ON THE BULK VISCOSITY IN THE COSMIC FLUID

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$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - 3H\zeta)h_{\mu\nu}$$

$$-2\eta\sigma_{\mu\nu} + Q_\mu U_\nu + Q_\nu U_\mu$$

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$$

Scalar expansion $\theta = 3H = U^\mu_{;\mu}$.

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - Hh_{\mu\nu}$$

$$\theta_{\mu\nu} = h^\alpha_\mu h^\beta_\nu U_{(\alpha;\beta)}$$

$$Q^\mu = -\kappa h^{\mu\nu} T_{,\nu}$$

Expansion tensor $\theta_{\mu\nu}$, shear tensor $\sigma_{\mu\nu}$.

If $Q_\mu = 0$, Shear viscosity $\eta = 0$:

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - 3H\zeta)h_{\mu\nu}$$

FRW metric:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$$

$$\Lambda = 0, \quad k = 0$$

Equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

Assume a simple equation of state

$$p = w\rho, \quad w = \mathrm{constant} \equiv -1 + \alpha$$

From 2015 Planck data: $w = -1.019^{+0.075}_{-0.080}$, $\Rightarrow$

$$\alpha_{\mathrm{min}} = -0.099, \quad \alpha_{\mathrm{max}} = +0.056$$

Quintessence region: $-1 < w < -1/3$,
Phantom region: $w < -1$

If the fluid starts out from the phantom region at $t = 0$, it will encounter a Big Rip when

$$t_s = \frac{2}{|\alpha|\sqrt{24\pi G\rho_0}}$$

where H_0 is the Hubble parameter at present time.

Softer variants of the future singularity:
Little Rip,
Pseudo Rip,
Quasi Rip.
(Caldwell, Nojiri, Odintsov, Frampton, Brevik, Wei...)
For a multicomponent fluid:

$$\rho = \sum_i \rho_i$$

ΛCDM model:

$$\Omega_{0m} + \Omega_\Lambda + \Omega_{0K} = 1,$$

where

$$\Omega_\Lambda = 0.6911 \pm 0.0062, \quad \Omega_{0m} = 0.3089 \pm 0.0062, \quad |\Omega_{0K}| < 0.005$$

Ansatz for the bulk viscosity:

$$\zeta(\rho) = \zeta_0 \left(\frac{H}{H_0}\right)^{2\lambda} = \zeta_0 \left(\frac{\rho}{\rho_0}\right)^{\lambda},$$

where $\lambda =$constant ≥ 0 and ζ_0 is present viscosity.
Most popular:

$$\lambda = \frac{1}{2},$$

$\Rightarrow$

$$\zeta \propto H \propto \sqrt{\rho},$$

or

$$\lambda = 1,$$

$\Rightarrow$

$$\zeta \propto H^2 \propto \rho.$$

Assume standard FRW metric, assuming $k = 0$,

$$ds^2 = -dt^2 + a^2(t)\, d\mathbf{x}^2. \tag{1}$$

Energy-momentum tensor for the whole fluid:

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - 3H\zeta)h_{\mu\nu},$$

where

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$$

Friedmann equations

$$H^2 = \frac{8}{3}\pi G\rho,$$

$$\dot{H} + \frac{3}{2}H^2 = -4\pi G[p - 3H\zeta(\rho)].$$

Conservation equations for energy and momentum

$$T^{\mu\nu}{}_{;\nu} = 0 \quad \Rightarrow \quad \dot{\rho} + 3H(\rho + p) = 9H^2\zeta \text{ when } \mu = 0.$$

