Global results in special geometry: a survey

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Outline of the lecture

Special geometry

Dimensional reduction

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An effective completeness criterion

Special geometry I

Scalar geometry

$$\mathcal{L} = -\sum g_{ij}(\phi^1, \ldots, \phi^n) h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j + \ldots$$

Physics Definition

Special geometry is the scalar geometry of supersymmetric field theories with 8 real supercharges ($N = 2$ theories).

One distinguishes between

- Affine special geometry/rigid supersymmetric theories
- Projective special geometry/supergravity theories.

Here we restrict to

- Minkowskian (rather than Euclidean or other) space-time signature.

Special geometry II

Scalar geometry of $N = 2$ theories depends on: space-time dimension d and field content: vector multiplets or hypermultiplets

Special geometries of rigid $N = 2$ supersymm. theories

d	vector multiplets	hypermultiplets
5	affine special real	hyper-Kähler
4	affine special Kähler	hyper-Kähler
3	hyper-Kähler	hyper-Kähler

Special geometries of $N = 2$ supergravity theories

d	vector multiplets	hypermultiplets
5	projective special real	quat. Kähler
4	projective special Kähler	quat. Kähler
3	quaternionic Kähler	quat. Kähler

Dimensional reduction

Dimensional reduction from 5 to 4 space-time dimensions

- ... of sugra coupled to VMs relates the corresponding scalar geometries by a construction called the supergravity r-map [DV] de Wit, Van Proeyen (CMP '92).
- The analogous construction for rigid theories is called the rigid r-map [CMMS] C.–, Mayer, Mohaupt, Saueressig (JHEP '04).

Dimensional reduction from 4 to 3 space-time dimensions

- ... of sugra coupled to VMs relates the corresponding scalar geometries by the supergravity c-map [FS] Ferrara, Sabharwal (NPB '90).
- Similarly, the scalar geometries of rigid $N = 2$ vector multiplets in 4 and 3 space-time dimensions are related by the rigid c-map [CFG] Cecotti, Ferrara, Girardello (IJMP '89).

Talk is based on various collaborations

[CDJL] C.–, Dyckmanns, Juengling, Lindemann, in preparation

[CDS] C.–, Dyckmanns, Suhr, in preparation

[CNS] C.–, Nardmann, Suhr (CAG, accepted 03/15), math.DG:1407.3251

[ACDM] Alekseevsky, C.– , Dyckmanns, Mohaupt (JGP '15)

[CDL] C.–, Dyckmanns, Lindemann (PLMS '14)

- [ACM] Alekseevsky, C.– , Mohaupt (CMP '13), [CHM] C.–, Han, Mohaupt (CMP '12), [CM] C.–, Mohaupt (JHEP '09), [ACD] Alekseevsky, C.– , Devchand (JGP '02), [C] C.– (TAMS '98)

Further related work

[D] Dyckmanns, PhD thesis Hamburg, 09/15

[MS] Macía, Swann (CMP '15)

[Hi13] Hitchin (CMP '13)

[APP] Alexandrov, Persson, Pioline (JHEP '11)

- [Hi09] Hitchin PIM '09, [Ha], Haydys (JGP '08), [RSV] Robles-Llana, Saueressig, Vandoren (JHEP '06), [AMTV] Antoniadis, Minasian, Theisen, Vanhove CQG '03, [F] Freed (CMP '99), [L] LeBrun (Duke '91).

Preservation of completeness under dimensional reduction

Theorem [CHM]

- The supergravity r-map associates a complete projective special Kähler manifold of dimension $2n + 2$ with every complete projective special real manifold of dimension n.
- The supergravity c-map associates a complete quaternionic Kähler manifold of dimension $4n + 4$ (of negative scalar curvature) with every complete projective special Kähler manifold of dimension $2n$.

Mathematical relevance

- Only a few constructions of complete quaternionic Kähler manifolds are know.
- Above constructions (and quantum corrections) yield explicit complete metrics if the completeness of the initial metrics is under control.

Projective special real manifolds I: extrinsic definition

Definition

A projective special real (PSR) manifold is a hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ s.t. $\exists$ homog. cubic polynomial h on $\mathbb{R}^{n+1}$ s.t.

i) $h = 1$ on $\mathcal{H}$ and

ii) $\partial^2 h$ is negative definite on $T\mathcal{H}$.

$\mathcal{H}$ is endowed with the Riemannian metric

$$g_{\mathcal{H}} = -\frac{1}{3}\iota^* \partial^2 h,$$

where $\iota : \mathcal{H} \to \mathbb{R}^{n+1}$ is the inclusion map.

