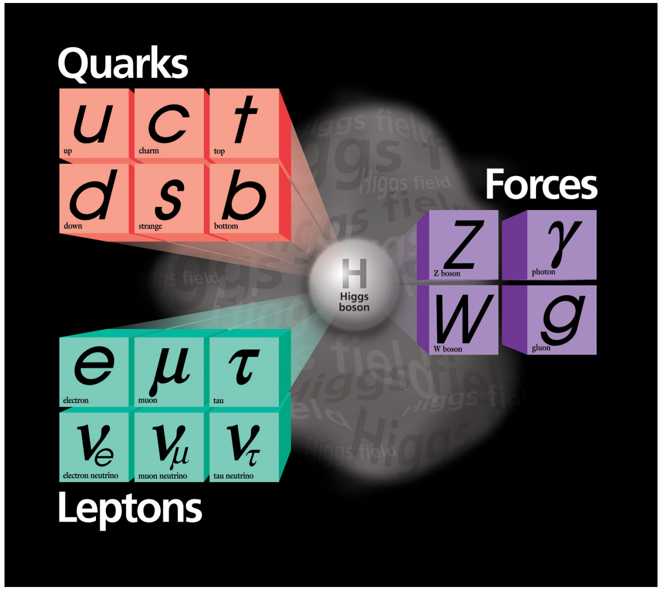
Quantum Field Theory and the Standard Model



José Ignacio Illana

Universidad de Granada

Outline

1. Quantum Field Theory: Gauge Theories
   - ▷ The symmetry principle
   - ▷ Quantization of gauge theories
   - ▷ Spontaneous Symmetry Breaking
2. The Standard Model
   - ▷ Gauge group and particle representations
   - ▷ The SM with one family: electroweak interactions
   - ▷ Electroweak SSB: Higgs sector, gauge boson and fermion masses
   - ▷ Additional generations: fermion mixings
   - ▷ Complete Lagrangian and Feynman rules
3. Phenomenology of the Electroweak Standard Model
   - ▷ Input parameters, experiments, observables, precise predictions
   - ▷ Global fits

1. Gauge Theories

The symmetry principle | free Lagrangian

- Lagrangian of a free fermion field $\psi(x)$:

$$\text{(Dirac)} \qquad \boxed{\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\not\partial - m)\psi} \quad \not\partial \equiv \gamma^\mu \partial_\mu \,, \quad \overline{\psi} = \psi^\dagger \gamma^0$$

$\Rightarrow$ Invariant under global U(1) phase transformations:

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}q\theta}\psi(x) \,, \quad q,\ \theta \text{ (constants)} \in \mathbb{R}$$

$\Rightarrow$ By Noether's theorem there is a conserved current:

$$j^\mu = q\, \overline{\psi}\gamma^\mu\psi \,, \quad \partial_\mu j^\mu = 0$$

and a Noether charge:

$$Q = \int \mathrm{d}^3x\ j^0, \quad \partial_t Q = 0$$

The symmetry principle | free Lagrangian

- A quantized free fermion field:

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1,2} \left(a_{\boldsymbol{p},s} u^{(s)}(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px} + b^{\dagger}_{\boldsymbol{p},s} v^{(s)}(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px} \right)$$

– is a solution of the Dirac equation (Euler-Lagrange):

$$(\mathrm{i}\partial\!\!\!/ - m)\psi(x) = 0 \,, \quad (p\!\!\!/ - m)u(\boldsymbol{p}) = 0 \,, \quad (p\!\!\!/ + m)v(\boldsymbol{p}) = 0 \,,$$

– is an operator from the canonical quantization rules (anticommutation):

$$\{a_{\boldsymbol{p},r}, a^{\dagger}_{\boldsymbol{k},s}\} = \{b_{\boldsymbol{p},r}, b^{\dagger}_{\boldsymbol{k},s}\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{k}) \delta_{rs} \,, \quad \{a_{\boldsymbol{p},r}, a_{\boldsymbol{k},s}\} = \cdots = 0 \,,$$

that annihilates/creates particles/antiparticles on the Fock space of fermions

The symmetry principle | free Lagrangian

- For a quantized free fermion field:

  $\Rightarrow$ Normal ordering for fermionic operators (H spectrum bounded from below):

$$: a_{\boldsymbol{p},r} a^\dagger_{\boldsymbol{q},s} : \equiv -a^\dagger_{\boldsymbol{q},s} a_{\boldsymbol{p},r} \ , \quad : b_{\boldsymbol{p},r} b^\dagger_{\boldsymbol{q},s} : \equiv -b^\dagger_{\boldsymbol{q},s} b_{\boldsymbol{p},r}$$

  $\Rightarrow$ The Noether charge is an operator:

$$: Q : = q \int \mathrm{d}^3x \ : \overline{\psi} \gamma^0 \psi : = q \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^\dagger_{\boldsymbol{p},s} a_{\boldsymbol{p},s} - b^\dagger_{\boldsymbol{p},s} b_{\boldsymbol{p},s} \right)$$

$$Q \ a^\dagger_{\boldsymbol{k},s} \left|0\right\rangle = +q \ a^\dagger_{\boldsymbol{k},s} \left|0\right\rangle \ \ (\text{particle}) \ , \quad Q \ b^\dagger_{\boldsymbol{k},s} \left|0\right\rangle = -q \ b^\dagger_{\boldsymbol{k},s} \left|0\right\rangle \ \ (\text{antiparticle})$$

The symmetry principle | gauge symmetry dictates interactions

- To make $\mathcal{L}_0$ invariant under local $\equiv$ gauge transformations of U(1):

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}q\theta(x)}\psi(x) \,, \quad \theta = \theta(x) \in \mathbb{R}$$

perform the minimal substitution:

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}eqA_\mu \quad \text{(covariant derivative)}$$

where a gauge field $A_\mu(x)$ is introduced transforming as:

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x) \quad \Leftarrow \quad \boxed{D_\mu\psi \mapsto \mathrm{e}^{-\mathrm{i}q\theta(x)}D_\mu\psi} \quad \overline{\psi}\not{D}\psi \text{ inv.}$$

$\Rightarrow$ The new Lagrangian contains interactions between ψ and A_μ:

$$\boxed{\mathcal{L}_{\mathrm{int}} = -eq\,\overline{\psi}\gamma^\mu\psi A_\mu} \quad \propto \begin{cases} \text{coupling} & e \\ \text{charge} & q \end{cases}$$

$$(= -e\,j^\mu A_\mu)$$

The symmetry principle | gauge invariance dictates interactions

- Dynamics for the gauge field ⇒ add gauge invariant kinetic term:

$$\text{(Maxwell)} \qquad \boxed{\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \qquad \Leftarrow \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \mapsto F_{\mu\nu}$$

- The full U(1) gauge invariant Lagrangian for a fermion field $\psi(x)$ reads:

$$\mathcal{L}_{\text{sym}} = \overline{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_1)$$

- The same applies to a complex scalar field $\phi(x)$:

$$\mathcal{L}_{\text{sym}} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The symmetry principle | non-Abelian gauge theories

- A general gauge symmetry group G is an *compact* N-dimensional Lie group

$$\mathrm{g} \in G\,, \quad \mathrm{g}(\boldsymbol{\theta}) = \mathrm{e}^{-\mathrm{i}T_a\theta^a}\,, \quad a = 1, \ldots, N$$

$$\theta^a = \theta^a(x) \in \mathbb{R}\,, \quad T_a = \text{Hermitian generators}\,, \quad [T_a, T_b] = \mathrm{i}f_{abc}T_c \quad \text{(Lie algebra)}$$

$\mathrm{Tr}\{T_a T_b\} \equiv \frac{1}{2}\delta_{ab}$ structure constants: $f_{abc} = 0$ Abelian

$f_{abc} \neq 0$ non-Abelian

⇒ *Unitary* finite-dimensional irreducible representations:

$g(\boldsymbol{\theta})$ represented by $U(\boldsymbol{\theta})$

$d \times d$ matrices : $U(\boldsymbol{\theta})$ [given by $\{T_a\}$ algebra representation]

$$d\text{-multiplet}: \quad \Psi(x) \mapsto \Psi'(x) = U(\boldsymbol{\theta})\Psi(x)\,, \quad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}$$

The symmetry principle | non-Abelian gauge theories

- Examples:

G	N	Abelian	
U(1)	1	Yes	
SU(n)	n^2-1	No	($n \times n$ matrices with det $= 1$)

  - U(1): 1 generator (q), one-dimensional irreps only
  - SU(2): 3 generators

    $f_{abc} = \epsilon_{abc}$ (Levi-Civita symbol)
    * Fundamental irrep ($d = 2$): $T_a = \frac{1}{2}\sigma_a$ (3 Pauli matrices)
    * Adjoint irrep ($d = N = 3$): $(T_a^{\text{adj}})_{bc} = -\mathrm{i}f_{abc}$
  - SU(3): 8 generators

    $f^{123} = 1$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$, $f^{147} = f^{156} = f^{246} = f^{247} = f^{345} = -f^{367} = \frac{1}{2}$
    * Fundamental irrep ($d = 3$): $T_a = \frac{1}{2}\lambda_a$ (8 Gell-Mann matrices)
    * Adjoint irrep ($d = N = 8$): $(T_a^{\text{adj}})_{bc} = -\mathrm{i}f_{abc}$

(for SU(n): f_{abc} totally antisymmetric)

The symmetry principle | non-Abelian gauge theories

- To make $\mathcal{L}_0$ invariant under local $\equiv$ gauge transformations of G:

$$\Psi(x) \mapsto \Psi'(x) = U(\boldsymbol{\theta})\Psi(x) \; , \quad \boldsymbol{\theta} = \boldsymbol{\theta}(x) \in \mathbb{R}$$

substitute the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}g\widetilde{W}_\mu \; , \quad \widetilde{W}_\mu \equiv T_a W_\mu^a$$

where a gauge field $A_\mu^a(x)$ per generator is introduced, transforming as:

$$\widetilde{W}_\mu(x) \mapsto \widetilde{W}'_\mu(x) = U\widetilde{W}_\mu(x)U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \quad \Leftarrow \quad \boxed{D_\mu\Psi \mapsto UD_\mu\Psi} \quad \overline{\Psi}\not{D}\Psi \text{ inv.}$$

$\Rightarrow$ The new Lagrangian contains **interactions** between Ψ and W_μ^a:

$$\boxed{\mathcal{L}_{\text{int}} = g\,\overline{\Psi}\gamma^\mu T_a \Psi W_\mu^a} \quad \propto \begin{cases} \text{coupling} & g \\ \text{charge} & T_a \end{cases}$$

$$(= g\, j_a^\mu W_\mu^a)$$

The symmetry principle | non-Abelian gauge theories

- Dynamics for the gauge fields ⇒ add gauge invariant kinetic terms:

(Yang-Mills) $$\mathcal{L}_{\rm YM} = -\frac{1}{2}{\rm Tr}\left\{\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right\} = -\frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu}$$

$$\widetilde{W}_{\mu\nu} \equiv D_\mu\widetilde{W}_\nu - D_\nu\widetilde{W}_\mu = \partial_\mu\widetilde{W}_\nu - \partial_\nu\widetilde{W}_\mu - {\rm i}g[\widetilde{W}_\mu,\widetilde{W}_\nu] \quad \Rightarrow \quad \widetilde{W}_{\mu\nu} \mapsto U\widetilde{W}_{\mu\nu}U^\dagger$$

$$\Rightarrow \quad W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu$$

⇒ $\mathcal{L}_{\rm YM}$ contains cubic and quartic self-interactions of the gauge fields W^a_μ:

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{a,\nu} - \partial^\nu W^{a,\mu})$$

$$\mathcal{L}_{\rm cubic} = -\frac{1}{2} g f_{abc}\,(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)W^{b,\mu}W^{c,\nu}$$

$$\mathcal{L}_{\rm quartic} = -\frac{1}{4} g^2 f_{abe} f_{cde}\, W^a_\mu W^b_\nu W^{c,\mu} W^{d,\nu}$$

Quantization of gauge theories — propagators

- The (Feynman) propagator of a scalar field:

$$D(x-y) = \langle 0|\, T\{\phi(x)\phi^\dagger(y)\}\,|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

  is a Green's function of the Klein-Gordon operator:

$$(\Box_x + m^2) D(x-y) = -\mathrm{i}\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{D}(p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

