Nielsen–Olesen cosmic strings, the Einstein–Bogomol'nyi equations, and algebraic geometry

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QUESTIONS:

When is a moduli space non-empty?

$\Longleftrightarrow$ existence and obstructions to the existence of solutions of (Euler–Lagrange) equations

Geometry of a moduli space?

USEFUL TOOLS (SOMETIMES):

Geometry of quotients of manifolds by Lie group actions in (∞- and finite-dimensional) symplectic and algebraic geometry

SPECIFIC PROBLEM OF THIS TALK:

Moduli of vortices coupled to gravity on a compact Riemann surface

Very specific 20-year-old problem — the same methods have been applied to other problems [Atiyah, Bott, Donaldson, Hitchin. . .]

1. Abelian vortices on a compact Riemann surface

Ginzburg–Landau theory of superconductivity on a surface

Physics:

- $F_A =$ field strength tensor of U(1)-connection A
- $\phi =$ electron wave-function density amplitude (Cooper pairs)
- action functional depends on an order parameter λ:

$$S(A, \phi) = \int_X \left(|F_A|^2 + |d_A \phi|^2 + \frac{\lambda}{4}(|\phi|^2 - \tau)^2 \right) \omega_X$$

Complex geometry:

- X compact Riemann surface
- ω_X **fixed** Kähler 2-form on X
- $L \longrightarrow X$ holomorphic line bundle
- $\phi \in H^0(X, L)$ holomorphic section of L

In the so-called *Bogomol'nyi phase* $\lambda = 1$,

Euler–Lagrange equations $\Longleftrightarrow$ vortex equation

Vortex equation

for a Hermitian metric h on L:

$$i\Lambda F_h + |\phi|_h^2 = \tau$$

- $F_h \in \Omega^2(X)$ curvature 2-form of Chern connection of h on L
- $\Lambda F_h = g^{j\bar{k}} F_{j\bar{k}} \in iC^\infty(X)$ contraction of F_h with ω_X
- $|\cdot|_h \in C^\infty(X)$ pointwise norm on L associated to h
- $\tau \in \mathbb{R}$ constant parameter

Vortex = solution of the vortex equation

Integrating, i.e. applying $\frac{1}{\mathrm{vol}(X)} \int_X (-)\omega_X$ to the vortex equation,

$$\boxed{\deg L + \|\phi\|_{L^2}^2 = \tau}$$

where $\deg L := \frac{2\pi}{\mathrm{vol}(X)} \int_X c_1(L)$, $\|\phi\|_{L^2}^2 := \frac{1}{\mathrm{vol}(X)} \int_X |\phi|_h^2 \omega_X$, so

$$\phi \neq 0 \iff \tau > \deg L$$
$$\phi = 0 \iff \tau = \deg L$$

Theorem

$$\text{Existence of vortices} \iff \tau \geq \deg L$$

- If $\phi = 0$, by Hodge theorem, existence $\iff \deg L = \tau$
- For $\phi \neq 0$, there are several proofs:
  - — Noguchi (1987, $\tau = 1$): direct proof using tools of analysis
  - — Bradlow (1990): reduces to Kazdan–Warner equation in Riemannian geometry
  - — García-Prada (1991): dimensional reduction of Hermitian Yang–Mills equation from 2 to 1 complex dimension
- Previous work by Taubes (1980) on $\mathbb{R}^2$, after work by Witten (1977) on $\mathbb{R}^{1,1}$.

We will now review the proof by García-Prada via dimensional reduction of Hermitian Yang–Mills equations

2. Hermitian Yang–Mills equation

Generalization of instanton equation to Kähler manifolds

Data: M compact Kähler manifold with $n = \dim_{\mathbb{C}} M$

ω_M **fixed** Kähler 2-form on M

$E \longrightarrow M$ holomorphic vector bundle

Hermitian Yang–Mills equation (HYM)

for a Hermitian metric H on E:

$$i\Lambda F_H = \mu(E)\,\mathrm{Id}_E$$

- F_H = curvature 2-form of Chern connection of H on E
- $\Lambda F_H = g^{j\bar{k}} F_{j\bar{k}} \colon E \longrightarrow E$ = contraction of F_H with ω_M

Taking traces in the equation and $\int_M (-)\,\mathrm{dvol}_M$:

$$\mu(E) = \text{slope of } E := \frac{\deg E}{\operatorname{rank} E}$$

where $\deg E = \frac{2\pi}{\mathrm{vol}(X)} \int_M c_1(E) \wedge \omega_M^{n-1}$.

