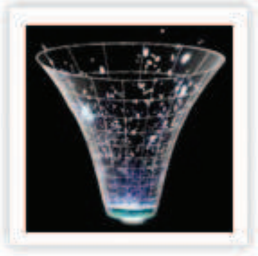
SIU Southern Illinois University CARBONDALE

NTNU Norwegian University of Science and Technology

CUD Zaragoza

Casimir Energies of fractal configurations

K. V. Shajesh, I. Brevik, P. Parashar and I. Cavero-Peláez



Cosmology and the Quantum Vacuum

Benasque, September 4th - 10th, 2016

Talk given by ICP

Fractal$\longleftrightarrow$ Self-similarity

We understand by fractal a geometrical figure, in which similar patterns recur at progressively smaller and/or bigger scales.

We show two/three self-similar configurations,

- δ-function plates positioned at points given by the series

$$\sum_{n=0}^{\infty} \frac{a}{2^n} \qquad \text{and} \qquad \sum_{n=0}^{\infty} 2^n a$$

- δ-function plates located at points of a Cantor set,

and calculate the Casimir energy in two independent ways.

K. V. Shajesh, I. Brevik, I. Cavero-Peláez and P. Parashar, "Self-similar plates: Casimir energies," Accepted to be published in Phys. Rev. D (PRD **94** XXXXX (2016)), arXiv:1607.00214 [hep-th].

Self-similar series

Example 1

$$x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots.$$

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As before:

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Notice that the sum is a negative number.

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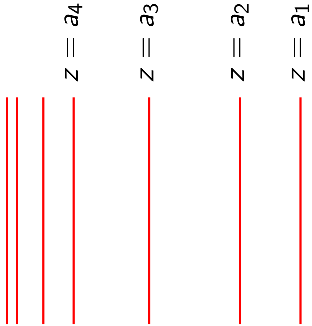
Example 3

$$x = 1 + 1 + 1 + 1 + \cdots \qquad \boxed{x = \zeta(0) = -\frac{1}{2}}$$

General sequence of δ-function plates



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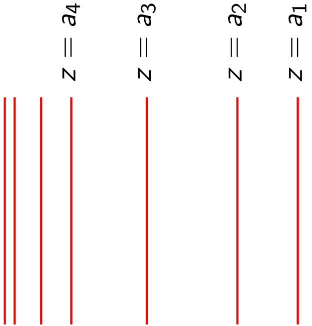

- Sequence of infinitely thin plates
- Interactions $\leftrightarrow$ scalar quantum field
- Potentials

$$V_i(\mathbf{x}) = \lambda_i \delta(z - a_i).$$

- Total energy per unit area:

$$\mathcal{E} = \mathcal{E}_0 + \sum_{i=1}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a)$$

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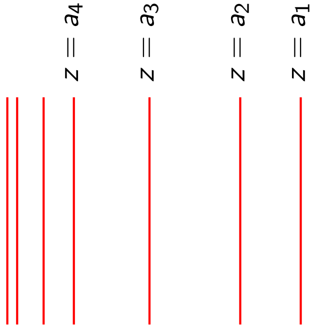

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- Total energy per unit area:

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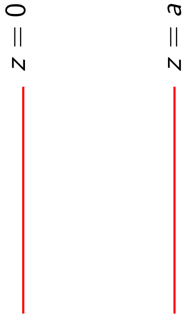
- $\mathcal{E}_0$. Energy of the vacuum (diverges)
- $\Delta\mathcal{E}_i = \mathcal{E}_i - \mathcal{E}_0$. Energies associated to the individual objects (diverges)
- $\Delta\mathcal{E}(a)$. Interaction energy per unit area of the entire set of plates (finite and dependent on the distance)

Example: Two δ-function plates



$$\mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}_1 + \Delta\mathcal{E}_2 + \Delta\mathcal{E}_{12}(a)$$

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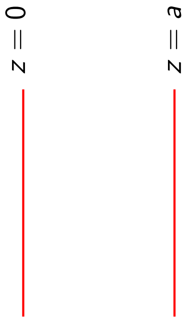


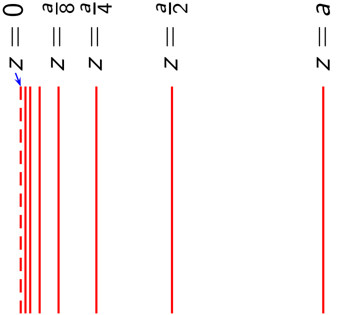
$$\mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}_1 + \Delta\mathcal{E}_2 + \Delta\mathcal{E}_{12}(a)$$

