THE DIRAC EQUATION IN A NON-RIEMANNIAN BACKGROUND

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UV/IFIC/CSIC - Taller de Altas Energias 2016

PURPOSE

- Study and understand the effects of non-metricity on fermions and calculate the generic corrections to the Hamiltonian for spinors in a torsion-free non-Riemannian spacetime.

- Calculate the corrections for the Hydrogen atom Hamiltonian in a torsion-free non-Riemannian spacetime.

Study of the Hydrogen atom in a Riemannian background by L.Parker.

VOLUME 44, NUMBER 23 PHYSICAL REVIEW LETTERS 9 JUNE 1980

One-Electron Atom in Curved Space-Time

Leonard Parker
Department of Physics, University of Wisconsin–Milwaukee, Milwaukee, Wisconsin 53201
(Received 7 March 1980)

A one-electron atom in a general curved space-time is considered. The Hamiltonian of the Dirac equation is found in Fermi normal coordinates, and expressions are obtained, to first order in the Riemann tensor, for the shifts in energy of the $1S_{1/2}$, $2S_{1/2}$, and $2P_{1/2}$ levels.

PACS numbers: 98.80.Dr, 04.90.+e, 31.10.+z

No studies in a non-Riemannian background have been done.

MECHANICS, LIGHT & GRAVITATION

Galileo

↓

Principle of Relativity

Realized through Galilean transformations

Absolute space and time for all observers

MECHANICS, LIGHT & GRAVITATION

Newton

↓

$$\vec{F} = m\ddot{\vec{x}}$$

$$\vec{g}_m = -G\frac{m\vec{r}}{r^3} \quad ; \quad \vec{F}_g(\vec{g}, m) = m\vec{g}$$

Principle of Relativity through

↓

Galilean transformations

MECHANICS, LIGHT & GRAVITATION

Newton

$$\vec{F} = m\ddot{\vec{x}}$$

$$\vec{g}_m = -G\frac{m\vec{r}}{r^3} \quad ; \quad \vec{F}_g(\vec{g}, m) = m\vec{g}$$

Maxwell

$$F = dA \quad ; \quad Div(F) = \mu_0 J$$

Principle of Relativity through

Galilean transformations

Lorentz transformations

Fundamental Problem

Galilean Transformations

+

Electromagnetism

↓

~~Principle of Relativity~~

MECHANICS, LIGHT & GRAVITATION

Einstein

↓

Keep Relativity principle
Speed of light is absolute for all observers

↓

Space and time are no longer absolute

↓

Special Relativity

Not compatible with Newtonian Gravity

↓

Equivalence Principle

↓

General Relativity

MECHANICS, LIGHT & GRAVITATION

$$S = \frac{4\pi G}{c^4}\int d^{\bar{4}}x\mathcal{R} \xrightarrow{\delta S = 0} R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Trajectories of freely falling particles are geodesics in spacetime
(Their Action is just given by the length of their worldlines)

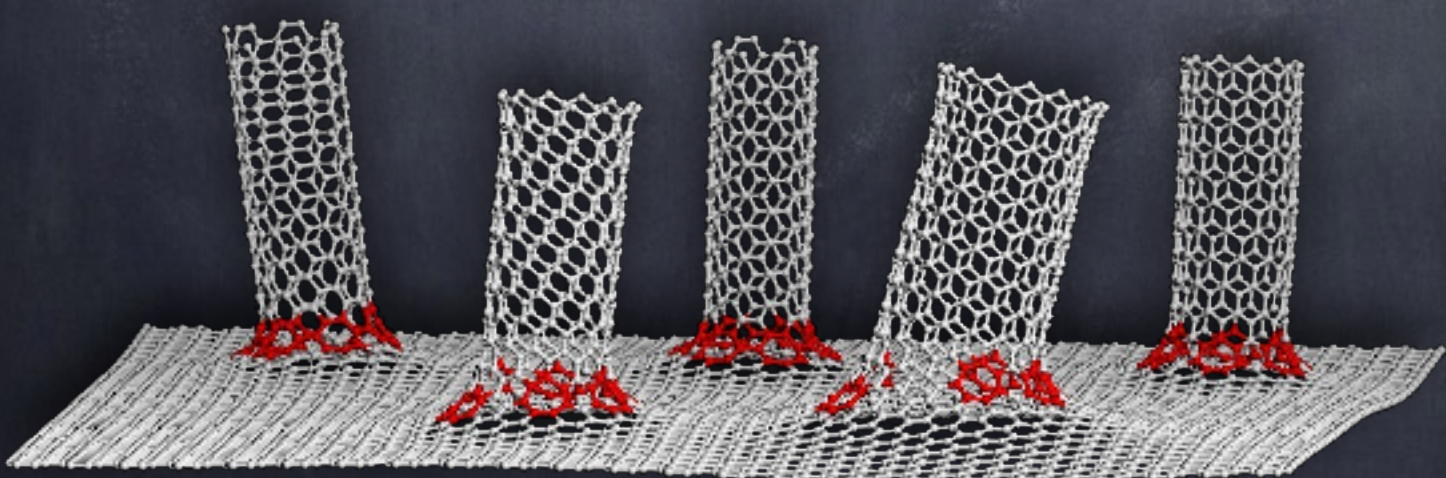
Spacetime becomes dynamical and is represented by a Riemannian manifold

