

Institut de Chimie de Strasbourg - Laboratoire de Chimie Quantique -Université de Strasbourg /CNRS

7th School and Workshop on TD-DFT, Benasque, Spain, 21.09.16

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Motivation and strategy

- Linear response TD-DFT, which is a time-dependent single-reference perturbation theory, does not treat quasi-degenerate states adequately.
- This is problematic for describing strongly correlated molecular systems (transition metal and actinide chemistry, avoided crossings, conical intersections, ...).
- For such systems, the standard quantum chemical approach is **multi-state** multireference perturbation theory (MS-CASPT2, QD-NEVPT2). In contrast to MP2, the zeroth-order wavefunctions are multiconfigurational. They are all described with the same set of orbitals that are optimized by state averaging.
- We want to develop a rigorous and computationally cheaper alternative to these methods by using DFT.
- State averaging can be performed in principle exactly in ensemble DFT (eDFT) for excited states.
- Multiconfigurational wavefunctions can be introduced in standard (ground-state) DFT by means of range separation.
- Therefore range-separated eDFT provides a rigorous framework for merging state-averaged multiconfigurational methods with DFT.

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Ensemble DFT

• Variational principle for an equi-ensemble (*Theophilou*): if Ψ and Ψ' are orthonormal then

 $\langle \Psi | \hat{H} | \Psi \rangle + \langle \Psi' | \hat{H} | \Psi' \rangle \ge E_0 + E_1$

• Generalization: for a given ensemble weight *w*,

$$(1-w)\langle\Psi|\hat{H}|\Psi\rangle + w\langle\Psi'|\hat{H}|\Psi'\rangle = (1-2w)\underbrace{\langle\Psi|\hat{H}|\Psi\rangle}_{\geq E_0} + w\left(\underbrace{\langle\Psi|\hat{H}|\Psi\rangle + \langle\Psi'|\hat{H}|\Psi'\rangle}_{\geq E_0+E_1}\right)$$

• *Gross-Oliveira-Kohn* (GOK) variational principle:

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for
$$0 \le w \le 1/2$$
, $(1-w)\langle \Psi | \hat{H} | \Psi \rangle + w \langle \Psi' | \hat{H} | \Psi' \rangle \ge E^w$

where E^w is the exact ensemble energy: $E^w = (1 - w)E_0 + wE_1 \rightarrow \omega = \frac{\mathrm{d}E^w}{\mathrm{d}w} = E_1 - E_0$

• E^{w} is a functional of the ensemble density $n^{w}(\mathbf{r}) = (1 - w)n_{0}(\mathbf{r}) + wn_{1}(\mathbf{r})$

Ensemble DFT

- Trial ensemble density matrix operator: $\hat{\gamma}^w = (1 w) |\Psi\rangle \langle \Psi| + w |\Psi'\rangle \langle \Psi'|$.
- The GOK variational principle can be written in a compact form as

$$E^{\boldsymbol{w}} \leq \operatorname{Tr}\left[\hat{\gamma}^{\boldsymbol{w}}\hat{H}\right].$$

• With
$$\hat{H} = \hat{T} + \hat{W}_{ee} + \int d\mathbf{r} \, \mathbf{v}_{ne}(\mathbf{r}) \hat{n}(\mathbf{r})$$
 it comes $E^{w} \leq \operatorname{Tr} \left[\hat{\gamma}^{w} (\hat{T} + \hat{W}_{ee}) \right] + \int d\mathbf{r} \, \mathbf{v}_{ne}(\mathbf{r}) n_{\hat{\gamma}^{w}}(\mathbf{r})$

where the trial ensemble density equals $n_{\hat{\gamma}^{w}}(\mathbf{r}) = \operatorname{Tr}\left[\hat{\gamma}^{w}\hat{n}(\mathbf{r})\right] = (1-w)n_{\Psi}(\mathbf{r}) + wn_{\Psi'}(\mathbf{r})$