$\mathcal{H}$ complete $:\iff$ $(\mathcal{H}, g_{\mathcal{H}})$ complete.

Affine special Kähler manifolds

Definition

A (pseudo-) Kähler manifold (M, g, J) is a (pseudo-) Riemannian manifold (M, g) endowed with a parallel skew-symm. cx. str. J.

Definition [F]

An affine special (pseudo-) Kähler manifold (M, J, g, ∇) is a (pseudo-) Kähler mf. (M, J, g) endowed with a flat torsionfree connection ∇ such that

(i) $\nabla\omega = 0$, where $\omega = g(\cdot, J\cdot)$,

(ii) $d^\nabla J = 0$, where J is considered as a 1-form with values in TM.

Conical and projective special Kähler manifolds

Definition [ACD, CM]

A conical affine special Kähler (CASK) manifold (M, J, g, ∇, ξ) is an affine special (pseudo-)Kähler manifold (M, J, g, ∇) endowed with a vector field ξ such that

(iii) $\nabla\xi = D\xi = \mathrm{Id}$, where D is the Levi Civita connection and

(iv) g is positive definite on $\mathcal{D} := \mathrm{span}\{\xi, J\xi\}$ and negative definite on $\mathcal{D}^{\perp}$.

$\Rightarrow$ ξ and $J\xi$ generate a hol. action of a 2-dim. Abelian Lie algebra. We will assume that the action lifts to a principal $\mathbb{C}^*$-action with the base $\bar{M} = M/\mathbb{C}^*$. Then $J\xi$ generates a free isometric and Hamiltonian S^1-action and $\bar{M}$ inherits a Kähler metric $\bar{g}$. $(\bar{M}, \bar{g})$ is called a projective special Kähler (PSK) manifold.

Extrinsic construction of special Kähler manifolds I

The ambient space

$V = (\mathbb{C}^{2n}, \Omega, \tau)$, $\Omega = \sum dz^i \wedge dw_i$, $\tau =$ cx. conjugation.
$\to$ pseudo-Hermitian form $\gamma := \sqrt{-1}\Omega(\cdot, \tau\cdot)$.

Definition

A holomorphic immersion $\phi : M \to V$ is called nondegenerate if $\phi^*\gamma$ is nondeg. It is called Lagrangian if $\phi^*\Omega = 0$ and $\dim M = n$.

Theorem [ACD]

- A nondeg. hol. Lagrangian immersion $\phi : M \to V$ induces an affine special pseudo-Kähler structure (J, g, ∇) on M.
- Every s.c. affine special (pseudo-) Kähler mf. (M, J, g, ∇) of dim. n admits a nondeg. Lagr. immersion $\phi : M \to V$ inducing (J, g, ∇) on M. The immersion is unique up to affine transformations with real symplectic linear part.

Extrinsic construction of special Kähler manifolds II

Example (affine special pseudo-Kähler domains)

Let F be a holomorphic function defined on a domain $M \subset \mathbb{C}^n$ such that the matrix

$$(N_{ij}) = (2\mathrm{Im}\, F_{ij}),$$

is nondeg, where $F_i = \frac{\partial F}{\partial z^I}$, $F_{ij} = \frac{\partial F}{\partial z^i \partial z^j}$ etc. Then

$$\phi : M \to V, \quad z = (z^1, \ldots, z^n) \mapsto (z, F_1, \ldots, F_n)$$

is a nondeg. Lagr. immersion and, thus, induces an affine special pseudo-Kähler structure (J, g, ∇) on M.

Definition

Affine special pseudo-Kähler manifolds as in the above example are called affine special pseudo-Kähler domains. The function F is called a holomorphic prepotential.

Extrinsic construction of special Kähler manifolds III

Since every Lagrangian submanifold of (V, Ω) is locally defined by equations $w_i = F_i(z)$, $i = 1, \ldots, n$, for some hol. function F and some choice of adapted coordinates (z^i, w_i), we obtain:

Corollary

Let (M, J, g, ∇) be an affine special pseudo-Kähler manifold. Then for every $p \in M$ there exists a neighborhood U isomorphic to an affine special pseudo-Kähler domain.