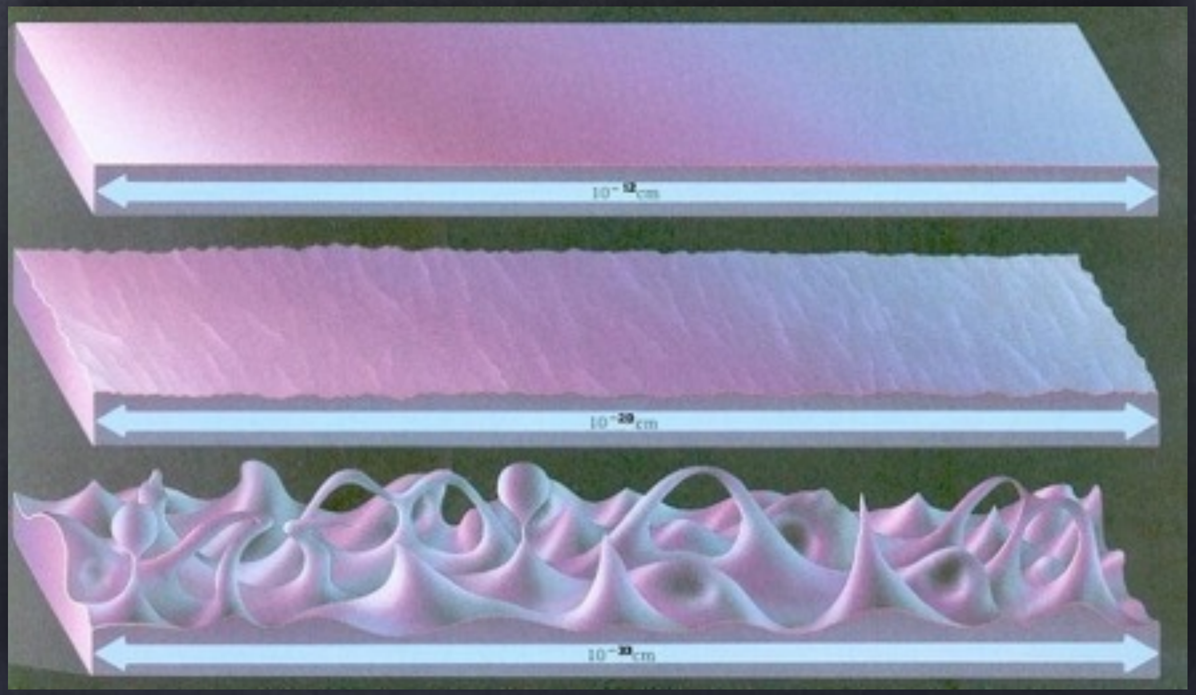
WHY CONSIDERING NON-METRICITY

- There was only Riemannian Geometry at the time: $\nabla g = 0$.
- Posible generalizations include non-metricity and torsion.
- Torsion is needed when taking into account fermions as sources of gravity (Einstein-Cartan theory)
- Non-metricity $\nabla_\mu g_{\alpha\beta} \equiv -Q_{\mu\alpha\beta}$ could be helpful in dealing with some problems.

WHY CONSIDERING NON-METRICITY

- f($\mathcal{R}$) theories could solve some singulartiy problems.
- It has been found that defects in a crystal structure can be described by an effective non-Riemannian geometry.
- Quantum fluctuations could be analog to defects in crystals and could be explained (in scales where GR breaks down) by a metric-affine effective field theory.





THE DIRAC EQUATION

Electromagnetism fails in predictng atomic spectra.