• Hohenberg–Kohn theorem for ensembles*: $E^w = \min_n$

$$E^{\boldsymbol{w}} = \min_{n} \left\{ F^{\boldsymbol{w}}[n] + \int d\mathbf{r} \, v_{\mathrm{ne}}(\mathbf{r}) n(\mathbf{r}) \right\}$$

where the ensemble Levy–Lieb functional equals

$$F^{\boldsymbol{w}}[n] = \min_{\hat{\gamma}^{\boldsymbol{w}} \to n} \left\{ \operatorname{Tr} \left[\hat{\gamma}^{\boldsymbol{w}} (\hat{T} + \hat{W}_{ee}) \right] \right\}$$

*E. K. U. Gross, L. N. Oliveira, and W. Kohn, Phys. Rev. A 37, 2809 (1988)

Kohn–Sham eDFT

• Kohn–Sham decomposition of the ensemble Levy–Lieb functional:

$$F^{\boldsymbol{w}}[n] = T_{\mathrm{s}}^{\boldsymbol{w}}[n] + E_{\mathrm{Hxc}}^{\boldsymbol{w}}[n]$$

where the ensemble non-interacting kinetic energy equals

$$T_{\rm s}^{\boldsymbol{w}}[n] = \min_{\hat{\gamma}^{\boldsymbol{w}} \to n} \left\{ \operatorname{Tr}\left[\hat{\gamma}^{\boldsymbol{w}} \hat{T} \right] \right\} = \operatorname{Tr}\left[\hat{\Gamma}_{\rm s}^{\boldsymbol{w}}[n] \hat{T} \right]$$

and
$$E_{\text{Hxc}}^{\boldsymbol{w}}[n] = E_{\text{H}}[n] + E_{\text{xc}}^{\boldsymbol{w}}[n]$$
 with $E_{\text{H}}[n] = \frac{1}{2} \int \int d\mathbf{r} d\mathbf{r}' \frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$

• Exact KS-eDFT variational ensemble energy expression:

$$E^{\boldsymbol{w}} = \min_{\hat{\gamma}^{\boldsymbol{w}}} \left\{ \operatorname{Tr} \left[\hat{\gamma}^{\boldsymbol{w}} \hat{T} \right] + E^{\boldsymbol{w}}_{\operatorname{Hxc}}[n_{\hat{\gamma}^{\boldsymbol{w}}}] + \int d\mathbf{r} \, v_{\operatorname{ne}}(\mathbf{r}) n_{\hat{\gamma}^{\boldsymbol{w}}}(\mathbf{r}) \right\}$$

*E. K. U. Gross, L. N. Oliveira, and W. Kohn, Phys. Rev. A 37, 2809 (1988)

• The minimizing density matrix operator is a non-interacting one,

$$\hat{\Gamma}_{s}^{w} = (1-w) |\Phi_{0}^{\mathrm{KS},w}\rangle \langle \Phi_{0}^{\mathrm{KS},w}| + w |\Phi_{1}^{\mathrm{KS},w}\rangle \langle \Phi_{1}^{\mathrm{KS},w}|,$$

which reproduces the exact physical ensemble density $n^w(\mathbf{r})$. It is obtained from the self-consistent KS-eDFT equations

$$\hat{T} + \int d\mathbf{r} \left(v_{\rm ne}(\mathbf{r}) + \frac{\delta E_{\rm Hxc}^{\boldsymbol{w}} \left[n_{\hat{\Gamma}_{\rm S}^{\boldsymbol{w}}} \right]}{\delta n(\mathbf{r})} \right) \hat{n}(\mathbf{r}) \left[|\Phi_i^{\rm KS, \boldsymbol{w}}\rangle = \mathcal{E}_i^{\rm KS, \boldsymbol{w}} |\Phi_i^{\rm KS, \boldsymbol{w}}\rangle, \quad i = 0, 1.$$