- The first three terms on the right diverge.
- If we impose Dirichlet boundary conditions, the interaction energy between the plates is

$$\Delta\mathcal{E}_{12}(a) = -\frac{\pi^2}{1440a^3}.$$

Self-similar δ-function plates at points $\frac{a}{2^i}$

Let's consider a geometric sequence of parallel plates $\frac{a}{2^i}$



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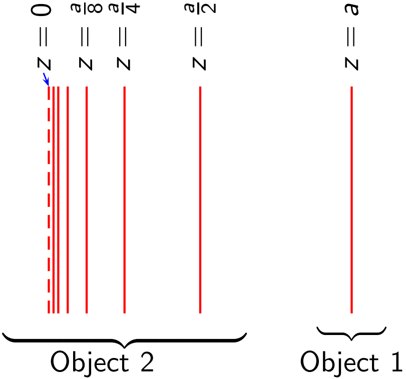

$$\mathcal{E}_0 + \sum_{i=1}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a)$$

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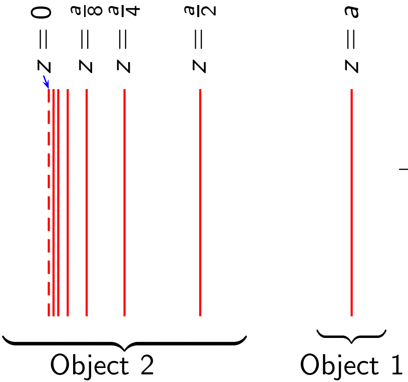

$$\mathcal{E}_0 + \sum_{i=1}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a) = \mathcal{E}_0 + \Delta\mathcal{E}_1$$

$$+ \left(\sum_{i=2}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a/2) \right) + \Delta\mathcal{E}_{12}(a),$$

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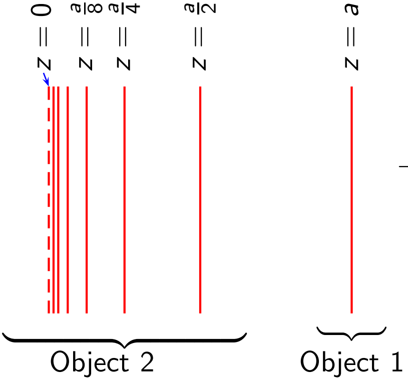

$$\mathcal{E}_0 + \sum_{i=1}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a) = \mathcal{E}_0 + \Delta\mathcal{E}_1$$

$$+ \left(\sum_{i=2}^{\infty} \Delta\mathcal{E}_i + \Delta\mathcal{E}(a/2) \right) + \Delta\mathcal{E}_{12}(a),$$

$$\Delta\mathcal{E}(a) = \Delta\mathcal{E}(a/2) + \Delta\mathcal{E}_{12}(a),$$

$\Delta\mathcal{E}(a) \rightarrow$ Total interaction energy
$\Delta\mathcal{E}(a/2) \rightarrow$ Interaction energy of the plates in Object 2
$\Delta\mathcal{E}_{12}(a) \rightarrow$ Interaction energy between objects 1 and 2

Self-similar δ-function plates. Dirichlet b.c.

The interaction energy is a function of only a:

$$\Delta\mathcal{E}(a) = \Delta\mathcal{E}(a/2) + \Delta\mathcal{E}_{12}(a),$$

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In 3+1 D the dimension of the energy per unit area is $[L]^{-3}$ allowing us to write

$$\Delta\mathcal{E}(a/2) = 2^3\,\Delta\mathcal{E}(a).$$

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The interaction energy is a function of only *a*:

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Taking this scaling into account, we get an expression similar in nature to the self-similar series on the first slide,

$$\boxed{\Delta\mathcal{E}(a) = 8\,\Delta\mathcal{E}(a) + \Delta\mathcal{E}_{12}(a)}$$

$$\Delta\mathcal{E}(a) = -\frac{1}{7}\Delta\mathcal{E}_{12}(a)$$

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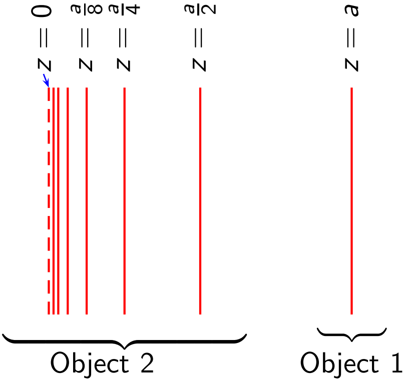

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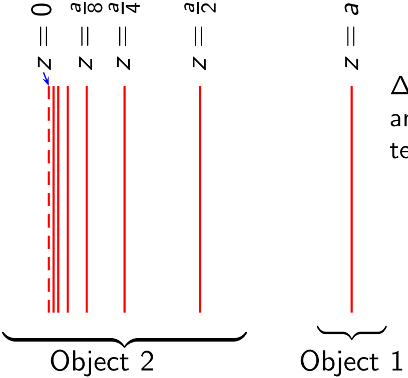

$$\Delta\mathcal{E}(a) = -\frac{1}{7}\Delta\mathcal{E}_{12}(a)$$

$\Delta\mathcal{E}_{12}(a)$ Interaction between Object 1 and Object 2, in general not easy to determine.