Note: if the conservation equation is imposed for the matter subsystem $i = m$, one gets for $\mu = 0$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 9H^2\zeta_m,$$

with ζ_m referring to the matter. Compare with the balance equations for energy for an interacting system consisting of matter and dark energy:

$$\dot{\rho}_{\rm m} + (\rho_{\rm m} + p_{\rm m})\theta = Q,$$

$$\dot{\rho}_{\rm de} + (\rho_{\rm de} + p_{\rm de})\theta = -Q,$$

Coupling due to viscosity. Viscosity chosen to be a function of the fluid as a whole:

$$\zeta = \zeta(\rho), \quad \text{eventually } \zeta = \zeta(H).$$

Useful definition

$$B \equiv 12\pi G \zeta_0$$

where ζ_0 is the present viscosity.
Relation to physical units:

$$\zeta_0 = B[\text{astro.units}] \times 1.15 \times 10^6 \text{ Pa s}.$$

Energy conservation equation can be written as

$$a\partial_a \rho(a) + 3[\rho(a) + p] = 3\zeta(\rho)\theta$$

We have so far made no assumption about the form of $\rho(a)$. For a general multicomponent fluid one can write $\rho = \sum_i \rho_i$.
If there is no viscosity, (i.e. $\zeta = 0$)

$$p = \sum_i w_i \rho_i, \qquad \textbf{assumption 1}$$

$$\rho_{\rm h}(a) = \sum_i \rho_{{\rm h}i}(a) = \sum_i \rho_{0i} a^{-3(w_i+1)},$$

where ρ_{0i} are the present densities ($a_0 = 1$). Including viscosity the general solution is a sum of a homogeneous and a particular one,

$$\rho(a) = \sum_i \rho_{{\rm h}i}(a) + \rho_{\rm p}(a,\zeta) = \sum_i \rho_{{\rm h}i}(a)\,[1 + u_i(a,\zeta)] = \sum_i \rho_{0i} a^{-3(w_i+1)}\,[1 + u_i(a,\zeta)]$$

$$\rho(a) = \rho_{\mathrm{h}}(a)\,[1 + u(a)]\,, \qquad \textbf{assumption 2}.$$

This assumption simplifies the formalism:

$$\frac{\partial u(a)}{\partial a} = 9\,\frac{\zeta(a)}{a\rho_h(a)}\sqrt{\frac{8\pi G}{3}\rho_h(a)\,[1 + u(a,\zeta)]}.$$

The solution is

$$\boxed{u(z,B,\lambda) = \begin{cases} \left[1 - (1-2\lambda)\,\dfrac{B}{H_0}\displaystyle\int_0^z \frac{1}{(1+z)\sqrt{\Omega}^{1-2\lambda}}\,dz\right]^{\frac{2}{1-2\lambda}} - 1 & \text{for} \lambda \neq \frac{1}{2}, \\ (1+z)^{-\frac{2B}{H_0}} & \text{for} \lambda = \frac{1}{2}, \end{cases}}$$

the redshift introduced through $a = 1/(1+z)$. The initial condition chosen such that $\rho(z=0, \zeta=0) = \sum_i \rho_{0i}$. Also introducing abbreviations

$$\Omega \equiv \sum_i \Omega_{0i}(1+z)^{3(1+w_i)} \qquad \text{where} \qquad \Omega_{0i} = \frac{\rho_{0i}}{\rho_c} \qquad \text{and} \qquad \rho_c = \frac{3\,H_0^2}{8\pi\,G}$$

and as defined previously; $B = 12\pi G\zeta_0$. Dimensionless Hubble parameter $E(z)$:

$$\boxed{E^2(z) = \Omega\,[1 + u(\lambda, \zeta_0)]}\ .$$

In the case of zero viscosity, the equation reduces to the first Friedmann equation on dimensionless form. The general solution is involved, but can be solved for the cases $\lambda = 1/2$ and $\lambda = 1$.

The experiments on the Hubble parameter go back to redshifts $z \sim 2.3$. This stretches deep into the matter dominated epoch. At redshift $z = 0.25$ dark energy becomes the main constituent. It is natural as a first approach to assume the universe consisting of dust ($w = 0$) and a constant dark energy term ($w = -1$). With $\rho(z) \to \rho_m(z) + \rho_{\mathrm{de}}$, we find

$$E^2(z) = [\Omega_{\mathrm{de}} + \Omega_{0\mathrm{m}}(1+z)^3]\,(1+u)\,,$$

We will not consider a one component fluid, but will assign a bulk viscosity to the fluid as a whole. This gives a natural transition into a one-component phenomenological description of the future cosmic fluid. In the following, we shall implement the three most used cases $\zeta = \mathrm{const}$, $\zeta \propto \sqrt{\rho}$ and $\zeta \propto \rho$ in order to estimate the magnitude of the viscosity ζ_0. Main point here to determine its impact on the future cosmic fluid.