• According to the Hellmann–Feynman theorem,

$$\omega = \frac{\mathrm{d}E^{w}}{\mathrm{d}w} = \mathcal{E}_{1}^{\mathrm{KS},w} - \mathcal{E}_{0}^{\mathrm{KS},w} + \left.\frac{\partial E_{\mathrm{xc}}^{w}[n]}{\partial w}\right|_{n=n^{w}}$$

• If the first excitation is a single electron excitation then

$$\omega = \varepsilon_{\mathrm{L}}^{w} - \varepsilon_{\mathrm{H}}^{w} + \left. \frac{\partial E_{\mathrm{xc}}^{w}[n]}{\partial w} \right|_{n=n^{w}} \quad \xrightarrow{w \to 0} \quad \omega = \varepsilon_{\mathrm{L}} - \varepsilon_{\mathrm{H}} + \underbrace{\left. \frac{\partial E_{\mathrm{xc}}^{w}[n^{0}]}{\partial w} \right|_{w=0}}_{w=0}$$

 Δ_{xc} : derivative discontinuity (DD)*

*M. Levy, Phys. Rev. A 52, R4313 (1995).



Md. M. Alam, S. Knecht, and E. Fromager, Phys. Rev. A 94, 012511 (2016).

Linear interpolation method (LIM)

- In the exact theory: $2(E^{w=1/2} E_0) = \omega = \frac{dE^w}{dw}$
- The WIDFA ensemble energy \tilde{E}^{w} has curvature.
- The WIDFA excitation energy obtained from

$$\mathrm{d}\tilde{E}^{\boldsymbol{w}}/\mathrm{d}\boldsymbol{w} = \tilde{\mathcal{E}}_{1}^{\mathrm{KS},\boldsymbol{w}} - \tilde{\mathcal{E}}_{0}^{\mathrm{KS},\boldsymbol{w}}$$

is weight-dependent (!)

• On the other hand, we have $\omega_{\text{LIM}} = 2\left(\tilde{E}^{w=1/2} - E_0\right)$

that is a well-defined approximate excitation energy, by analogy with the fundamental gap problem *

T. Stein, J. Autschbach, N. Govind, L. Kronik, and R. Baer, J. Phys. Chem. Lett. 3, 3740 (2012).



B. Senjean, S. Knecht, H. J. Aa. Jensen, and E. Fromager, Phys. Rev. A 92, 012518 (2015).



B. Senjean *et al.*, *Phys. Rev. A* **92**, 012518 (2015).

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Z-h. Yang et al., Phys. Rev. A 90, 042501 (2014).



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• Exact range-separated ground-state energy expression (*Savin*):

$$E_0 = \min_{\Psi} \left\{ \langle \Psi | \hat{T} + \hat{W}_{ee}^{lr,\mu} + \hat{V}_{ne} | \Psi \rangle + E_{Hxc}^{sr,\mu} [n_{\Psi}] \right\}$$

- The minimizing wavefunction Ψ₀^μ is the ground state of a long-range interacting system whose density equals the exact ground-state density n₀.
- Ψ_0^{μ} fulfils the self-consistent equation

$$\left(\hat{T} + \hat{W}_{\rm ee}^{\rm lr,\mu} + \hat{V}_{\rm ne} + \int d\mathbf{r} \; \frac{\delta E_{\rm Hxc}^{\rm sr,\mu}}{\delta n(\mathbf{r})} [n_{\Psi_0^{\mu}}] \, \hat{n}(\mathbf{r}) \right) |\Psi_0^{\mu}\rangle = \mathcal{E}_0^{\mu} |\Psi_0^{\mu}\rangle$$

• standard KS-DFT is recovered when $\mu = 0$

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- pure WFT is recovered when $\mu \to +\infty$
- Short-range functionals: srLDA, srPBE, ... (Savin, Toulouse, Gori-Giorgi, Stoll, Goll, Scuseria, ...)
- Long-range-interacting wave function calculation: HF-srDFT, FCI-srDFT, ...