Recall the **Donaldson–Uhlenbeck–Yau Theorem**:

It is a correspondence between:

- gauge theory: Hermitian Yang–Mills equation on E
- algebraic geometry: polystability of E

Definition (Mumford–Takemoto)

E is **stable** if $\mu(E') < \mu(E)$ for all coherent subsheaves $E' \subsetneq E$.

E is **polystable** if $E \cong \oplus E_i$ with E_i stable of the same slope.

Theorem (Donaldson, Uhlenbeck–Yau, 1986–87)

$$\exists \text{ Hermitian Yang–Mills metric on } E \iff E \text{ is polystable.}$$

- For $n = 1$: Narasimhan–Seshadri (1965), Donaldson (1983)
- The HYM equation and the proof have symplectic meaning.

3. Dimensional reduction of HYM to vortices

Work by García-Prada 1991 (previous work by Witten 1977; Taubes 1980)

Come back to pair (L, ϕ) over compact Riemann surface X:

- Associate a rank 2 holomorphic vector bundle E over $X \times \mathbb{P}^1$:

$$0 \longrightarrow p^*L \longrightarrow E \longrightarrow q^*\mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0$$

$\mathbb{P}^1 := \mathbb{CP}^1$, $p\colon X \times \mathbb{P}^1 \to X$ and $q\colon X \times \mathbb{P}^1 \to \mathbb{P}^1$ projections

- By Künneth formula, these extensions are parametrized by ϕ:

$$\begin{aligned} \operatorname{Ext}^1(q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*L) &\cong H^1(X \times \mathbb{P}^1, p^*L \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \\ &\cong H^0(X, L) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(X, L) \ni \phi, \end{aligned}$$

using Serre duality $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \cong \mathbb{C}$.

SU(2)-**action**:

- SU(2) acts on $X \times \mathbb{P}^1$:
  - — on X: trivially
  - — on $\mathbb{P}^1$: via $\mathbb{P}^1 \cong \mathrm{SU}(2)/\mathrm{U}(1)$
- SU(2) acts trivally on $H^0(X, L) \cong \mathrm{Ext}^1(q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*L) \Longrightarrow$ holomorphic extension E is SU(2)-invariant.
- SU(2)-invariant Kähler metric on $X \times \mathbb{P}^1$:

$$\omega_\tau = p^*\omega_X \oplus \frac{4}{\tau} q^*\omega_{\mathbb{P}^1}$$

where $\tau > 0$ and $\omega_{\mathbb{P}^1}$ = Fubini–Study metric.

Theorem (existence of vortices on $L \longrightarrow X$)

$$\begin{tikzcd}
\exists\ \tau\text{-vortex on } X \arrow[r, leftrightarrow, dotted] \arrow[d, Leftrightarrow, "\text{García-Prada}"'] & \tau > \deg L \arrow[d, Leftrightarrow, "\text{García-Prada}"] \\
\exists\,\mathrm{SU}(2)\text{-inv. } \omega_\tau\text{-HYM metric} \arrow[r, Leftrightarrow, "\text{Donaldson–Uhlenbeck–Yau}"] & E\ \omega_\tau\text{-polystable}
\end{tikzcd}$$

Some generalizations: non-abelian vortices

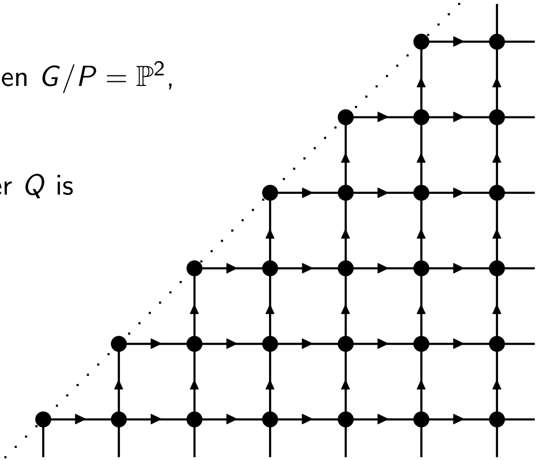
- **Higher rank** [Bradlow 1991]:
  Replace $L \to X$ by higher-rank holomorphic vector bundle $E \to X$, and X by a compact Kähler manifold $\Longrightarrow$ study gauge equations for pairs (E, ϕ) with $\phi \in H^0(X, E)$.
  Define τ-stability for (V, ϕ) and show equivalence with existence of solutions of a 'non-abelian τ-vortex equation'.
- **Holomorphic chains** [___ & O. a García-Prada, 2001]:
  SU(2)-equivariant holomorphic vector bundles on $X \times \mathbb{P}^1$ are equivalent to 'holomorphic chains'