- The propagator of a fermion field:

$$S(x-y) = \langle 0|\, T\{\psi(x)\overline{\psi}(y)\}\,|0\rangle = (\mathrm{i}\partial\!\!\!/_x + m) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

  is a Green's function of the Dirac operator:

$$(\mathrm{i}\partial\!\!\!/_x - m) S(x-y) = \mathrm{i}\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{S}(p) = \frac{\mathrm{i}}{p\!\!\!/ - m + \mathrm{i}\epsilon}$$

- BUT the propagator of a gauge field cannot be defined unless $\mathcal{L}$ is modified:

(e.g. modified Maxwell) $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$$

Euler-Lagrange: $$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \Rightarrow \quad \left[g^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\mu = 0$$

– In momentum space the propagator is the inverse of:

$$-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \quad \Rightarrow \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$\Rightarrow$ Note that $(-k^2 g^{\mu\nu} + k^\mu k^\nu)$ is singular!

$\Rightarrow$ One may argue that $\mathcal{L}$ above will not lead to Maxwell equations …
unless we fix a (Lorenz) gauge where:

$$\partial^\mu A_\mu = 0 \quad \Leftarrow \quad A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \Lambda \;\; \text{with} \;\; \partial^\mu\partial_\mu\Lambda \equiv -\partial^\mu A_\mu$$

Quantization of gauge theories | gauge fixing (Abelian case)

- The extra term is called Gauge Fixing:

$$\boxed{\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2}$$

$\Rightarrow$ modified $\mathcal{L}$ equivalent to Maxwell Lagrangian just in the gauge $\partial^\mu A_\mu = 0$

$\Rightarrow$ the ξ-dependence always cancels out in physical amplitudes

- Several choices for the gauge fixing term (simplify calculations): R_ξ gauges

('t Hooft-Feynman gauge) $\quad \xi = 1: \quad \widetilde{D}_{\mu\nu}(k) = -\dfrac{\mathrm{i}g_{\mu\nu}}{k^2 + \mathrm{i}\epsilon}$

(Landau gauge) $\quad \xi = 0: \quad \widetilde{D}_{\mu\nu}(k) = \dfrac{\mathrm{i}}{k^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + \dfrac{k_\mu k_\nu}{k^2}\right]$

Quantization of gauge theories | gauge fixing (non-Abelian case)

- For a non-Abelian gauge theory, the gauge fixing terms:

$$\mathcal{L}_{\text{GF}} = -\sum_a \frac{1}{2\xi_a}(\partial^\mu W_\mu^a)^2$$

allow to define the propagators:

$$\widetilde{D}_{\mu\nu}^{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + (1 - \xi_a)\frac{k_\mu k_\nu}{k^2}\right]$$

BUT, unlike the Abelian case, this is not the end of the story ...

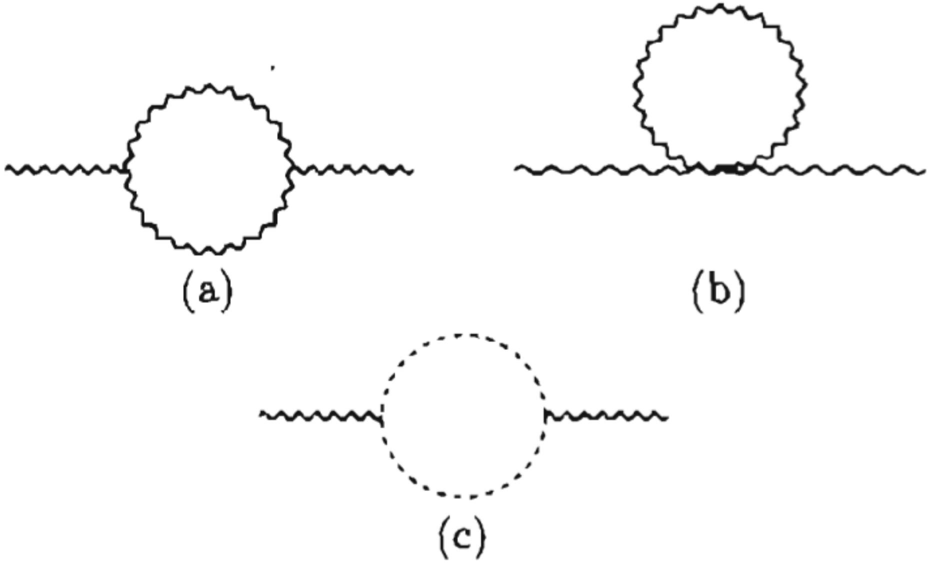
Quantization of gauge theories | Faddeev-Popov ghosts