Remark

Similar results hold for conical and projective special Kähler manifolds. CASK manifolds are realized as conical hol. nondeg. Lagrangian immersions. The corresponding prepotential is defined on a $\mathbb{C}^*$-invariant domain $M \subset \mathbb{C}^n$ and is required to be homogeneous of degree 2 and to satisfy: $\sum N_{ij} z^i \bar{z}^j > 0$ and the real symmetric matrix (N_{ij}) has signature $(1, n-1)$ on M.

Extrinsic construction of special Kähler manifolds IV

Example (complex hyperbolic space as PSK domain)

$$F = \frac{i}{4}\left((z^0)^2 - \sum_{j=1}^{n}(z^j)^2\right)$$

on $M = \{|z^0|^2 - \sum_{j=1}^{n}|z^j|^2 > 0\} \subset \mathbb{C}^{n+1}$ is a prepot. for a CASK domain (M, J, g, ∇, ξ). The corresponding PSK domain is $\mathbb{C}H^n$.

The supergravity r-map I: from projective special real to projective special Kähler manifolds

- The sugra r-map can be described as follows [CHM]:
- Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a PSR mf. and h the corresponding cubic polynomial.
- Then $U = \mathbb{R}^{>0} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$ is an open cone.
- We endow it with the Riem. metric

$$g_U = -\frac{1}{3}\partial^2 \ln h,$$

  isometric to the product metric $dt^2 + g_{\mathcal{H}}$ on $\mathbb{R} \times \mathcal{H}$
- and finally the domain $\bar{M} = U \times \mathbb{R}^{n+1}$ with the Riem. metr.

$$g_{\bar{M}} := \frac{3}{4} \sum_{a,b=1}^{n+1} g_{ab}(dx^a dx^b + dy^a dy^b), \quad g_{ab} := g_U\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right).$$

The supergravity r-map II

Theorem

(i) $(\bar{M}, g_{\bar{M}})$ *defined above is projective special Kähler with respect to the cx. structure J defined by the embedding*

$$\bar{M} = U \times \mathbb{R}^{n+1} \to \mathbb{C}^{n+1}, \quad (x, y) \mapsto y + ix.$$

(ii) *The natural inclusions $\mathcal{H} \subset U \cong U \times \{0\} \subset \bar{M}$ are totally geodesic.*

- The correspondence $\mathcal{H} \mapsto (\bar{M}, J, g_{\bar{M}})$ is the supergravity r-map.
- It maps PSR mfs. of dim. n to PSK mfs. of (real) dim. 2n+2.

The supergravity c-map I: from projective special Kähler to quaternionic Kähler manifolds

- The supergravity c-map metric (or Ferrara-Sabharwal metric) g_{FS} resulting from dim. reduction of sugra coupled to vector multiplets from 4 to 3 space-time dimensions was computed in [FS]. The QK property was also proven in [Hi09].
- Here we follow [CHM]: In the case of a PSK domain $(\bar{M}, g_{\bar{M}})$ of dim. $2n$ the metric g_{FS} has the following structure:

$$g_{FS} = g_{\bar{M}} + g_G,$$

  where g_G is a family of left-invariant Riemannian metrics on $G = \mathrm{Iwa}(\mathrm{SU}(n+2,1))$ depending on $p \in \bar{M}$.
- In particular, g_{FS} is defined on the product $\bar{N} := \bar{M} \times G$.
- The inclusion $\bar{M} \cong \bar{M} \times \{e\} \subset \bar{N}$ is totally geodesic.

The supergravity c-map II

The explicit form of the family of metrics $(g_G(p))_{p\in M}$:

$$\frac{1}{4\phi^2}d\phi^2 + \frac{1}{4\phi^2}\left(d\tilde{\phi} + \sum(\zeta^i d\tilde{\zeta}_i - \tilde{\zeta}_i d\zeta^i)\right)^2 + \frac{1}{2\phi}\sum \mathcal{I}_{ij}(p)d\zeta^i d\zeta^j$$

$$+\frac{1}{2\phi}\sum \mathcal{I}^{ij}(p)\left(d\tilde{\zeta}_i + \sum \mathcal{R}_{ik}(p)d\zeta^k\right)\left(d\tilde{\zeta}_j + \sum \mathcal{R}_{j\ell}(p)d\zeta^\ell\right),$$

- where $(\phi, \tilde{\phi}, \zeta^1, \ldots, \zeta^{n+1}, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_{n+1}) : G \to \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ is a global coord. system on $G \cong \mathbb{R}^{2n+4}$ and
- $\mathcal{R}_{ij}$, $\mathcal{I}_{ij}$ are real and imaginary parts of

$$\bar{F}_{ij} + \sqrt{-1}\frac{\sum N_{ik}z^k \sum N_{j\ell}z^\ell}{\sum N_{kl}z^k z^\ell},$$

- determined by the prepot. F of the underlying CASK dom.
- $\mathcal{I} = (\mathcal{I}_{ij}) > 0$. Hence $(\mathcal{I}^{ij}) = \mathcal{I}^{-1}$ is defined and $g_G > 0$.

