Exercise 1: Covariant derivative

Prove that the term $\overline{\Psi}\not{D}\Psi$ where the covariant derivative is given by:

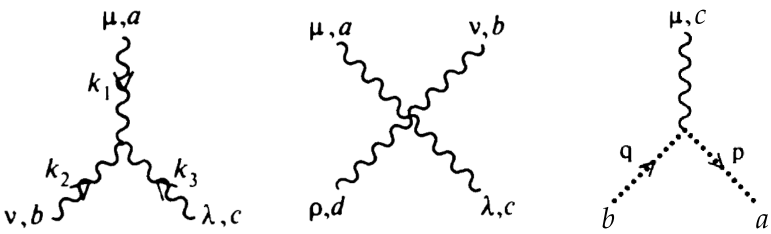
$$D_\mu = \partial_\mu - \mathrm{i}g\widetilde{W}_\mu \ , \quad \widetilde{W}_\mu = T_a W_\mu^a$$

is invariant under gauge transformations:

$$\Psi \mapsto \Psi' = U\Psi \ , \quad U = \exp\{-\mathrm{i}T_a\theta^a(x)\}$$
$$\widetilde{W}_\mu \mapsto \widetilde{W}'_\mu = U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger$$

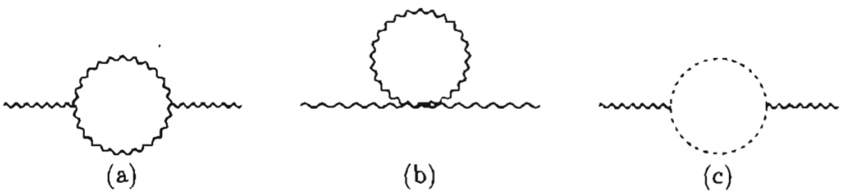
Exercise 2: Feynman rules of general non-Abelian gauge theories

Obtain the Feynman rules for cubic and quartic self-interactions among gauge fields in a general non-Abelian gauge theory, as well as those for the interactions of Faddeev-Popov ghosts with gauge fields:



Exercise 3: Faddeev-Popov ghosts and gauge invariance

Consider the 1-loop self-energy diagrams for non-Abelian gauge theories in the figure. Calculate the diagrams in the 't Hooft-Feynman gauge and show that the sum does not have the tensor structure $g_{\mu\nu}k^2 - k_\mu k_\nu$ required by the gauge invariance of the theory unless diagram (c) involving ghost fields is included.



Hint: Take Feynman rules from previous excercise and use dimensional regularization. It is convenient to use the Passarino-Veltman tensor decomposition of loop integrals:

$$\frac{\mathrm{i}}{16\pi^2}\{B_0,\, B_\mu,\, B_{\mu\nu}\} = \mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{\{1,\, q_\mu,\, q_\mu q_\nu\}}{q^2(q+k)^2}$$

where $\quad B_0 = \Delta_\epsilon + \text{finite}$

$$B_\mu = k_\mu B_1 \ , \quad B_1 = -\frac{\Delta_\epsilon}{2} + \text{finite}$$

$$B_{\mu\nu} = g_{\mu\nu} B_{00} + k_\mu k_\nu B_{11}\ , \quad B_{00} = -\frac{k^2}{12}\Delta_\epsilon + \text{finite}\ , \quad B_{11} = \frac{\Delta_\epsilon}{3} + \text{finite}$$

with $\Delta_\epsilon = 2/\epsilon - \gamma + \ln 4\pi$ and $D = 4 - \epsilon$. You may check that the ultraviolet divergent part has the expected structure or find the final result in terms of scalar integrals, that for this configuration of masses and momenta read:

$$B_1 = -\frac{1}{2} B_0\ , \quad B_{00} = -\frac{k^2}{4(D-1)} B_0\ , \quad B_{11} = \frac{D}{4(D-1)} B_0\ .$$

Do not forget a symmetry factor (1/2) in front of (a) and (b), and a factor (-1) in (c).