  $$E_m \xrightarrow{\phi_m} E_{m-1} \xrightarrow{\phi_{m-1}} \cdots \xrightarrow{\phi_1} E_0$$

  $\Longrightarrow$ useful to understand topology of moduli of Higgs bundles.
- **Holomorphic quiver bundles** [___ & O. García-Prada, 2003]:
  G-equivariant holomorphic vector bundles on $X \times G/P$, for a flag manifold G/P, are equivalent to holomorphic $(Q, \mathcal{R})$-bundles, for a quiver with relations $(Q, \mathcal{R})$ depending on $P \subset G$.
  $\Longrightarrow$ correspondence between stability and quiver vortex equations.

Example: when $G/P = \mathbb{P}^2$,

- the quiver Q is



- the relations $\mathcal{R}$ are 'commutative diagrams'.

4. The Kähler–Yang–Mills equations

Goal: apply dimensional reduction to HYM coupled to gravity

Data:

M	compact (Kählerian) complex manifold with $\dim_{\mathbb{C}} M = n$
$E \longrightarrow M$	holomorphic vector bundle over M

The Kähler–Yang–Mills equations (KYM)

for a Kähler metric g on M and a Hermitian metric H on E:

$$i\Lambda_g F_H = \mu(E)\,\mathrm{Id}_E$$
$$S_g - \alpha \Lambda_g^2 \operatorname{Tr} F_H^2 = C$$

- S_g scalar curvature of g
- $\operatorname{Tr} F_H^2 \in \Omega^4(M)$, so contraction $\Lambda_g^2 \operatorname{Tr} F_H^2 \in C^\infty(M)$
- $\alpha > 0$ coupling constant
- $C \in \mathbb{R}$ determined by the topology

The Kähler–Yang–Mills equations were introduced in:

- M. García-Fernández, *Coupled equations for Kähler metrics and Yang–Mills connections*. PhD Thesis. ICMAT, Madrid, 2009, `arXiv:1102.0985 [math.DG]`
- ___, M. García-Fernández and O. García-Prada, *Coupled equations for Kähler metrics and Yang–Mills connections*, Geometry and Topology **17** (2013) 2731–2812

As far as we know, the equations have no the physical as a coupling of Yang–Mills fields to gravity, but have a symplectic origin.

The symplectic origin of the KYM equations

Let E be C^∞ complex vector bundle over M and fix:

H Hermitian metric on E

ω symplectic form on M

Define two ∞-dimensional manifolds:

$\mathcal{J} := \{$complex structures $J : TM \to TM$ on $M\}$

$\mathcal{A} := \{$unitary connections A on $(E, H)\}$

Define $\mathcal{P} :=$ set of pairs $(J, A) \in \mathcal{J} \times \mathcal{A}$ such that:

- (M, J, ω) is Kähler
- A induces a holomorphic structure $\bar{\partial}_A$ on E over (M, J)

$\mathcal{J}$ and $\mathcal{A}$ have canonical symplectic structures $\omega_{\mathcal{J}}$ and $\omega_{\mathcal{A}}$.

Symplectic form on $\mathcal{P}$: $\omega_\alpha := (\omega_{\mathcal{J}} + \alpha\omega_{\mathcal{A}})|_{\mathcal{P}}$ for fixed $\alpha \neq 0$

Group action:

Fujiki–Donaldson:

- $\mathcal{H} := \{\text{Hamiltonian symplectomorphisms } (M,\omega) \to (M,\omega)\}$
- Symplectic action of group $\mathcal{H}$ on $(\mathcal{J}, \omega_{\mathcal{J}})$ has moment map $\mu_{\mathcal{J}} : \mathcal{J} \to (\mathrm{Lie}\,\mathcal{H})^*$ such that

$$\mu_{\mathcal{J}}(J) = 0 \Longleftrightarrow S_{J,\omega} = \text{constant}$$

Hamiltonian extended gauge group $\widetilde{\mathcal{G}}$:
$\widetilde{\mathcal{G}} := \{\text{automorphisms } g \text{ of } (E,H) \text{ covering elements } \check{g} \text{ of } \mathcal{H}\}$.