Cosmological Evolution

Cosmic time	*scale factor a*	*Era*	*Redshifts*
$t = 13.8$ Gy	1	Present	0
9.8 Gy$< t <$ 13.8 Gy	$a(t) = e^{H_0 t}$	DE dominance	-
$t = 9.8$ Gy	0.75	onset of DE dominance	0.25
47 ky$< t <$ 9.8 Gy	$a(t) \propto t^{2/3}$	matter dominance	-
$t = 47$ ky	$1.2 \cdot 10^{-4}$	onset of matter dominance	3400
$t <$ 47 ky	$a(t) \propto t^{1/2}$	radiation dominance	-
$t = 10^{-10}$ s	$1.7 \cdot 10^{-15}$	electroweak phase transition	-
10^{-44}s$< t < 10^{-10}$s	$a(t) \propto t^{1/2}$	Possible inflation or bounce	-
$t < 10^{-44}$	$1.7 \cdot 10^{-32}$	Planck time	$5.9 \cdot 10^{31}$

Solving the integral for $u(z)$ in the three different cases, one gets

$$E(z) = \begin{cases} \sqrt{\Omega(z)}\left[1 - \frac{2B}{3H_0\sqrt{\Omega_{\rm de}}} arctanh\left(\sqrt{\frac{\Omega(z)}{\Omega_{\rm de}}}\right) + I_0\right] \ \textbf{when}\ \zeta = \text{const.}, \\ \sqrt{\Omega(z)}(1+z)^{-\frac{B}{H_0}} \ \textbf{when}\ \zeta = \zeta_0\left(\frac{\rho}{\rho_0}\right)^{1/2}, \\ \frac{\sqrt{\Omega}}{\sqrt{1 + \frac{2B}{3H_0}\left[\sqrt{\Omega}\left(1 - \frac{\sqrt{\Omega_{\rm de}}}{\sqrt{\Omega}} arctanh\sqrt{1 + \frac{\Omega_{\rm 0m}}{\Omega_{\rm de}}(1+z)^3}\right)\right] + C}} \ \textbf{when}\ \zeta \propto \rho. \end{cases}$$

where we have rewritten the expressions in terms of relative densities. The integration constants are determined by the initial condition $E(z=0) = \Omega(z=0) \equiv \Omega_0 = 1$.
Minimizes

$$\chi^2_{\mathrm{H}}(H_0, \zeta) = \sum_{i=1}^{N} \frac{\left[H^{\mathrm{th}}(z_i; H_0, \zeta) - H^{\mathrm{obs}}(z_i)\right]^2}{\sigma^2_{\mathrm{H},i}}$$

through a non-linear least square procedure. Here N is the number of data points, $H^{\mathrm{th}}(z_i)$ is the theoretical Hubble parameter value at redshift z_i, $H^{\mathrm{obs}}(z_i)$ is the observed value at redshift z_i and $\sigma^2_{\mathrm{H},i}$ is the variance in observation i.

$$H_0 = 67.74 \ \mathrm{km\,s^{-1}\,Mpc^{-1}} \quad , \quad \Omega_{0\mathrm{m}} = 0.3089, \quad \Omega_{\mathrm{de}} = 0.6911$$

Summary of Model fitting

Model for ζ	Adjusted R^2 $[-]$	Fit-value for B $(\mathrm{km\ s^{-1}\ Mpc^{-1}})$	95%CI
$\zeta =$const.	0.9601	0.6873	(-2.788, 4.163)
$\zeta \propto \rho^{1/2} \propto H$	0.9604	0.7547	(-1.706, 3.215)
$\zeta \propto \rho \propto H^2$	0.9609	0.5906	(-0.8498, 2.031)

Table: Results of the different models.