↓

Bohr Quantization Rules $\left\{ \begin{array}{l} \text{Predicts Hydrogen spectrum} \\ \text{Contradicts classical physics} \end{array} \right.$

↓

Quantum Mechanics

THE DIRAC EQUATION

Electromagnetism fails in predictng atomic spectra.

↓

Bohr Quantization Rules $\begin{cases}\text{Predicts Hydrogen spectrum}\\ \text{Contradicts classical physics}\end{cases}$

↓

The Dirac Equation ⟵ Relativistic wavefunction equation

$$(\gamma^\mu \partial_\mu + m)\psi = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$$

$$H_D^o \psi \equiv i\partial_0 \psi \longrightarrow H_D^o = \gamma^0\left(i\gamma^k\partial_k + m\right)$$

Hermiticity $\longrightarrow$ $(\gamma^0)^\dagger = \gamma^0 \;\; ; \;\; (\gamma^k)^\dagger = -\gamma^k$

THE DIRAC EQUATION

$$\boxed{(\gamma^\mu \partial_\mu + m)\psi = 0} \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$$

The background space does not need to be Minkowski!!

(What about description by non-inertial observers?)

GENERALIZED DIRAC EQUATION

- The lorentz group is now a local symmetry group.
- Spinor fields are elements under which the spin rep. of the Lorentz group $\mathcal{SO}(1,3)$ acts.
- $\underline{\gamma}^\mu$ is an element of the vector rep.of $\mathcal{SO}(1,3)$ whose components are elements of the spin rep. of the Lorentz group.

↓

Covariant Dirac equation

$$(\underline{\gamma}^\mu \nabla_\mu + m)\psi = 0 \qquad \{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = 2g^{\mu\nu}\mathbb{I}$$

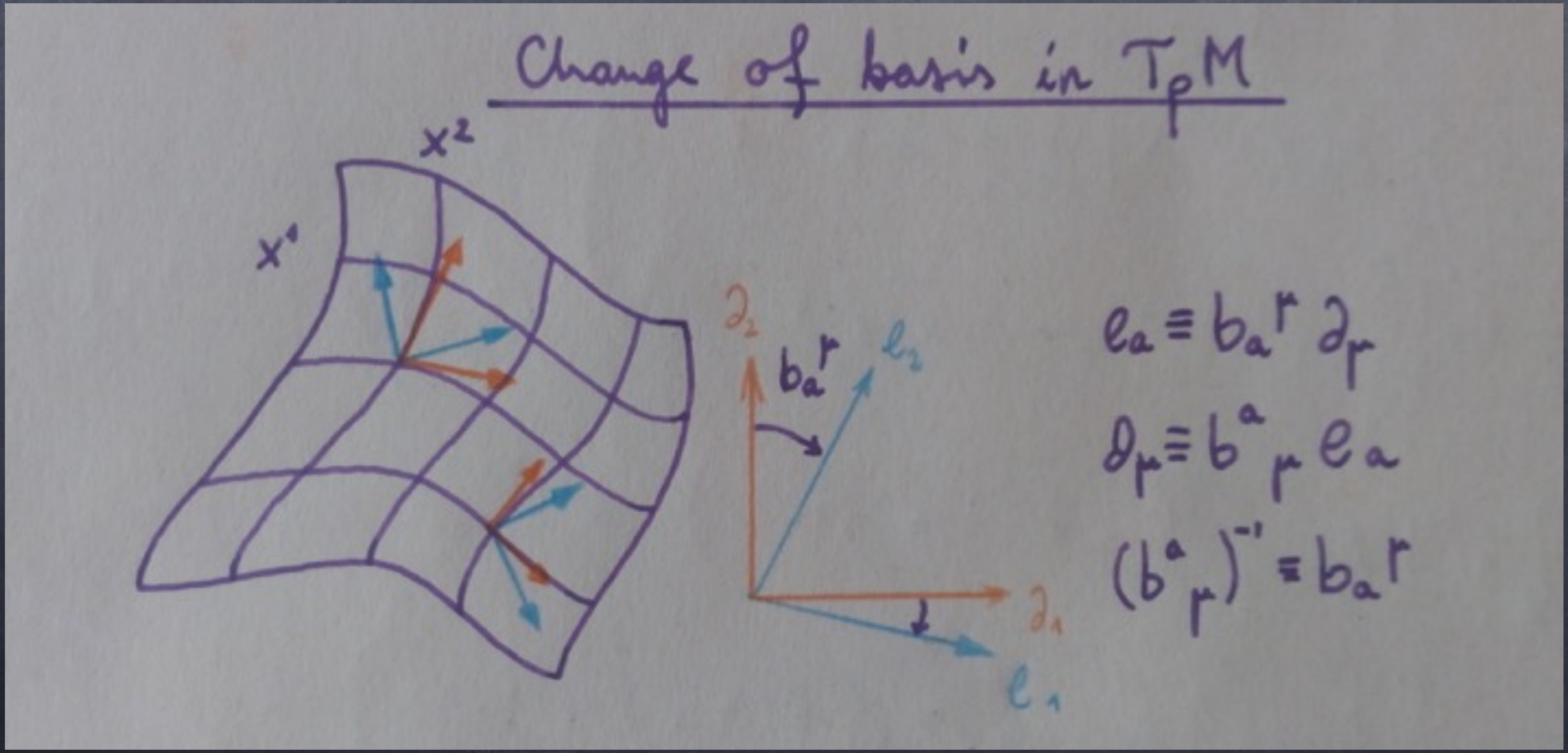
$$\nabla_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi \longrightarrow \Gamma_\mu \text{ is the spinor connaction.}$$