$$\begin{array}{ccc} (E,H) & \xrightarrow{g} & (E,H) \\ \downarrow & & \downarrow \\ (M,\omega) & \xrightarrow{\check{g}} & (M,\omega) \end{array}$$

Action of group $\widetilde{\mathcal{G}}$ on $\mathcal{P} \subset \mathcal{J} \times \mathcal{A}$

Proposition

- $\widetilde{\mathcal{G}}$-action on $(\mathcal{P}, \omega_\alpha)$ has moment map $\mu_\alpha : \mathcal{P} \to (\mathrm{Lie}\, \widetilde{\mathcal{G}})^*$ s.t.

$$\mu_\alpha^{-1}(0) = \{\text{solutions of the KYM equations}\}.$$

- For $\alpha > 0$, $(\mathcal{P}, \omega_\alpha)$ has a $\widetilde{\mathcal{G}}$-invariant Kähler structure.
- Moduli space $\mathcal{M}_\alpha := \{\text{solutions of KYM equations}\}/\widetilde{\mathcal{G}}$ is Kähler (away from singularities) for $\alpha > 0$.

Remarks:

- We recover the HYM equation, while the equation $S_g =$ constant (Donaldson–Tian–Yau theory) is deformed.
- Equations 'decouple' for $\dim_{\mathbb{C}} M = 1$ (as $F_H^2 = 0$ in this case).

Programme: Study existence of solutions of KYM equations

- Very difficult problem!
- In the paper we give a conjecture involving geodesic stability.

5. Gravitating vortex equations

Data:

X	compact Riemann surface
$L \longrightarrow X$	holomorphic line bundle
$\phi \in H^0(X, L)$	holomorphic section

Let E be the SU(2)-equivariant rank 2 holomorphic vector bundle over $X \times \mathbb{P}^1$ determined by (L, ϕ):

$$0 \longrightarrow p^* L \longrightarrow E \longrightarrow q^* \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0$$

Proposition. SU(2)-invariant solutions of the KYM equations on $E \longrightarrow X \times \mathbb{P}^1$ are equivalent to solutions of:

Gravitating vortex equations

for a Kähler metric g on X and a Hermitian metric h on L:

$$i\Lambda_g F_h + |\phi|_h^2 - \tau = 0$$
$$S_g + \alpha(\Delta_g + \tau)(|\phi|_h^2 - \tau) = c$$

Gravitating vortex = solution of the gravitating vortex equations

Einstein–Bogomol'nyi equations & cosmic strings

- **Einstein–Bogomol'nyi equations** $\overset{\text{def}}{\Longleftrightarrow}$ gravitating vortex equations **with $c = 0$**

- Solutions of the Einstein–Bogomol'nyi equations $\Longleftrightarrow$ Nielsen–Olesen cosmic strings (1973) in the Bogomol'nyi phase i.e. solutions of coupled Abelian Einstein–Higgs equations, in the Bogomol'nyi phase, on $\mathbb{R}^{1,1} \times X$ independent of variables in $\mathbb{R}^{1,1}$.

- Cosmic strings are a model (by spontaneous symmetry breaking) for topological defects in the early universe.

- $\alpha = 2\pi G$, $G > 0$ is universal gravitation constant

Physics literature: Linet (1988), Comtet–Gibbons (1988), Spruck–Yisong Yang (1995), Yisong Yang (1995)...

Gravitating vortex equations:

$$i\Lambda_g F_h + |\phi|_h^2 - \tau = 0$$
$$S_g + \alpha(\Delta_g + \tau)(|\phi|_h^2 - \tau) = c$$

- $\tau > 0$, $\alpha > 0$ real parameters
- c is determined by the topology

Combining integration of the two gravitating vortex equations:

$$c = \frac{2\pi}{\mathrm{vol}_g(X)}\chi(X) - \alpha\tau \deg L$$

Therefore the Einstein–Bogomol'nyi equations (i.e. $c = 0$) can only have solutions on the Riemann sphere (as $\alpha, \tau, \deg L \geq 0$):

$$c = 0 \Longrightarrow \chi(X) > 0 \Longrightarrow X = \mathbb{P}^1$$

6. Existence of solutions

Theorem (Yisong Yang, 1995, 1997)