- Add Faddeev-Popov ghost fields $c_a(x)$, $a = 1, \ldots, N$:

$$\boxed{\mathcal{L}_{\rm FP} = (\partial^\mu \bar{c}_a)(D_\mu^{\rm adj})_{ab} c_b = (\partial^\mu \bar{c}_a)(\partial_\mu c_a - g f_{abc} c_b W_\mu^c)} \quad \Leftarrow \quad D_\mu^{\rm adj} = \partial_\mu - {\rm i} g T_c^{\rm adj} W_\mu^c$$

Computational trick: anticommuting scalar fields, just in loops as virtual particles

$\Rightarrow$ Faddeev-Popov ghosts needed to preserve gauge symmetry:



$$= {\rm i}(g_{\mu\nu} k^2 - k_\mu k_\nu)\Pi(k^2)$$

with

$$\widetilde{D}_{ab}(k) = \frac{{\rm i}\delta_{ab}}{k^2 + {\rm i}\epsilon} \qquad [(-1) \text{ sign for closed loops! (like fermions)}]$$

Quantization of gauge theories | complete Lagrangian

- Then the complete quantum Lagrangian is

$$\boxed{\mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}}$$

⇒ Note that in the case of a massive vector field

$$\text{(Proca)} \qquad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu$$

it is not gauge invariant

– The propagator is:

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\epsilon} \left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2} \right)$$

Spontaneous Symmetry Breaking | discrete symmetry

- Consider a real scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 \quad \text{invariant under} \quad \phi \mapsto -\phi$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2) + V(\phi)$$

$$V = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

$\mu^2,\ \lambda \in \mathbb{R}$ (Real/Hermitian Hamiltonian) and $\lambda > 0$ (existence of a ground state)

(a) $\mu^2 > 0$: min of $V(\phi)$ at $\phi_{\text{cl}} = 0$

(b) $\mu^2 < 0$: min of $V(\phi)$ at $\phi_{\text{cl}} = v \equiv \pm\sqrt{\frac{-\mu^2}{\lambda}}$, in QFT $\langle 0|\,\phi\,|0\rangle = v \neq 0$ (VEV)

– A quantum field must have $v = 0$

$a\,|0\rangle = 0$

$$\Rightarrow \qquad \phi(x) \equiv v + \eta(x)\,, \quad \langle 0|\,\eta\,|0\rangle = 0$$

Spontaneous Symmetry Breaking | discrete symmetry

- At the quantum level, the same system is described by $\eta(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4 \quad \text{not invariant under} \quad \eta \mapsto -\eta$$

⇒ Lesson:

$\mathcal{L}(\phi)$ had the symmetry but the parameters can be such that the ground state of the Hamiltonian is not symmetric (Spontaneous Symmetry Breaking)

⇒ Note:

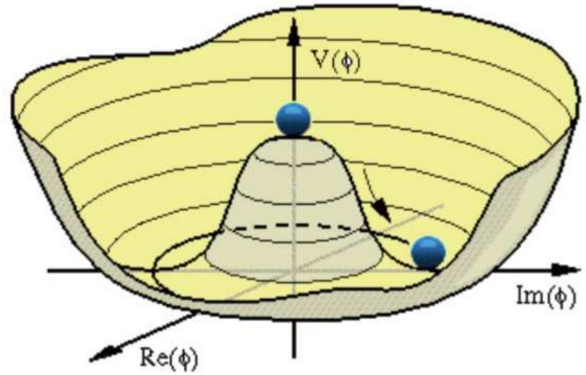
One may argue that $\mathcal{L}(\eta)$ exhibits an explicit breaking of the symmetry. However this is not the case since the coefficients of terms η^2, η^3 and η^4 are determined by just two parameters, λ and v (remnant of the original symmetry)