Exercise 4: Propagator of a massive vector boson field

Consider the Proca Lagrangian of a massive vector boson field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu\ , \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ .$$

Show that the propagator of A_μ is

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right]$$

Exercise 5: Propagator of a massive gauge field

Consider the U(1) gauge invariant Lagrangian $\mathcal{L}$ with gauge fixing $\mathcal{L}_{\mathrm{GF}}$:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \\
\mathcal{L}_{\mathrm{GF}} &= -\frac{1}{2\xi} (\partial_\mu A^\mu - \xi M_A \chi)^2\ , \quad \text{with} \quad D_\mu = \partial_\mu + \mathrm{i} e A_\mu\ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned}$$

where $M_A = ev$ after spontanteous symmetry breaking ($\mu^2 < 0$, $\lambda > 0$) when the complex scalar field ϕ acquires a VEV and is parameterized by

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \varphi(x) + \mathrm{i}\chi(x)]\ , \quad \mu^2 = -\lambda v^2.$$

Show that the propagators of φ, χ and the gauge field A_μ are respectively

$$\begin{aligned}
\widetilde{D}^{\varphi}(k) &= \frac{\mathrm{i}}{k^2 - M_\varphi^2 + \mathrm{i}\epsilon} \quad \text{with } M_\varphi^2 = -2\mu^2 = 2\lambda v^2 \\
\widetilde{D}^{\chi}(k) &= \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\epsilon}\ , \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]
\end{aligned}$$

Exercise 6: The conjugate Higgs doublet

Show that $\Phi^c \equiv i\sigma_2 \Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$ transforms under SU(2) like $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, with $\phi^- = (\phi^+)^*$. What are the weak isospins, hypercharges and electric charges of ϕ^0, ϕ^{0*}, ϕ^+, ϕ^-? *Hint*: Use the property of Pauli matrices: $\sigma_i^* = -\sigma_2 \sigma_i \sigma_2$.

Exercise 7: Lagrangian and Feynman rules of the Standard Model

Try to reproduce the Lagrangian and the corresponding Feynman rules of as many Standard Model interactions as you can. Of particular interest/difficulty are [VVV] and [VVVV].

Exercise 8: Z pole observables at tree level

Show that

(a) $\Gamma(f\bar{f}) \equiv \Gamma(Z \to f\bar{f}) = N_c^f \frac{\alpha M_Z}{3}(v_f^2 + a_f^2)\ , \quad N_c^f = 1\ (3)$ for $f =$ lepton (quark)

(b) $\sigma_{\text{had}} = 12\pi \frac{\Gamma(\mathrm{e^+e^-})\Gamma(\text{had})}{M_Z^2 \Gamma_Z^2}$

(c) $A_{FB} = \frac{3}{4} A_f\ ,$ with $A_f = \frac{2 v_f a_f}{v_f^2 + a_f^2}$

Exercise 9: Higgs partial decay widths at tree level

Show that

(a) $\Gamma(H \to f\bar{f}) = N_c^f \frac{G_F M_H}{4\pi\sqrt{2}} m_f^2 \left(1 - \frac{4m_f^2}{M_H^2}\right)^{3/2}\ , \quad N_c^f = 1\ (3)$ for $f =$ lepton (quark)

(b) $\Gamma(H \to W^+W^-) = \frac{G_F M_H^3}{8\pi\sqrt{2}} \sqrt{1 - \frac{4M_W^2}{M_H^2}} \left(1 - \frac{4M_W^2}{M_H^2} + \frac{12 M_W^4}{M_H^4}\right)$

$$\Gamma(H \to ZZ) = \frac{G_F M_H^3}{16\pi\sqrt{2}} \sqrt{1 - \frac{4M_Z^2}{M_H^2}} \left(1 - \frac{4M_Z^2}{M_H^2} + \frac{12 M_Z^4}{M_H^4}\right)$$