Let $D = \sum n_i p_i$ be an effective divisor on $\mathbb{P}^1$ corresponding to a pair (L, ϕ) s.t. $c = 0$ and $N := \sum n_i < \tau$.
Then the Einstein–Bogomol'nyi equations on $(\mathbb{P}^1, L, \phi)$ have solutions if

$$n_i < \frac{N}{2} \text{ for all } i. \qquad (*)$$

A solution also exists if $D = \frac{N}{2} p_1 + \frac{N}{2} p_2$, with $p_1 \neq p_2$ and N even.

Yang (1995) mentions $(*)$ *"is a technical restriction on the local string number. It is not clear at this moment whether it may be dropped"*, but we will show $(*)$ comes from **geometry**.

Yang's proof: apply conformal transformations

Fix metrics g_0 on X and h_0 on L and solve for $g = e^{2u} g_0$ and $h = e^{2f} h_0 \Longrightarrow$ the gravitating vortex equations are equivalent to equations for $f, u \in C^\infty(X)$:

$$\Delta_{g_0} f + e^{2u}(e^{2f}|\phi|^2_{h_0} - \tau) = -\frac{2\pi \deg L}{\mathrm{vol}_{g_0}(X)}$$

$$\Delta_{g_0}(u + \alpha e^{2f} - 2\alpha\tau f) + c(1 - e^{2u}) = 0$$

$c = 0 \Longrightarrow u = \text{const.} - \alpha e^{2f} + 2\alpha\tau f \Longrightarrow$ plug u in the first equation. Yang applies the continuity method to solve the resulting Kazhdan–Warner type equation, finding it suffices to assume

$$n_i < \frac{N}{2} \text{ for all } i, \qquad (*)$$

or $D = \frac{N}{2} p_1 + \frac{N}{2} p_2$, with $p_1 \neq p_2$ and N even.

7. Obstruction to the existence of solutions and Algebraic Geometry (GIT)

GIT=Geometric Invariant Theory (Mumford, ICM 1962)

Striking fact: Yang's "technical restriction" has an **algebro-geometric meaning**, for the natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{Sym}^N \mathbb{P}^1 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(N))$ (*binary quantics* [Sylvester 1882]):

$$n_i < \frac{N}{2} \text{ for all } i \iff D \in \mathrm{Sym}^N \mathbb{P}^1 \text{ is GIT stable}$$

$$D = \frac{N}{2} p_1 + \frac{N}{2} p_2 \iff D \in \mathrm{Sym}^N \mathbb{P}^1 \text{ is strictly GIT polystable}$$

Theorem (___, M. García-Fernández, O. García-Prada, 2015)

The converse of Yang's theorem also holds:

existence of cosmic strings $\iff$ GIT-polystability.

In fact, the converse ($\Longrightarrow$) holds more generally for gravitating vortices on $X = \mathbb{P}^1$ (i.e. c may be non-zero).
The proof relies on the following symplectic and algebro-geometric constructions.

The symplectic origin of the gravitating vortex equations

Fix: C^∞ compact surface X and C^∞ line bundle L over X

h Hermitian metric on L

ω symplectic form on X

Define ∞-dimensional manifolds:

$\mathcal{J} := \{$Kähler complex structures $J\colon TX \to TX$ on $(X, \omega)\}$

$\mathcal{A} := \{$unitary connections A on $(L, h)\}$

$\Gamma := \Gamma(L) = \{C^\infty$ global sections ϕ of $L \to X\}$

$\dim_{\mathbb{R}} X = 2 \implies$ $A \in \mathcal{A}$ are in bijection with the holomorphic structures $\bar{\partial}_A$ on L over (X, J)

$$\mathcal{T} := \left\{ \begin{array}{c} \text{triples } T = (J, A, \phi) \in \mathcal{J} \times \mathcal{A} \times \Gamma \\ \text{s.t. } \phi \text{ is holomorphic w.r.t. } J \text{ and } \bar{\partial}_A \end{array} \right\}$$

$\mathcal{J}, \mathcal{A}$ and Γ have canonical symplectic structures $\omega_{\mathcal{J}}, \omega_{\mathcal{A}}$ and ω_Γ.