Spontaneous Symmetry Breaking | continuous symmetry

- Consider a complex scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \quad \text{invariant under U(1):} \quad \phi \mapsto e^{-iq\theta}\phi$$

$$\lambda > 0,\ \mu^2 < 0: \quad \langle 0|\,\phi\,|0\rangle \equiv \frac{v}{\sqrt{2}}, \quad |v| = \sqrt{\frac{-\mu^2}{\lambda}}$$



Take $v \in \mathbb{R}^+$. In terms of quantum fields:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + i\chi(x)], \quad \langle 0|\,\eta\,|0\rangle = \langle 0|\,\chi\,|0\rangle = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \lambda v^2\eta^2 - \lambda v\eta(\eta^2+\chi^2) - \frac{\lambda}{4}(\eta^2+\chi^2)^2 + \frac{1}{4}\lambda v^4$$

⇒ The actual quantum Lagrangian $\mathcal{L}(\eta,\chi)$ is not invariant under U(1)

U(1) broken ⇒ one scalar field remains massless: $m_\eta = \sqrt{2\lambda}\,v$, $m_\chi = 0$

Note: if $ve^{i\alpha}$ (complex) replace η by $(\eta\cos\alpha - \chi\sin\alpha)$ and χ by $(\eta\sin\alpha + \chi\cos\alpha)$

Spontaneous Symmetry Breaking | continuous symmetry

- Another example: consider a real scalar SU(2) triplet $\Phi(x)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi^\mathsf{T})(\partial^\mu \Phi) - \frac{1}{2}\mu^2 \Phi^\mathsf{T}\Phi - \frac{\lambda}{4}(\Phi^\mathsf{T}\Phi)^2 \quad \text{inv. under SU(2):} \quad \Phi \mapsto \mathrm{e}^{-\mathrm{i}T_a\theta^a}\Phi$$

that for $\lambda > 0,\ \mu^2 < 0$ acquires a VEV $\langle 0|\,\Phi^\mathsf{T}\Phi\,|0\rangle = v^2 \qquad (\mu^2 = -\lambda v^2)$

Assume $\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ v + \varphi_3(x) \end{pmatrix}$ and define $\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)$

$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) + \frac{1}{2}(\partial_\mu \varphi_3)(\partial^\mu \varphi_3) - \lambda v^2 \varphi_3^2 - \lambda v(2\varphi^\dagger\varphi + \varphi_3^2)\varphi_3 - \frac{\lambda}{4}(2\varphi^\dagger\varphi + \varphi_3^2)^2 + \frac{1}{4}\lambda v^4$$

$\Rightarrow$ Not symmetric under SU(2) but invariant under U(1):

$$\varphi \mapsto \mathrm{e}^{-\mathrm{i}q\theta}\varphi \quad (q = \text{arbitrary}) \qquad\qquad \varphi_3 \mapsto \varphi_3 \quad (q = 0)$$

SU(2) broken to U(1) $\Rightarrow 3 - 1 = 2$ broken generators

$\Rightarrow$ 2 (real) scalar fields (= 1 complex) remain massless: $m_\varphi = 0$, $m_{\varphi_3} = \sqrt{2\lambda}\, v$

Spontaneous Symmetry Breaking | continuous symmetry

⇒ Goldstone's theorem: [Nambu '60; Goldstone '61]

The number of massless particles (Nambu-Goldstone bosons) is equal to the number of spontaneously broken generators of the symmetry

$$\text{Hamiltonian symmetric under group } G \quad \Rightarrow \quad [T_a, H] = 0\,, \quad a = 1, \ldots, N$$

$$\text{By definition: } H\,|0\rangle = 0 \quad \Rightarrow \quad H(T_a\,|0\rangle) = T_a H\,|0\rangle = 0$$

– If $|0\rangle$ is such that $T_a\,|0\rangle = 0$ for all generators

⇒ non-degenerate minimum: *the* vacuum

– If $|0\rangle$ is such that $T_{a'}\,|0\rangle \neq 0$ for some (broken) generators a'

⇒ degenerate minimum: chose one (*true* vacuum) and $e^{-iT_{a'}\theta^{a'}}\,|0\rangle \neq |0\rangle$

⇒ excitations (particles) from $|0\rangle$ to $e^{-iT_{a'}\theta^{a'}}\,|0\rangle$ cost no energy: massless!