GENERALIZED DIRAC EQUATION

Vierbein $\longrightarrow$

$$\mathbf{e}_a \equiv b_a{}^{\mu}\partial_{\mu} \quad ; \quad b_a{}^{\nu} b^a{}_{\mu} = \delta^{\nu}{}_{\mu}$$

$$\partial_{\mu} \equiv b^a{}_{\mu}\mathbf{e}_a \quad ; \quad b_b{}^{\mu} b^a{}_{\mu} = \delta^a{}_b$$



GENERALIZED DIRAC EQUATION

$$(\underline{\gamma}^{\mu}\nabla_{\mu}+m)\psi=0 \qquad \{\underline{\gamma}^{\mu},\underline{\gamma}^{\nu}\}=2g^{\mu\nu}\mathbb{I}$$

Vierbein $\longrightarrow$

$$\mathbf{e}_a \equiv b_a{}^{\mu}\partial_{\mu} \quad ; \quad b_a{}^{\nu}b^a{}_{\mu}=\delta^{\nu}{}_{\mu}$$

$$\partial_{\mu} \equiv b^a{}_{\mu}\mathbf{e}_a \quad ; \quad b_b{}^{\mu}b^a{}_{\mu}=\delta^a{}_b$$

Choice of verbein: $\longrightarrow g_{ab}\equiv b_a{}^{\mu}b_b{}^{\nu}g_{\mu\nu}=\eta_{ab}$

$\downarrow$

$$\gamma^a \equiv b^a{}_{\mu}\underline{\gamma}^{\mu}$$

$$\{\gamma^a,\gamma^b\}=2g^{ab}\mathbb{I}=2\eta^{ab}\mathbb{I}$$

THE SPINOR CONNECTION

$$\left[b_a{}^{\mu}\gamma^a\left(\partial_\mu + \Gamma_\mu\right) + m\right]\psi = 0 \qquad \{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}$$

The spinor connection and the vierbein accounts for the effects of the background geometry on fermions.

Rep. theory of Lie groups

↓

$$\Gamma_\mu = -\frac{i}{2}\sigma_{ab} b^a{}_\nu g^{\nu\rho}\nabla_\mu b^b{}_\rho$$

THE SPINOR CONNECTION

$$\left[b_a{}^{\mu}\gamma^a\left(\partial_\mu + \Gamma_\mu\right) + m\right]\psi = 0 \qquad \{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}$$

The spinor connection and the vierbein accounts for the effects of the background geometry on fermions.

Dirac álgebra and manipulations explicit the role of non-metricity:

$$\Gamma_\mu = \frac{1}{4}\gamma^a\gamma^b b_a{}^{\nu} g_{\nu\rho}\nabla_\mu b_b{}^{\rho} - \frac{1}{8}Q_{\mu a}{}^{a}\mathbb{I}$$

Riemannian Non-Riemannian

THE SPINOR CONNECTION

Dirac álgebra and manipulations explicit the role of non-metricity:

$$\Gamma_\mu = \underbrace{\frac{1}{4}\gamma^a\gamma^b b_a{}^\nu g_{\nu\rho}\nabla_\mu b_b{}^\rho}_{\text{Riemannian}} - \underbrace{\frac{1}{8}Q_{\mu a}{}^a\mathbb{I}}_{\text{Non-Riemannian}}$$