Symplectic form on $\mathcal{T}$: $\omega_\alpha := (\omega_{\mathcal{J}} + \alpha\omega_{\mathcal{A}} + \alpha\omega_\Gamma)|_{\mathcal{T}}$
(for fixed $\alpha \neq 0$)

The symplectic origin of the gravitating vortex equations

The **Hamiltonian extended gauge group** is
$\widetilde{\mathcal{G}} := \{$automorphisms g of (L, h) covering elements $\check{g}$ of $\mathcal{H}\}$

$$\begin{array}{ccc} (L,h) & \xrightarrow{g} & (L,h) \\ \downarrow & & \downarrow \\ (X,\omega) & \xrightarrow{\check{g}} & (X,\omega) \end{array}$$

where $\mathcal{H} := \{$Hamiltonian symplectomorphisms $(X, \omega) \to (X, \omega)\}$.
The group $\widetilde{\mathcal{G}}$ acts on $\mathcal{T} \subset \mathcal{J} \times \mathcal{A} \times \Gamma$.

Proposition

- $\widetilde{\mathcal{G}}$-action on $(\mathcal{T}, \omega_\alpha)$ has moment map $\mu_\alpha \colon \mathcal{T} \to (\mathrm{Lie}\,\widetilde{\mathcal{G}})^*$ s.t.
$$\mu_\alpha^{-1}(0) = \{\text{gravitating vortices}\}.$$
- For $\alpha > 0$, $(\mathcal{T}, \omega_\alpha)$ has a $\widetilde{\mathcal{G}}$-invariant Kähler structure.
- Moduli space $\mathcal{M}_{\alpha,\tau} := \{\text{gravitating vortices}\}/\widetilde{\mathcal{G}}$ is Kähler for $\alpha > 0$.

Geodesics on space of metrics

Fix: volume $0 < \mathrm{vol}_X \in \mathbb{R}$ of oriented C^∞ surface X

$I := (J, \bar{\partial}_A)$=holomorphic structures on X and L

Vary $b = (\omega, h)$ in space

$$B_I := \left\{ \begin{array}{c} \text{pairs } (\omega, h) \text{ with } h\text{=Hermitian metric on } E, \\ \omega\text{=volume form, with total volume } \mathrm{vol}_X, \\ \text{s.t. } (X, J, \omega) \text{ is Kähler} \end{array} \right\}$$

Theorem (___, M. García-Fernández, O. García-Prada, *G&T*, 2013)

B_I is a symmetric space, i.e. it has an affine connection ∇ s.t.

- torsion $T_\nabla = 0$
- $\nabla R_\nabla = 0$, where R_∇ is the curvature

Geodesic equations for a curve $b_t = (\omega_t, h_t)$ on (B_I, ∇), with $\omega_t = \omega_0 + dd^c\varphi_t$, $d\dot{\varphi}_t = \eta_{\dot{\varphi}_t} \lrcorner \omega_t$ (i.e. $\eta_{\dot{\varphi}_t}$:=Hamiltonian vector field of $\dot{\varphi}_t$):

$$dd^c(\ddot{\varphi}_t - (d\dot{\varphi}_t, d\dot{\varphi}_t)_{\omega_t}) = 0,$$

$$\ddot{h}_t - 2J\eta_{\dot{\varphi}_t} \lrcorner d_{h_t}\dot{h}_t + iF_{h_t}(\eta_{\dot{\varphi}_t}, J\eta_{\dot{\varphi}_t}) = 0.$$

Geodesic stability

- For each $b = (\omega, h)$, we have a group $\widetilde{\mathcal{G}}_b$ and a Kähler $\widetilde{\mathcal{G}}_b$-manifold
$$\mathcal{T}_b = \{\text{triples } T = (J, \bar{\partial}_A, \phi) \text{ compatible with } b = (\omega, h)\},$$
with moment map $\mu_b : \mathcal{T}_b \to (\mathrm{Lie}\, \widetilde{\mathcal{G}}_b)^*$.
- Define 1-**form** σ_T on B_I, for $I = (J, \bar{\partial}_A)$ and $T = (J, \bar{\partial}_A, \phi)$, by $\sigma_T(v) := \langle \mu_b(T), v \rangle$ for $v \in T_b B_I \cong \mathrm{Lie}\, \widetilde{\mathcal{G}}_b$.
- Along a geodesic ray b_t on B_I, $\dfrac{d}{dt}\sigma_T(\dot{b}_t) \geq 0$.