Range-separated ensemble DFT

• **Range separation** of the ensemble Levy–Lieb functional^{*a*,*b*}:

where $F^{\mathrm{lr},\mu,w}[n] = \min_{\hat{\gamma}^w \to n} \left\{ \mathrm{Tr} \left[\hat{\gamma}^w (\hat{T} + \hat{W}^{\mathrm{lr},\mu}_{\mathrm{ee}}) \right] \right\}.$

• **Exact** range-separated expression for the ensemble energy:

$$F^{\boldsymbol{w}}[n] = F^{\mathrm{lr},\mu,\boldsymbol{w}}[n] + E^{\mathrm{sr},\mu,\boldsymbol{w}}_{\mathrm{Hxc}}[n]$$

$$E^{w} = (1 - w) \langle \Psi_{0}^{\mu, w} | \hat{T} + \hat{W}_{ee}^{lr, \mu} + \hat{V}_{ne} | \Psi_{0}^{\mu, w} \rangle + w \langle \Psi_{1}^{\mu, w} | \hat{T} + \hat{W}_{ee}^{lr, \mu} + \hat{V}_{ne} | \Psi_{1}^{\mu, w} \rangle + E_{Hxc}^{sr, \mu, w} [n^{w}],$$

where the auxiliary ground- and first-excited-state wavefunctions fulfil the **self-consistent** equations

$$\left(\hat{T} + \hat{W}_{ee}^{lr,\mu} + \hat{V}_{ne} + \int d\mathbf{r} \frac{\delta E_{Hxc}^{sr,\mu,w}[n^w]}{\delta n(\mathbf{r})} \hat{n}(\mathbf{r})\right) |\Psi_i^{\mu,w}\rangle = \mathcal{E}_i^{\mu,w} |\Psi_i^{\mu,w}\rangle, \qquad i = 0, 1$$

and reproduce the **exact physical ensemble density** $n^{w}(\mathbf{r}) = (1 - w)n_{\Psi_{0}^{\mu},w}(\mathbf{r}) + w n_{\Psi_{1}^{\mu},w}(\mathbf{r}).$

^a E. Pastorczak, N. I. Gidopoulos, and K. Pernal, Phys. Rev. A 87, 062501 (2013).

^o O. Franck and E. Fromager, *Mol. Phys.* **112**, 1684 (2014).





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WIDFA, curvature and ghost interaction error

- Where does the curvature of the WIDFA ensemble energy come from ?
- In range-separated eDFT, the short-range Hartree energy is quadratic in w (unless $n_0(\mathbf{r}) = n_1(\mathbf{r})$):

$$E_{\rm H}^{{\rm sr},\mu}[n^w] = (1-w)^2 E_{\rm H}^{{\rm sr},\mu}[n_0] + w^2 E_{\rm H}^{{\rm sr},\mu}[n_1] + w(1-w) \int \int d\mathbf{r} d\mathbf{r}' \boxed{n_0(\mathbf{r})n_1(\mathbf{r}')} \frac{\operatorname{erfc}(\mu|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}.$$

- In addition, an unphysical short-range **"ghost interaction"** (GI) is introduced.
- This is a well-known problem in KS-eDFT^{*} ($\mu = 0$ limit).
- Both curvature and GI errors are removed in the exact theory by the complementary weight-dependent ensemble short-range xc functional $E_{xc}^{sr,\mu,w}[n]$.
- This is not the case at the WIDFA level since the weight-independent ground-state short-range xc functional is used.

*N. I. Gidopoulos, P. G. Papaconstantinou, and E. K. U. Gross, Phys. Rev. Lett. 88, 033003 (2002).