- Concerning the Dirac structure, the non-Riemannian term is analog to a as a gauge field:

$$\Gamma_\mu^{EM} = -iqA_\mu\mathbb{I}$$

GENERALIZED DIRAC EQUATION

$$\left[b_a{}^{\mu}\gamma^a\left(\partial_\mu + \Gamma_\mu\right) + m\right]\psi = 0$$

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}$$

$$H_D\psi \equiv i\partial_0\psi$$

$$H_I \equiv H_D - H_D^o$$

GENERALIZED DIRAC EQUATION

$$\left[b_a{}^{\mu}\gamma^a\left(\partial_\mu + \Gamma_\mu\right) + m\right]\psi = 0 \qquad \{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}$$

$$H_D\psi \equiv i\partial_0\psi \qquad H_I \equiv H_D - H_D^o$$

The general interaction Hamiltonian has the Dirac structure:

$$H_I = B\mathbb{I} + U_c\gamma^c + S_{ab}\gamma^a\gamma^b + T_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d$$

B, U_c, S_{ab}, T_{abcd} Contain all the geometric informaton

NON-RIEMANNIAN ELECTROMAGNETISM

The corrections to the Hydrogen Hamiltonian also come from the electromagnetic field in curved spaces.

Torsion-free spaces + Lorenz gauge $\nabla_\mu A^\mu = 0$

↓

Maxwell Equations

$$g^{\mu\sigma}\nabla_\mu\nabla_\sigma A_\alpha - R_\alpha{}^\sigma A_\sigma = -\mu_0 J_\alpha - Q_\mu{}^{\mu\sigma}\nabla_\sigma A_\alpha + Q^{\mu\sigma}{}_\alpha F_{\sigma\mu} - \left(\nabla_\mu Q_\alpha{}^{\mu\sigma}\right) A_\sigma - Q_\alpha{}^{\mu\sigma}\nabla_\mu A_\sigma + P^{\mu\sigma}{}_{\mu\alpha} A_\sigma$$

NON-RIEMANNIAN ELECTROMAGNETISM

- Corrections from the Electromagnetic Hamiltonian.

$$A_\mu = A_\mu^{(0)} + A_\mu^{(1)}$$

Flat solution Correction

$A_\mu^{(1)}$ vanishing in the flat limit.

Computation of $A_\mu^{(1)}$ needs a choice of coordinates.

EXPANSION IN TERMS OF $R_{\alpha\beta\mu\nu}$ AND $Q_{\alpha\mu\nu}$

Highest order observable effects are of order lower than
$\mathcal{O}({R_{k\beta\mu\nu}}^2, {Q_{l\rho\sigma}}^2, Q_{k\mu\nu}R_{k\beta\mu\nu}, \partial_\lambda R_{k\beta\mu\nu}, \partial_{\lambda\rho}Q_{k\mu\nu})$

$$g_{\mu\nu} = \eta_{\mu\nu} + Q_{k\mu\nu}y^k - \frac{1}{2}\left(\partial_m Q_{k\mu\nu} + \frac{1}{3}P_{\nu m\mu k} + \frac{2}{3}R_{\mu k\nu m}\right)y^k y^m$$

Non-Riemannian

↓

Expansion of ${b_a}^\mu$, $\Gamma^\alpha_{\mu\nu}$ and Γ_μ.

↓

Finding the corrections in the Hydrogen atom Hamiltonian

CORRECTIONS FOR THE HYDROGEN CASE

The Hydrogen atom case has a spinor connection:

$$\Gamma_\mu = \frac{1}{4}\gamma^a\gamma^b b_a{}^\nu g_{\nu\rho}\nabla_\mu b_b{}^\rho - \left(\frac{1}{8}Q_{\mu a}{}^a + iqA_\mu\right)\mathbb{I}$$

The interaction hamiltonian is the difference between the curved and flat Hydrogen Hamiltonians.

↓

Modification of B and S_{0k}

$$H_I = B\mathbb{I} + U_c\gamma^c + S_{ab}\gamma^a\gamma^b + T_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d$$

CONSERVED CURRENT AND PERTURBATION THEORY

Perturbation theory $\longrightarrow$ Need for defining a well behaved scalar product in spinor space.