The supergravity c-map III

Geometric interpretation of the fiber metric

- $(G, g_G(p))$ is isometric to $\mathbb{C}H^{n+2}$.
- The principal part of

$$g_G = \frac{1}{4\phi^2} d\phi^2 + \frac{1}{4\phi^2}\left(d\tilde{\phi} + \sum(\zeta^i d\tilde{\zeta}_i - \tilde{\zeta}_i d\zeta^i)\right)^2 + \frac{1}{2\phi} g_G^{pr}$$

  is related to the CASK domain $\pi : M \to \bar{M}$ as follows:
- M has a can. realization as a Lagrangian cone in $V = (\mathbb{C}^{2n+2}, \Omega, \gamma)$, where $g_M = \mathrm{Re}\,\gamma|_M$ is induced.
- Therefore we have a hol. map $\bar{M} \to Gr_0^{1,n}(V) = Sp(\mathbb{R}^{2n+2})/U(1,n)$, $p \mapsto L_p$.
- Composing it with the $Sp(\mathbb{R}^{2n+2})$-equivariant embedding

$$Gr_0^{1,n}(V) \to Sym^1_{2,2n}(\mathbb{R}^{2n+2}) = SL(2n+2,\mathbb{R})/SO(2,2n)$$

  we obtain $p \mapsto (g_{IJ}(p)) \in Sym^1_{2,2n}(\mathbb{R}^{2n+2})$.

The supergravity c-map IV

Geometric interpretation of the fiber metric continued

- In fact, $\sum g_{IJ}(p)dq^I dq^J = g_M(\tilde{p})$, $\forall \tilde{p} \in \pi^{-1}(p)$, where $(q^I)_{I=1,\ldots,2n+2}$ are conical affine Darboux coordinates.
- Next we change the indefinite scalar product $(g_{IJ}(p))$ to $(\hat{g}_{IJ}(p)) > 0$ by means of an $Sp(\mathbb{R}^{2n+2})$-equivariant diffeo. $\psi : F_0^{1,n}(V) \to F_0^{n+1,0}(V)$ from Griffiths to Weil flags.
- In the case of the CY_3 moduli space this is related to the switch from Griffiths to Weil intermediate Jacobians [C,Hi09]
- This corresponds to switching the sign of the indefinite metric g_M on the negative definite distribution $\mathcal{D}^\perp$.
- We show that the cx. symm. matrix $\mathcal{R} + i\mathcal{J} \in Sym_{n+1,0}(\mathbb{C}^{n+1})$ corresponds to the pos. def. Lagrangian subspace L' defined by $\psi(\ell, L) = (\ell, L')$, where $L = L_p$ and $\ell = p = \mathbb{C}\tilde{p}$. This proves $\mathcal{J} > 0$.
- Finally we prove that $g_G^{pr}(p) = \sum \hat{g}^{IJ}(p)dq_I dq_J$, where $(q_I) = (\tilde{\zeta}_i, \zeta^j)$.

The supergravity c-map V

Concluding remarks

- The c-map can be obtained as an application of an indefinite version of Haydys HK/QK-correspondence [ACM], see [Ha,APP,Hi13,MS] for related work.
- This can be used to give a proof of the QK property for an explicit 1-parameter deformation of the c-map metric [ACDM], known as the one-loop correction (on next slide).
- In the general case, when the PSK mf. $\bar{M}$ is covered by PSK domains, we show that the local Ferrara-Sabharwal metrics are consistent and define a QK mf. $\bar{N}$ which fibers over $\bar{M}$ as a bundle of groups with totally geodesic can. section $\bar{M} \hookrightarrow \bar{N}$.
- This shows that the supergravity c-map is globally defined for every PSK mf.