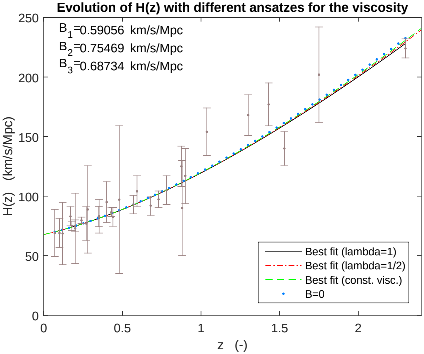
On the value of ζ_0

From experiments (Wang and Meng 2014, Velten and Schwarz 2012, Sasidharan and Mathew 2015)

$$10^4 \text{ Pa s} < \zeta_0 < 10^6 \text{ Pa s}$$

As the viscosity coefficients appear in connection with first order modification to thermodynamical equilibrium, one expects that the pressure modification caused by the bulk viscosity should be much smaller than the equilibrium pressure. With critical density ($\rho_c \sim 10^{-26}$ kg m^{-3}) as a measure of the present day energy in the universe, and with w to be of order unity, this leads to

$$|p| = |w\rho| \gg |H\zeta_0| \qquad \rightarrow \qquad |\zeta_0| \ll 10^8 \text{Pa s}$$


Evolution of H(z) with different ansatzes for the viscosity
B_1=0.59056 km/s/Mpc
B_2=0.75469 km/s/Mpc
B_3=0.68734 km/s/Mpc
Best fit (lambda=1)
Best fit (lambda=1/2)
Best fit (const. visc.)
B=0
H(z) (km/s/Mpc)
z (-)
0
50
100
150
200
250
0
0.5
1
1.5
2

LATE UNIVERSE: CALCULATION OF THE RIP TIME

Let $t = 0$ refer to the present time, and let $\zeta = \zeta(\rho)$ refer to the cosmological fluid as a whole. With $k = 0$ and $\Lambda = 0$ we obtain

$$\dot{H} + \frac{3}{2}\alpha H^2 - 12\pi G\zeta(\rho) = 0,$$

which can be rewritten in terms of the density as

$$\dot{\rho} + \sqrt{24\pi G}\,\alpha\rho^{3/2} - 24\pi G\zeta(\rho)\rho = 0.$$

The solution is

$$t = \frac{1}{\sqrt{24\pi G}} \int_{\rho_0}^{\rho} \frac{d\rho}{\rho^{3/2}\left[-\alpha + \sqrt{24\pi G}\,\zeta(\rho)/\sqrt{\rho}\right]}.$$

We will consider two models for the bulk viscosity:

Case 1: $\zeta =$ constant. Assume ζ equal to its present value,

$$\zeta = \zeta_0 = 10^5 \ \mathrm{Pa\ s}.$$

It corresponds to the viscosity time

$$t_\mathrm{c} = \frac{c^2}{12\pi G\zeta_0} = 3.58 \times 10^{20} \ \mathrm{s}.$$

The rip time is in this case

$$t_\mathrm{s} = t_\mathrm{c} \ln\left(1 + \frac{2}{|\alpha|\theta_0 t_\mathrm{c}}\right),$$

($\alpha < 0$ must be negative to lead to a big rip). Choose

$$\alpha = -0.05.$$

With $\theta_0 = 6.60 \times 10^{-18}$ s^{-1}:

$$t_\mathrm{s} = 6.00 \times 10^{18} \ \mathrm{s} = 190 \ \mathrm{Gy},$$

which is much larger than the age 13.8 Gy of our present universe.

Case 2: $\zeta \propto \sqrt{\rho}$. Take

$$\zeta(\rho) = \zeta_0 \sqrt{\tilde{\rho}}, \quad \tilde{\rho} = \rho/\rho_0,$$

with ζ_0 as above. Then

$$t = \frac{1}{\sqrt{24\pi G}} \, \frac{2}{-\alpha + \zeta_0 \sqrt{24\pi G/\rho_0}} \left(\frac{1}{\sqrt{\rho_0}} - \frac{1}{\sqrt{\rho}} \right).$$