Flat Dirac equation $\longrightarrow$ $j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi$ is conserved.

$$\Downarrow$$

$$d_t Q \equiv \frac{d}{dt} \int d^3x j^0 = \frac{d}{dt} \int d^3x \psi^\dagger \psi = 0$$

Q (or j^μ) defines a well behaved scalar product:

$$(\psi, \phi) \equiv \int d^3x \phi^\dagger \psi \longrightarrow \frac{d(\psi, \psi)}{dt} = 0$$

CONSERVED CURRENT AND PERTURBATION THEORY

$$\left.\begin{array}{r}\nabla_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi \quad , \psi \in \mathbf{Sp} \\ \bar{\psi} \equiv \psi^\dagger \gamma^0 \in \mathbf{Sp}^* (\text{dual of } \mathbf{Sp})\end{array}\right\} \longrightarrow \nabla_\mu \bar{\psi} = (\partial_\mu + \Gamma_\mu)\bar{\psi}$$

$$\downarrow$$

$$\nabla_\mu j^\mu \equiv \nabla_\mu \bar{\psi} \underline{\gamma^\mu} \psi = 0 \longleftrightarrow \boxed{\nabla_\mu \underline{\gamma^\mu} = 0}$$

$$\nabla_\mu \underline{\gamma^\alpha} = \frac{1}{2} Q_{\mu\nu}{}^{\alpha} \underline{\gamma^\nu}$$

$$\boxed{\nabla_\mu j^\mu = 0 \longleftrightarrow Q_{\mu\nu}{}^{\mu} = 0 \quad \nu = 0, 1, 2, 3}$$

Restriction in the possible non-metric spaces conserving fermion charges.

CONSERVED CURRENT AND PERTURBATION THEORY

$$\nabla_\mu j^\mu = 0 \longleftrightarrow Q_{\mu\nu}{}^{\mu} = 0 \quad \nu = 0, 1, 2, 3$$

- This condition may be satisfied, restricting the possible effective geometries.
- Arbitrary effective geometries would allow C violation scenarios.
- Definition of a proper perturbation theory in arbitrary spaces needs to be further studied.

CONCLUSIONS

- General corrections to the Hamiltonian due to non-metricity have been computed. Also for the Hydrogen atom case.

- Non-metricity entering the spinor connection as a SM gauge field is an interesting feature to considered.

- Definition of appropriate perturbation theory in non-metric backgrounds has to be worked. Then the corrections to the Hydrogen energy levels computed.

- Restrictions from the conservation of $j^\mu = \psi\dagger\underline{\gamma}^0\underline{\gamma}^\mu\psi$ in the possible effective spacetime geometries have to be analized.

- Possible scenarios of C violation in arbitraryly non-metric backgrounds should be studied.

THANK YOU

Generalized Dirac Equation

$$\left[b_a{}^{\mu}\gamma^a\left(\partial_\mu + \Gamma_\mu\right) + m\right]\psi = 0$$

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}$$

$$H_D\psi \equiv i\partial_0\psi \qquad H_I \equiv H_D - H_D^o$$

$$H_D = -i\Gamma_0 - i{g^{00}}^{-1}\underline{\gamma}^0\underline{\gamma}^k\left(\partial_k + \Gamma_k\right) - i{g^{00}}^{-1}\underline{\gamma}^0 m$$

$$H_I = -\,i\Gamma_0 - i{g^{00}}^{-1}b_a{}^0\delta_b{}^k\gamma^a\gamma^b\Gamma_k - i\left({g^{00}}^{-1}b_a{}^0 + \delta_a{}^0\right)\delta_b{}^k\gamma^a\gamma^b\partial_k - $$
$$-\,i\left({g^{00}}^{-1}b_a{}^0 + \delta_a{}^0\right)\gamma^a m$$

FERMI COORDINATES

- Corrections in terms of curvature and non-metricity
- Conditions for observation

$\Bigg\} \longrightarrow$ Choice of coordinates

Fermi Coordinates

- Adapted to the atom's worldline $\rightarrow$ natural for observing
- Geodesic coordinates $\rightarrow$ Simplify the expansion in terms of $Q_{\alpha\mu\nu}$ and $R_{\alpha\beta\mu\nu}$

EXPANSION OF THE GENERIC CORRECTIONS

$$H_I = B\mathbb{I} + U_c\gamma^c + S_{ab}\gamma^a\gamma^b + T_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d$$