One-loop correction of the FS-metric

Consider the FS-metric associated with a PSK domain $\bar{M}$. The following symmetric tensor field is called one-loop correction of the FS-metric [RSV]:

$$
\begin{aligned}
g_{FS}^{c} &= \frac{\phi + c}{\phi} g_{\bar{M}} + \frac{1}{4\phi^2}\frac{\phi + 2c}{\phi + c} d\phi^2 \\
&+ \frac{1}{4\phi^2}\frac{\phi + c}{\phi + 2c}\left(d\tilde{\phi} + \sum(\zeta^j d\tilde{\zeta}_j - \tilde{\zeta}_j d\zeta^j) + ic(\bar{\partial} - \partial)\mathcal{K}\right)^2 \\
&+ \frac{1}{2\phi}\sum dq_a \hat{g}^{ab} dq_b + \frac{2c}{\phi^2} e^{\mathcal{K}} \left|\sum(X^j d\tilde{\zeta}_j + F_j(X) d\zeta^j)\right|^2,
\end{aligned}
$$

where $c \in \mathbb{R}$, $X^j = z^j/z^0$ and

$$
\mathcal{K} = -\log\left(\sum X^i N_{ij} \bar{X}^j\right)
$$

is the Kähler potential for the projective special Kähler metric $g_{\bar{M}}$.

Simplest example of a one-loop corrected QK metric

Example

For $\bar{M} = pt$, i.e. $F = \frac{i}{2}(z^0)^2$, we have:

$$
\begin{aligned}
g^c &= \frac{1}{4\phi^2}\left(\frac{\phi+2c}{\phi+c}d\phi^2 + \frac{\phi+c}{\phi+2c}(d\tilde{\phi} + \zeta^0 d\tilde{\zeta}_0 - \tilde{\zeta}_0 d\zeta^0)^2\right. \\
&\quad \left. + 2(\phi+2c)((d\tilde{\zeta}_0)^2 + (d\zeta^0)^2)\right),
\end{aligned}
$$

with g^0 the complex hyperbolic plane metric and g^c complete for $c \geq 0$.

Completeness of the one-loop corrected QK metric

Theorem [CDS,D]

- Let $(\bar{M}, \bar{g})$ be a PSK manifold with regular boundary behaviour. Then the corresponding one-loop deformation g^c_{FS} is a family of complete QK metrics for $c \geq 0$.
- Let $(\bar{M}, \bar{g})$ be a complete PSK manifold with cubic prepotential. Then the corresponding one-loop deformation g^c_{FS} is a family of complete QK metrics for $c \geq 0$.

Corollary [CDS,D]

All symmetric QK manifolds in the image of the c-map can be deformed in this way by complete QK manifolds.

Remark

The only symmetric QK manifold of noncp. type which is not in the image of the c-map is quaternionic hyperbolic space. Its metric is also know to admit deformations by complete QK metrics [L].

Classification of complete PSR curves and surfaces

Theorem [CHM]

There are only 2 complete PSR curves (up to equivalence):

i) $\{(x, y) \in \mathbb{R}^2 | x^2 y = 1, \, x > 0\}$,

ii) $\{(x, y) \in \mathbb{R}^2 | x(x^2 - y^2) = 1, \, x > 0\}$.

Theorem [CDL]

There are only 5 discrete examples and a 1-parameter family of complete PSR surfaces:

a) $\{(x, y, z) \in \mathbb{R}^3 | xyz = 1, \, x > 0, y > 0\}$,

b) $\{(x, y, z) \in \mathbb{R}^3 | x(xy - z^2) = 1, \, x > 0\}$,

c) $\{(x, y, z) \in \mathbb{R}^3 | x(yz + x^2) = 1, \, x < 0, y > 0\}$,

d) $\{(x, y, z) \in \mathbb{R}^3 | z(x^2 + y^2 - z^2), \, z < 0\}$,

e) $\{(x, y, z) \in \mathbb{R}^3 | x(y^2 - z^2) + y^3 = 1, \, y < 0, x > 0\}$,

f) $\{\cdots | y^2 z - 4x^3 + 3xz^2 + bz^3 = 1, \, z < 0, 2x > z\}$, $b \in (-1, 1)$.

Classification of complete PSR manifolds with reducible cubic polynomial

Theorem[CDJL]

Every complete PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$, $n \geq 2$, for which h is reducible is linearly equivalent to exactly one of the following:

a) $\{x_{n+1}(\sum_{i=1}^{n-1} x_i^2 - x_n^2) = 1,\ x_{n+1} < 0, x_n > 0\}$,

b) $\{(x_1 + x_{n+1})(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 + x_{n+1} < 0\}$,

c) $\{x_1(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 < 0, x_{n+1} > 0\}$,

d) $\{x_1(x_1^2 - \sum_{i=2}^{n+1} x_i^2) = 1, \quad x_1 > 0\}$.