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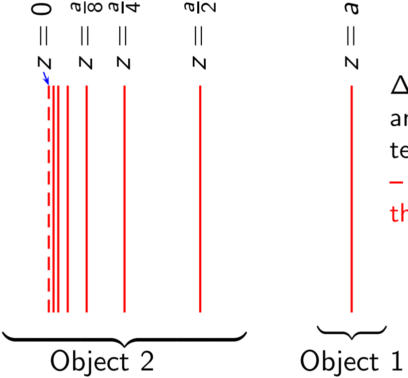

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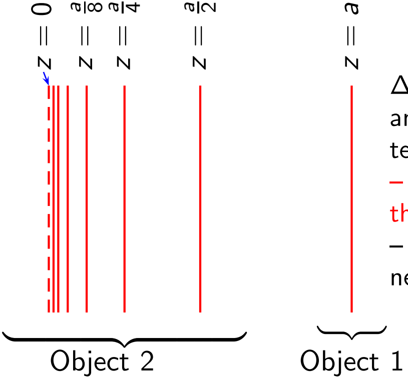

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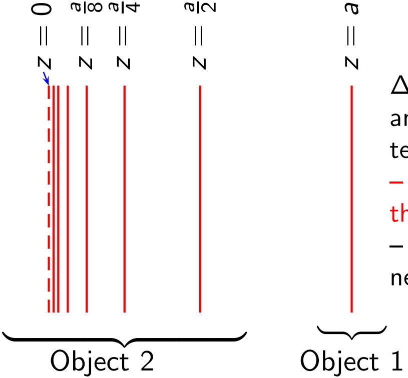
– Dirichlet boundary conditions between the plates is assumed.

– Then, each plate interacts only with its neighbors, and the interaction energy is

$$\Delta\mathcal{E}_{ij}(L) = -\frac{\pi^2}{1440L^3}$$

Self-similar δ-function plates at points $\frac{a}{2^i}$



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$\Delta\mathcal{E}_{12}(a)$ is the interaction energy between plates at $z = \frac{a}{2}$ and $z = a$,

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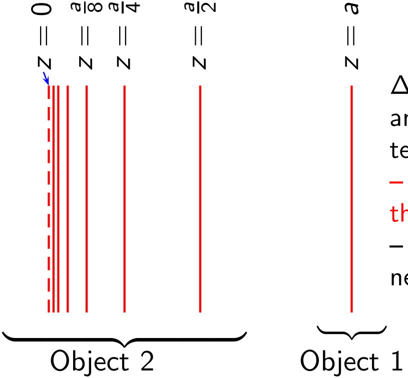

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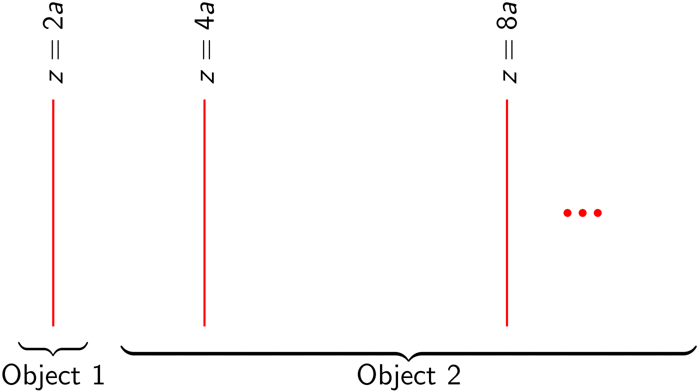
Casimir interaction energy per unit area for our set of plates

$$\boxed{\Delta\mathcal{E}(a) = +\frac{8}{7}\frac{\pi^2}{1440 a^3}}$$

(positive sign)

Self-similar δ-plates at points $2^i a$



$$\Delta\mathcal{E}(2a) = \Delta\mathcal{E}(4a) - \frac{\pi^2}{1440(2a)^3}$$

Where we have assumed Dirichlet boundary conditions.

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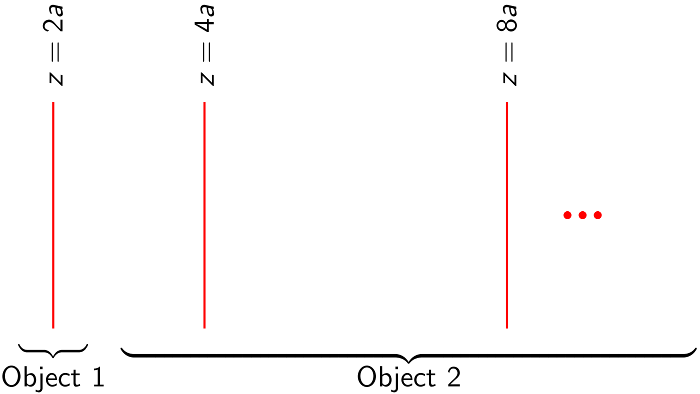

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Dimensional arguments lead us to, $\Delta\mathcal{E}(4a) = \frac{1}{2^3}\Delta\mathcal{E}(2a)$.

