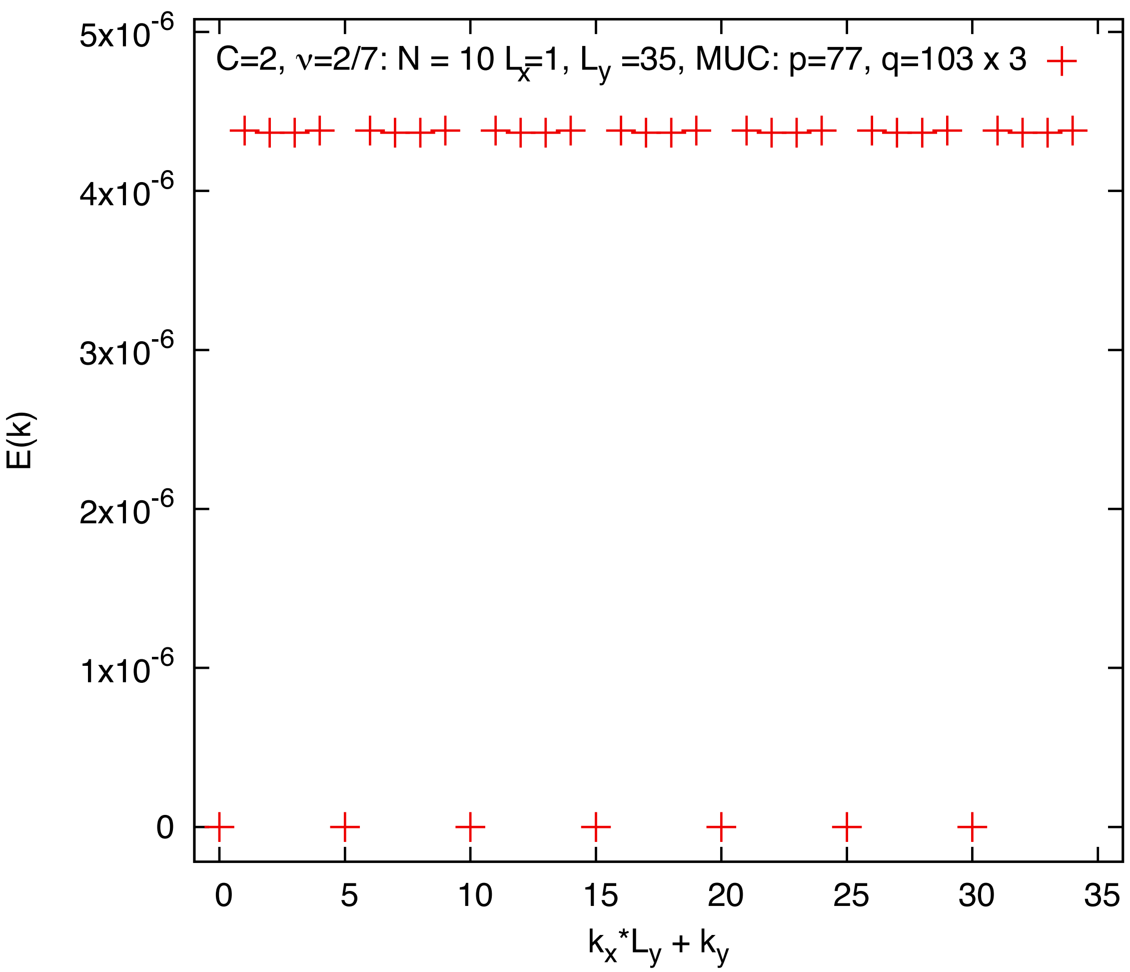
- flat band limit stabilises states, but finite size effects in N remain important

cf C=1 case in Bauer et al., arxiv:1504.07185; B. Andrews, GM, to be published.

Fermionic FCI in Hofstadter bands

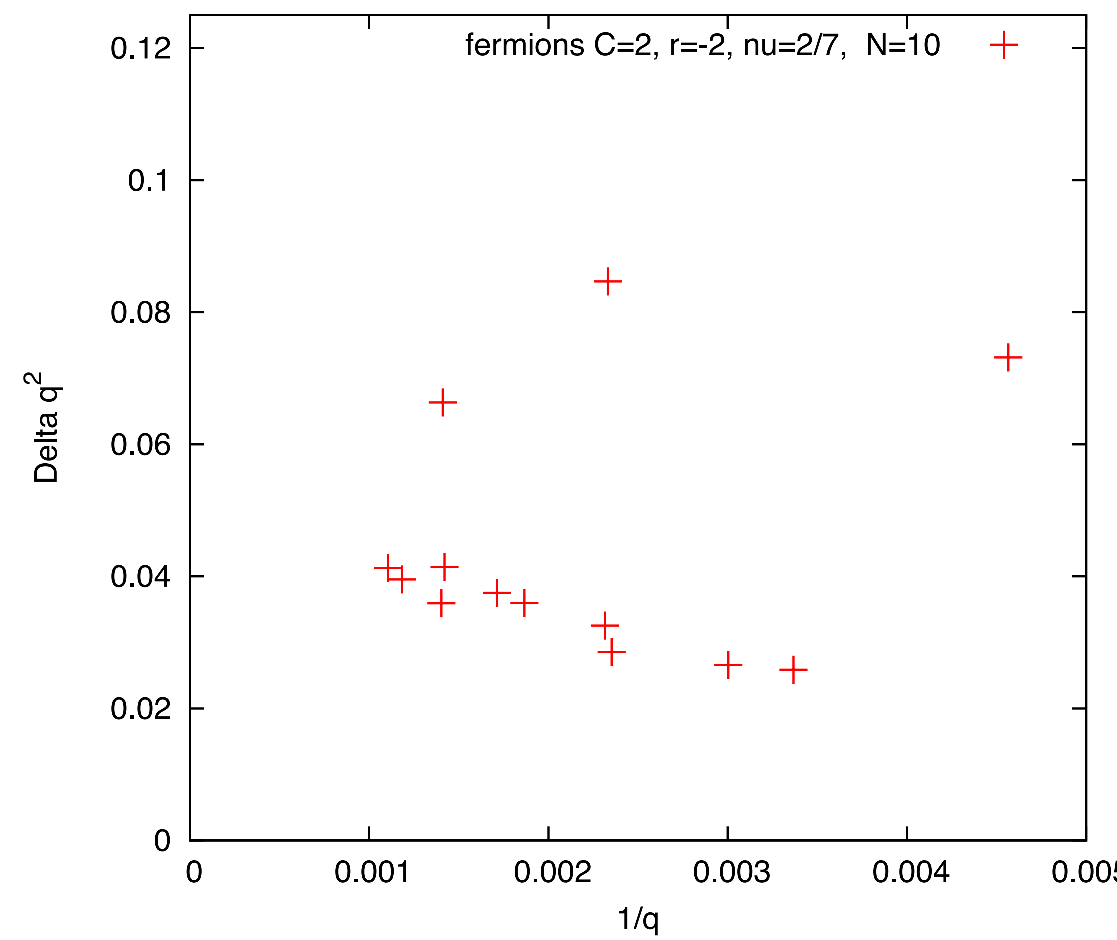
Good evidence for fermionic states, too: e.g. C=2, r=-2, ν=2/7:

example spectrum



ground state degeneracy d=7

flat band limit (N=10)



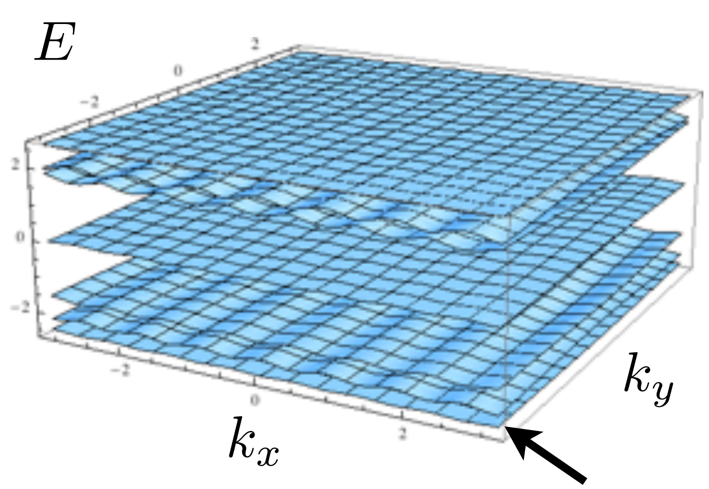
flatter bands

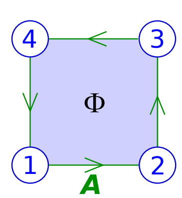
other competing states also present - e.g. generalisations of Read-Rezayi
see e.g., poster by Jörg Behrmann (k-body interactions)

Universality of Predictions

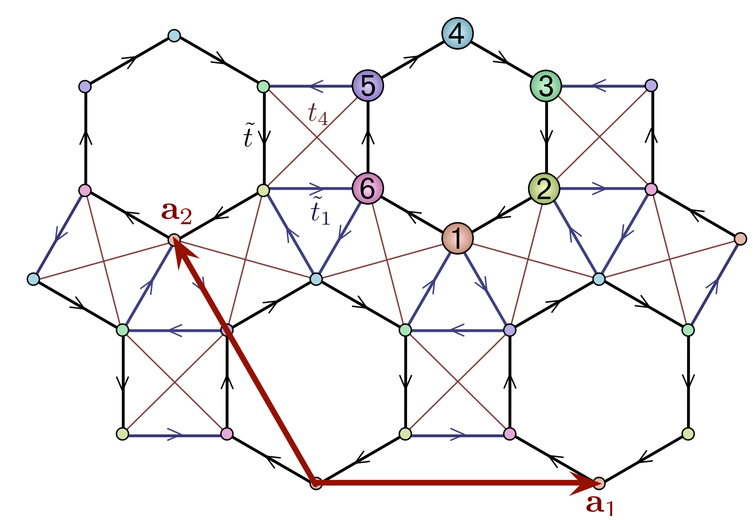
- Again, argue with adiabatic deformations:

- Hofstadter generates bands of any Chern #





- can deform to any other model…



- adiabatic connection for single bands as long as

$$C_1 = C_2$$

⇒ flux attachment provides candidate states for all Chern bands $\nu = \dfrac{r}{r|Ck|+1}$

Conclusions

The FQHE can be stabilised in general topological flat bands:

- Artificial magnetic fields ‘work as advertised’: same phases achieved
- Systems emulating Landau-levels closely work best
- Allows many new platforms for exploring quantum Hall physics

- New states exist in $|C|>1$ Chern bands: we predicted the series

$$\nu = \frac{r}{r|Ck|+1}$$

[k odd (even) for bosons (fermions)]

- Series includes a Bosonic Integer QHE in C=2 bands

- Numerical evidence matches the predictions (bosons, contact int.):
  - correct GS degeneracy
  - robust gap

GM & NR Cooper, PRL (2009) + (2015); T. Scaffidi & GM PRL (2012), T. Jackson, GM, R. Roy, Nat. Comm. (2015)