- Under the q-map (composition of r- and c-map), these are mapped to complete QK manifolds of co-homogeneity ≤ 1.
- The series d) is mapped to a series of complete QK manifolds of co-homogeneity 1.

Completeness of centroaffine hypersurfaces

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a centroaffine hypersurface with positive definite centroaffine metric g.

We are interested in the relation between

1) closedness,
2) Euclidian completeness and
3) completeness (with respect to g).

Under natural assumptions:
3) $\implies$ 1) $\iff$ 2).

Main problem:

Prove that 1) $\implies$ 3) in some interesting cases.

Example: Theorem (Cheng and Yau, CPAM '89)

1) $\implies$ 3) if $\mathcal{H}$ is an affine sphere, i.e. if $\nabla^g \nu = 0$.

Completeness of PSR manifolds and QK manifolds

Theorem [CNS]

A PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is complete if and only if $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed.

Corollary

Let $\mathcal{H}$ be a locally strictly convex component of the level set $\{h = 1\}$ of a homogeneous cubic polynomial h on $\mathbb{R}^{n+1}$. Then $\mathcal{H}$ defines a complete quaternionic Kähler metric of negative scalar curvature on $\mathbb{R}^{4n+8}$.

Applications

Using the Corollary we can construct many new explicit complete QK manifolds and even families depending on an arbitrary number of parameters, including multi-parameter defos of symm. spaces [CDJL]. On top one can add 1 parameter by one-loop defo [CDS].

Sketch of proof of the theorem I

- Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a Euclidian complete centroaffine hypersurface with positive definite centroaffine metric g.
- We have to show that $\mathcal{H}$ is complete if $\mathcal{H} \subset \{h = 1\}$ for a homogeneous cubic polynomial h. Let us not assume this yet.
- Consider the open cone $U = \mathbb{R}^{>0} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$ and let $k \in \mathbb{R}^*$.

Lemma 1

- There exists a unique smooth homogeneous function $h : U \to \mathbb{R}$ of degree k such that $h|_{\mathcal{H}} = 1$.
- For every hyperplane E tangent to $\mathcal{H}$ the intersection $B := U \cap E \subset E$ is a bounded convex domain.
- $$\varphi : B \to \mathcal{H}, \quad x \mapsto h(x)^{-1/k} x,$$
  is a parametrization of $\mathcal{H}$.

Sketch of proof of the theorem II

Lemma 2

In the above parametrization the centroaffine metric is given

$$g = -\frac{1}{k\bar{h}}\partial^2 \bar{h} + \frac{k-1}{(k\bar{h})^2} d\bar{h}^2,$$

where $\bar{h}$ denotes the restriction of h to B and ∂ denotes the flat connection of the affine space $E \supset B$.

Lemma 3

Let $k > 0$. Assume that there exists $\varepsilon \in (0, k)$ such that $f = \sqrt[k-\varepsilon]{\bar{h}}$ is concave. Then $\mathcal{H}$ is complete.

Sketch of pf. of Lemma 3

A calculation shows

$$g = \frac{k-\varepsilon}{f}\left(-\frac{1}{k}\partial^2 f\right) + \frac{\varepsilon}{(k-\varepsilon)(k\bar{h})^2} d\bar{h}^2 \geq \underbrace{\frac{\varepsilon}{k^2(k-\varepsilon)}}_{C:=}(d\ln\bar{h})^2.$$

Sketch of proof of the theorem III

Let $\gamma : I = [0, T) \to B,\ T \in (0, \infty]$, be a curve which is not contained in any compact subset of B and $I \ni t_i \to T$.

- Then $h(\gamma(t_i)) \to 0$ and the previous estimate implies

$$\begin{aligned} L(\gamma) &\geq L(\gamma|_{[0,t_i]}) \geq C \int_0^{t_i} \left| \frac{d}{dt} \ln h \circ \gamma \right| dt \geq C \left| \int_0^{t_i} \frac{d}{dt} \ln h \circ \gamma dt \right| \\ &= C |\ln h(\gamma(t_i)) - \ln h(\gamma(0))| \to \infty \quad \square \end{aligned}$$

Lemma 4
If h is a cubic polynomial then $\sqrt{h}$ is concave

Lemma 4 shows that the assumptions of Lemma 3 are satisfied with $(k, \epsilon) = (3, 1)$. This finishes the proof of the theorem. $\square$