The expansion of the coefficients is:

$$U_0 = -\frac{i}{2}\left[Q_l^{00}y^l + \left(\frac{1}{2}\partial l Q_m^{00} + \frac{1}{3}R_{0m0l}\right)y^l y^m\right]$$

$$U_k = -\frac{i}{2}\left[Q_{lk}{}^0 y^l + \left(\frac{1}{2}\partial l Q_{mk}{}^0\right) - \frac{1}{3}R_{km0l}y^l y^m\right]$$

$$S_{k0} = -\frac{i}{8}\left(-\frac{1}{2}\partial_0 Q_{m0k} + R_{0k0m}\right)y^m$$

$$S_{00} = -\frac{i}{8}\left(-\frac{1}{2}\partial_0 Q_{m00} + R_{000m}\right)y^m$$

$$S_{kl} = -\frac{i}{2}\left[\left(-\frac{1}{8}\partial_0 Q_{mlk} + \frac{1}{4}R_{lk0m}\right)y^m + \left(Q_{mk}{}^0 y^m + \left(\frac{1}{2}\partial_n Q_{mk}{}^0 - \frac{1}{3}R_{kn0m}\right)y^m y^n\right)\partial_l\right]$$

$$S_{0k} = -\frac{i}{2}\left[\left(-\frac{1}{8}\partial_0 Q_{mk0} + \frac{1}{4}R_{0k0m}\right)y^m - \frac{1}{4}\left(Q_{ka}{}^a + \partial_l Q_{ka}^a y^l\right) + \left(Q_l{}^{00}y^l + \left(\frac{1}{2}\partial_l Q_m{}^{00} + \frac{1}{3}R_{0l0m}\right)y^m y^l\right)\partial k\right]$$

$$B = \frac{i}{8}\partial_k Q_{0a}{}^a y^k$$

$$T_{kbcd} = T_{a0cd} = 0 \quad ; \quad T_{0kcd} \equiv -\frac{i}{8}\left[Q_{kcd} + \frac{1}{2}\left((\partial_m Q_{kcd} + \partial_k Q_{mcd}) + R_{dckm}\right)y^m\right]$$

CORRECTIONS FOR THE HYDROGEN CASE

The flat solution is the Coulomb field centered in the nucleus

$$A_\mu^{(0)} = qr^{-1}\delta^0{}_\mu$$

Introducing the flat solution in the Maxwell equations one gets a differential equation for the correction $A_\mu^{(1)}$

$$\delta^{ij}\partial ij A_0^{(1)} = -\, qr^{-5}\left[A_{ijk}y^iy^jy^k + B_{ijkl}y^iy^jy^kyl\right] + -$$
$$+ Cqr^{-1} - qr^{-3}\left[D_{ij}y^iy^j + E_iy^i\right]$$

$$\delta^{ik}\partial ik A_j^{(1)} = -qr^{-3}\left[H_{jik}y^iy^k + K_{ji}y^i\right] - qr^{-1}L_j$$

Corrections for the hydrogen case

$$A_0^{(1)} = \left[\frac{1}{12}A_{klm}y^k y^l y^m + \frac{1}{18}B_{klmn}y^k y^l y^m y^n\right] r^{-3} + \left[\frac{1}{12}A_{klm}\left(\delta^{kl}y^m + \delta^{km}y^l + \delta^{ml}y^k\right) - \right.$$
$$- \frac{1}{36}B_{klmn}\left(\delta^{kl}y^m y^n + \delta^{km}y^l y^n + \delta^{kn}y^l y^m + \delta^{lm}y^k y^n + \delta^{ln}y^k y^n + \delta^{mn}y^k y^l\right) +$$
$$\left. + \frac{1}{4}D_{kl}y^k y^l + \frac{1}{2}E_k y^k\right] r^{-1} + \left[\frac{1}{18}B_{klmn}\left(\delta^{kl}\delta^{mn} + \delta^{km}\delta^{ln} + \delta^{kn}\delta^{lm} + \delta^{lm}\delta^{kn} + \right.\right.$$
$$\left.\left. + \delta^{ln}\delta^{km} + \delta^{nm}\delta^{kl}\right) - \frac{1}{2}C - \frac{1}{4}D_{kl}\delta^{kl}\right] r$$

$$A_k^{(1)} = \left[\frac{1}{4}H_{klm}y^k y^l y^m + \frac{1}{2}K_{kn}y^k y^l\right] r^{-1} + \left[\frac{1}{2}L_k - \frac{1}{4}H_{klm}\delta^{lm}\right] r$$

All tensors appearing are functions of $R_{\alpha\beta\mu\nu}$, $Q_{\alpha\mu\nu}$ and $\partial_\beta Q_{\alpha\mu\nu}$ in the nucleus.