Exact exchange in range-separated eDFT

• Alternative decomposition of the exact ensemble short-range xc functional:

$$E_{\mathrm{xc}}^{\mathrm{sr},\mu,w}[n] = \underbrace{\mathrm{Tr}\left[\hat{\Gamma}^{\mu,w}[n]\hat{W}_{\mathrm{ee}}^{\mathrm{sr},\mu}\right]}_{-E_{\mathrm{H}}^{\mathrm{sr},\mu}[n]} - E_{\mathrm{H}}^{\mathrm{sr},\mu}[n] + E_{\mathrm{c,md}}^{\mathrm{sr},\mu,w}[n]$$

explicitly linear in *w*

where $\hat{\Gamma}^{\mu,w}[n]$ is the ensemble long-range interacting density matrix operator with density n (rather than the usual non-interacting KS ensemble density matrix operator), hence the name multideterminantal (md) exact exchange.

• This decomposition leads to the exact energy expression

 $E^{\boldsymbol{w}} = \operatorname{Tr}\left[\hat{\Gamma}^{\mu,\boldsymbol{w}}\hat{H}\right] + E^{\operatorname{sr},\mu,\boldsymbol{w}}_{\mathrm{c,md}}[n_{\hat{\Gamma}^{\mu,\boldsymbol{w}}}]$

where $\hat{\Gamma}^{\mu, w}$ reproduces the exact physical ensemble density $n^{w}(\mathbf{r})$.

• In the $\mu = 0$ limit, we obtain an ensemble Hartree–Fock-like energy (calculated with KS-eDFT orbitals) complemented by a density-functional correlation energy.

Md. M. Alam, S. Knecht, and E. Fromager, Phys. Rev. A 94, 012511 (2016).

GI correction in range-separated eDFT: practical calculation

- We use the WIDFA long-range interacting ensemble density matrix operator γ^{µ,w}, thus avoiding the (more rigorous) use of optimized effective potential (OEP) techniques.
- We use the ground-state LDA¹ for the complementary md correlation functional:

 $E_{\mathrm{c,md}}^{\mathrm{sr},\mu,w}[n] \to E_{\mathrm{c,md}}^{\mathrm{sr},\mu}[n]$

• Thus we obtain an approximate GI-corrected (GIC) range-separated ensemble $energy^2$

$$\tilde{E}_{\text{GIC}}^{\mu,\boldsymbol{w}} = \text{Tr}\left[\hat{\gamma}^{\mu,\boldsymbol{w}}\hat{H}\right] + E_{\text{c,md}}^{\text{sr},\mu}[n_{\hat{\gamma}^{\mu},\boldsymbol{w}}]$$

• Excitation energies are then computed with the LIM, hence the name GIC-LIM for the method.

¹ S. Paziani, S. Moroni, P. Gori-Giorgi, and G. B. Bachelet, Phys. Rev. B 73, 155111 (2006).

² Md. M. Alam, S. Knecht, and E. Fromager, Phys. Rev. A 94, 012511 (2016).



FIG. 2. Weight dependence of the WIDFA and GIC ensemble energies in HeH⁺ for $\mu = 0$ and $\mu = 0.4a_0^{-1}$. The FCI and LIM (dashed lines) are also shown. Energies are shifted by their values at w = 0 for ease of comparison.



FIG. 3. Relative error with respect to the FCI obtained for the LIM (dash-dotted lines) and GIC-LIM (solid lines) excitation energies when varying μ . Inset: Zoom-in on the range $0 \le \mu \le 2.0a_0^{-1}$. Excitations in the stretched HeH⁺ ($1^1\Sigma^+ \rightarrow 2^1\Sigma^+$) and H...H ($1^1\Sigma_g^+ \rightarrow 2^1\Sigma_g^+$) molecules correspond to a charge transfer and a double excitation, respectively.