Obstruction: *if $\exists$ smooth geodesic ray b_t on (B_I, ∇) on B_I s.t.*
$$\lim_{t\to\infty} \sigma_T(\dot{b}_t) < 0,$$
then $\mu_b^{-1}(0)$ is empty, i.e. $\nexists$ gravitating vortices $T = (J, \bar{\partial}_A, \phi)$ on $b = (\omega, h)$.

Definition

A triple $T = (J, \bar{\partial}_A, \phi)$ is *geodesically (semi)stable* if
$$\lim_{t\to\infty} \sigma_T(\dot{b}_t) > 0 \, (\geq 0)$$
for every non-constant geodesic ray b_t $(0 \leq t < \infty)$ on (B_I, ∇).

Converse of Yang's theorem. *$\exists$ gravitating vortex on (L, ϕ) over $\mathbb{P}^1$ corresponding to effective divisor $D = \sum n_i p_i \implies D \in \mathrm{Sym}^N \mathbb{P}^1$ GIT polystable for $\mathrm{SL}(2, \mathbb{C})$-action (with $N = \sum n_i$).*

Proof. Fix triple $T = (J, \bar{\partial}_A, \phi)$ and pair of metrics $b_0 = (\omega_0, h_0) \in B_I$.

- Line bundle $L = \mathcal{O}_{\mathbb{P}^1}(N)$ is $\mathrm{SL}(2, \mathbb{C})$-linearized $\implies$ each $\zeta \in \mathfrak{sl}(2, \mathbb{C})$ determines a geodesic ray b_t on B_I, given by pull-back along 1-PS $g_t = \exp(t\zeta) \in \mathrm{SL}(2, \mathbb{C})$:
$$b_t = (\omega_t, h_t) := (g_t^* \omega_0, g_t^* h_0).$$
- Since g_t fixes $I := (J, \bar{\partial}_A)$, i.e. $g_t \in \mathrm{Aut}(X_J, L_{\bar{\partial}_A})$,
$$\begin{aligned}\sigma_T(\dot{b}_t) &= \langle \mu_{b_t}(J, A, \phi), \dot{b}_t \rangle = \langle \mu_{b_0}(g_t \cdot (J, A, \phi)), \dot{b}_0 \rangle \\ &= \langle \mu_{b_0}(J, A, g_t \cdot \phi), \zeta_2 \rangle,\end{aligned}$$
where $\zeta = \zeta_1 + i\zeta_2$, with $\zeta_1, \zeta_2 \in \mathfrak{su}(2)$.
- The theorem follows now essentially because by a (long) direct calculation, if $\exists \lim_{t \to \infty} g_t \cdot \phi$, then
$$\begin{aligned}\lim_{t \to \infty} \sigma_T(\dot{b}_t) &= \lim_{t \to \infty} \langle \mu_{b_0}(J, A, g_t \cdot \phi), \zeta_2 \rangle \\ &\sim \alpha(N - \tau) \text{ Hilbert–Mumford weight in GIT.}\end{aligned}$$

8. Some open problems

— **Geometry of moduli spaces:**

Conjecture (___, M. García-Fernández, O. García-Prada, 2015)

$$\begin{array}{c}\text{Moduli of gravitating}\\ \text{vortices on } \mathbb{P}^1\end{array} = \begin{array}{c}\mathrm{Sym}^N \mathbb{P}^1 /\!\!/ \mathrm{SL}(2,\mathbb{C})\\ \text{(GIT quotient)}\end{array}$$

— **Supersymmetric extensions of Einstein–Bogomol'nyi equations:**

- Consider supersymmetric nonlinear σ-model with target manifold $\mathcal{M}$.
- If $\mathcal{M}$ is Kähler, σ-model admits coupling to $\mathcal{N} = 1$ SUGRA.
- **Geometrically**, gauged non-linear σ-model coupled to gravity is obtained by replacing $L \longrightarrow X$ by fibre bundle $\mathcal{M} \longrightarrow X$ with typical fibre given by a Hamiltonian Kähler G-manifold $\Longrightarrow$

PDEs couple moment map for space of holomorphic sections $\phi\colon X \longrightarrow \mathcal{M}$ to G-connection and Kähler metric on X.