CORRECTIONS FOR THE HYDROGEN CASE

Modified coefficients:

$$B = \frac{i}{8}\partial_k Q^a_{0a} y^k + Ze\left[\frac{1}{12}A_{klm}y^k y^l y^m + \frac{1}{18}B_{klmn}y^k y^l y^m y^n\right] r^{-3} +$$

$$- Ze\left[\frac{1}{12}A_{klm}\left(\delta^{kl}y^m + \delta^{km}y^l + \delta^{ml}y^k\right) - \frac{1}{36}B_{klmn}\left(\delta^{kl}y^m y^n + \delta^{km}y^l y^n + \delta^{kn}y^l y^m +\right.\right.$$

$$\left.\left. + \delta^{lm}y^k y^n + \delta^{ln}y^k y^n + \delta^{mn}y^k y^l\right) + \frac{1}{4}D_{kl}y^k y^l + \frac{1}{2}E_k y^k\right] r^{-1} - Ze\left[\frac{1}{18}B_{klmn}\left(\delta^{kl}\delta^{mn} +\right.\right.$$

$$\left.\left. + \delta^{km}\delta^{ln} + \delta^{kn}\delta^{lm} + \delta^{lm}\delta^{kn} +\right) + \delta^{ln}\delta^{km} + \delta^{nm}\delta^{kl}\right) - \frac{1}{2}C - \frac{1}{4}D_{kl}\delta^{kl}\Big] r$$

$$S_{0k} = -\frac{i}{2}\left[\left(-\frac{1}{8}\partial_0 Q_{mk0} + \frac{1}{4}R_{0k0m}\right) y^m - \frac{1}{4}\left(Q_{ka}{}^a + \partial_l Q^a_{ka} y^l\right) +\right.$$

$$\left. + \left(Q_l{}^{00} y^l + \left(\frac{1}{2}\partial_l Q_m{}^{00} + \frac{1}{3}R_{0l0m}\right) y^m y^l\right)\partial_k\right] +$$

$$+ Ze\left[\frac{1}{4}H_{klm}y^k y^l y^m + \frac{1}{2}K_{kn}y^k y^l\right] r^{-1} + Ze\left[\frac{1}{2}L_k - \frac{1}{4}H_{klm}\delta^{lm}\right] r$$

CORRECTIONS FOR THE HYDROGEN CASE

Order zero terms

$$B = \frac{i}{8}\partial_k Q^a_{0a} y^k - Ze\left[\frac{1}{12}A_{klm}y^k y^l y^m + \frac{1}{18}B_{klmn}y^k y^l y^m y^n\right] r^{-3} +$$

$$-Ze\left[\frac{1}{12}A_{klm}(\delta^{kl}y^m + \delta^{km}y^l + \delta^{ml}y^k) - \frac{1}{36}B_{klmn}\left(\delta^{kl}y^m y^n + \delta^{km}y^l y^n + \delta^{kn}y^l y^m + \right.\right.$$

$$\left.\left. + \delta^{lm}y^k y^n + \delta^{ln}y^k y^n + \delta^{mn}y^k y^l\right) + \frac{1}{4}D_{kl}y^k y^l + \frac{1}{2}E_k y^k\right] r^{-1} - Ze\left[\frac{1}{18}B_{klmn}\left(\delta^{kl}\delta^{mn} + \right.\right.$$

$$\left.\left. + \delta^{km}\delta^{ln} + \delta^{kn}\delta^{lm} + \delta^{lm}\delta^{kn} + \right) + \delta^{ln}\delta^{km} + \delta^{nm}\delta^{kl}\right) - \frac{1}{2}C - \frac{1}{4}D_{kl}\delta^{kl}\right] r$$

$$S_{0k} = -\frac{i}{2}\left[\left(-\frac{1}{8}\partial_0 Q_{mk0} + \frac{1}{4}R_{0k0m}\right)y^m - \frac{1}{4}\left(Q_{ka}{}^a + \partial_l Q^a_{ka}y^l\right) + \right.$$

$$\left. + \left(Q_l{}^{00}y^l + \left(\frac{1}{2}\partial_l Q_m{}^{00} + \frac{1}{3}R_{0l0m}\right)y^m y^l\right)\partial_k\right] +$$

$$+ Ze\left[\frac{1}{4}H_{klm}y^k y^l y^m + \frac{1}{2}K_{kn}y^k y^l\right] r^{-1} + Ze\left[\frac{1}{2}L_k - \frac{1}{4}H_{klm}\delta^{lm}\right] r$$