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Extrapolation technique in range-separated eDFT

- The ground-state range-separated energy \tilde{E}^{μ} is μ -dependent in practice since approximate short-range xc functionals are used.
- This energy reduces to the exact ground-state energy E_0 when $\mu \to +\infty$.
- Taylor expansion¹ for large μ :

$$\tilde{E}^{\mu} = E_0 + \frac{a}{\mu^2} + \mathcal{O}\left(\frac{1}{\mu^3}\right) \longrightarrow \mu \frac{\partial \tilde{E}^{\mu}}{\partial \mu} = -\frac{2a}{\mu^2} + \mathcal{O}\left(\frac{1}{\mu^3}\right)$$

thus leading to

$$\tilde{E}^{\mu} + \underbrace{\frac{\mu}{2} \frac{\partial \tilde{E}^{\mu}}{\partial \mu}}_{\mathcal{I}} = E_0 + \mathcal{O}\left(\frac{1}{\mu^3}\right)$$

extrapolation correction

- This extrapolation scheme can be applied to WIDFA range-separated ensemble energies. Its combination with LIM gives extrapolated LIM (ELIM) excitation energies².
- ¹ A. Savin, J. Chem. Phys. 140, 18A509 (2014).

² B. Senjean, E. D. Hedegård, M. M. Alam, S. Knecht, and E. Fromager, Mol. Phys. 114, 968 (2016).

Extrapolation technique in range-separated eDFT

- The GIC range-separated ensemble energy $\tilde{E}_{\text{GIC}}^{\mu,w}$ converges faster^{1,2} (as $1/\mu^3$) towards the exact ensemble energy E^w when $\mu \to +\infty$.
- Therefore, in this case, the extrapolation scheme reads

$$\underbrace{\tilde{E}_{\text{GIC}}^{\mu,w} + \frac{\mu}{3} \frac{\partial \tilde{E}_{\text{GIC}}^{\mu,w}}{\partial \mu}}_{= E^w + \mathcal{O}\left(\frac{1}{\mu^4}\right)$$

extrapolated GIC (EGIC) energy

• Combining EGIC with LIM gives EGIC-LIM excitation energies.

¹ Md. M. Alam, S. Knecht, and E. Fromager, Phys. Rev. A 94, 012511 (2016).
² Md. M. Alam, S. Knecht, and E. Fromager, to be submitted, (2016).



FIG. 6. LIM and GIC-LIM $1^1S \rightarrow 2^1S$ excitation energies obtained in He with and without extrapolation corrections when varying μ . Comparison is made with the FCI. See text for further details.

Conclusions and outlook

- Range-separated eDFT provides a rigorous framework for combining state-averaged multiconfigurational methods with DFT.
- Self-consistent implementation at the long-range FCI level in the DALTON program package.
- Long-term project: use state-averaged CASSCF rather than FCI → state-averaged CASDFT method !
- A linear interpolation method (LIM) for computing excitation energies in KS-eDFT has been proposed. The key idea is to use total ensemble energies rather than orbital energies.
- LIM is also applicable to range-separated eDFT.

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- A ghost-interaction correction (GIC) has been proposed in the context of range-separated eDFT.
- Very promising results have been obtained when combining GIC with LIM, even at the KS-eDFT level (μ = 0 limit).
- When range separation is used, the accuracy of GIC-LIM can be further improved by means of an extrapolation correction.
- LIM can be extended to higher excitations (linear interpolations between equiensembles up the multiplet of interest)
- We currently apply range-separated eDFT to the modeling of conical intersections.

Collaborators

- Hans Jørgen Aa. Jensen (*Odense, Denmark*)
- Stefan Knecht (*ETH*, *Zürich*, *Switzerland*)
- Bruno Senjean (*PhD student, Strasbourg, France*)
- Md. Mehboob Alam (*post-doc*, *Strasbourg*, *France*)
- Erik D. Hedegård (ETH, Zürich, Switzerland)





• LabEx "Chimie des Systèmes Complexes"

project title: "Multi-configuration density-functional theories for excited states"



- ANR jeune chercheur (**MCFUNEX** project)
- École doctorale des sciences chimiques de Strasbourg

2nd International summer School in electronic structure Theory: electron correlation in Physics and Chemistry, *Aussois (Savoie, France), June 18th -July 1st, 2017.*



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