Remarkable property: it permits a big rip singularity even if the fluid is initially in the quintessence region $\alpha > 0$. Condition only that

$$-\alpha + \zeta_0 \sqrt{24\pi G/\rho_0} > 0.$$

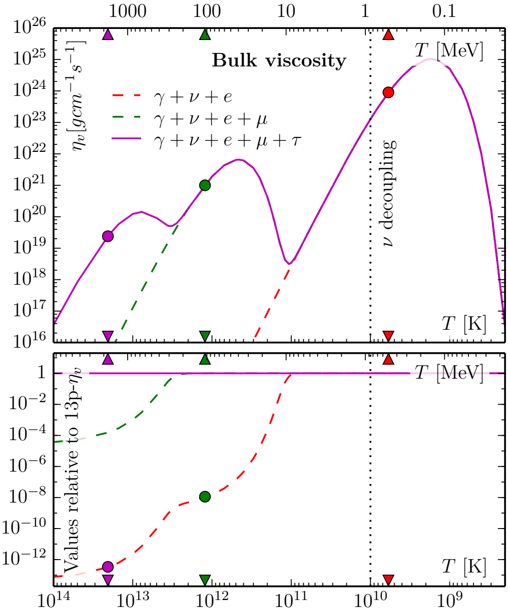
If this condition holds, the universe runs into a singularity ($\rho = \infty$) at a finite rip time

$$t_{\mathrm{s}} = \frac{1}{\sqrt{24\pi G \rho_0}} \, \frac{2}{-\alpha + (\zeta_0/c^2)\sqrt{24\pi G/\rho_0}}.$$

Identifying ρ_0 with the critical energy density $\rho_c = 2 \times 10^{-26}$ kg/m^3 (assuming h parameter equal to 0.7), we can write the rip time in the form

$$t_{\rm s} = \frac{2}{-\alpha + 0.0056} \times 10^{17}\ \mathrm{s},$$

showing the delicate dependence upon α. If the universe starts from the quintessence region, it may run into the big rip if $\alpha < 0.0056$, thus very small. If the universe starts from the phantom region, it will always encounter the singularity. In the special case when $\alpha = 0$ we obtain $t_{\rm s} = 3.6 \times 10^{19}$ s, greater than the previous expression for the constant viscosity case. If $\alpha = -0.05$ we find $t_{\rm s} = 3.59 \times 10^{18}$ s=114 Gy.



Bulk viscosity
$\gamma+\nu+e$
$\gamma+\nu+e+\mu$
$\gamma+\nu+e+\mu+\tau$
$\eta_v [gcm^{-1}s^{-1}]$
ν decoupling
T [MeV]
T [K]
1000
100
10
1
0.1
10^{26}
10^{25}
10^{24}
10^{23}
10^{22}
10^{21}
10^{20}
10^{19}
10^{18}
10^{17}
10^{16}
Values relative to 13p-η_v
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{14}
10^{13}
10^{12}
10^{11}
10^{10}
10^{9}

CONCLUSION

• Assumptions:
Viscous, isotropic Friedmann universe with $k = 0$.
Equation of state $p = \sum_i w_i \rho_i$, with $w_i =$ constant for all components in the fluid.
Three options:
(i) $\zeta =$const, (ii) $\zeta \propto \sqrt{\rho}$, and (iii) $\zeta \propto \rho$.
• Differences between the predictions from various viscosity models are small. Even the ansatz $\zeta =$constant reproduces experimental data quite well. These models underpredict $H(z)$ for large redshifts. In the literature, the ansatz $\zeta \propto \sqrt{\rho}$, is widely accepted.
• Present bulk viscosity ζ_0 lies within an interval, from 10^4 to 10^6 Pa s,
• Future universe:choosing $\zeta_0 = 10^5$ Pa s. Big rip singularity in the far future. Rip time t_s. With α defined as $\alpha = w + 1$, even the case $\zeta = \zeta_0 =$const allows big rip to occur, if α is negative, i.e., lying in the phantom region. Of special interest is the case $\zeta \propto \sqrt{\rho}$: the fate of the universe is critically dependent on the magnitude of α. If $\alpha < 0$, the big rip is inevitable. If $\alpha > 0$ (the quintessence region), the big rip can also occur if α is very small, less than about 0.005. This possibility of sliding through the phantom divide was actually pointed out several years ago (Brevik and Gorbunova 2005), but can now be better quantified. Typical rip times are found to lie roughly in the interval from 100 to 200 Gy.