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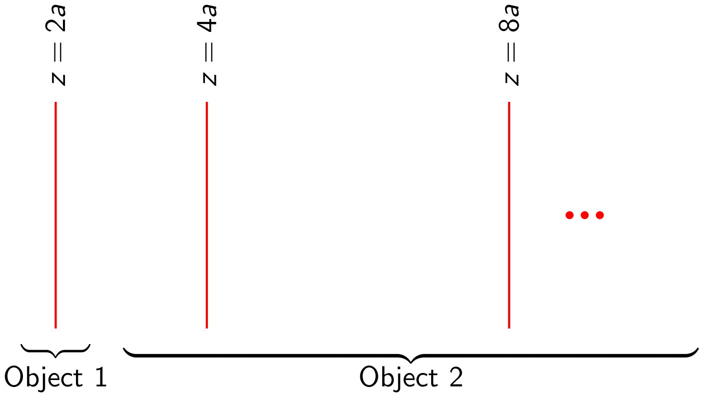

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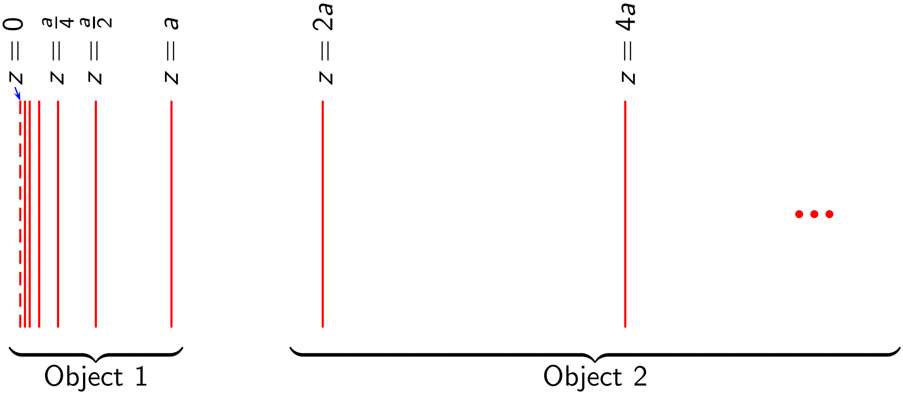
We immediately learn that

$$\boxed{\Delta\mathcal{E}(2a) = -\frac{1}{7}\frac{\pi^2}{1440a^3}} \qquad \text{(negative sign)}$$

Both sequences of plates

We consider now plates at positions $\frac{a}{2^i}$ and $2^i a$ which extend now to the whole space,



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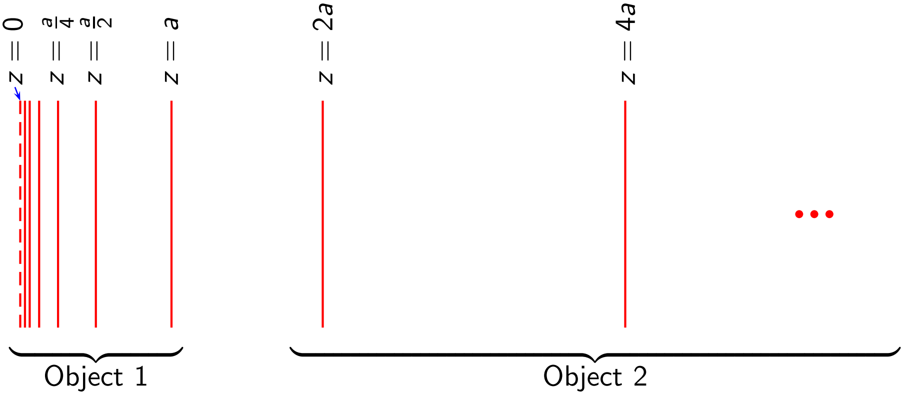

$$\Delta\mathcal{E}_{total}(a) = \Delta\mathcal{E}_{object1} + \Delta\mathcal{E}_{object2} + \Delta\mathcal{E}_{12}$$

Both sequences of plates

The interaction energy between the two objects defined above is

$$\Delta\mathcal{E}_{total}(a) = \Delta\mathcal{E}_{object1} + \Delta\mathcal{E}_{object2} + \Delta\mathcal{E}_{12}.$$

We have already calculated every term using Dirichlet b.c.,

$$\begin{aligned}
\Delta\mathcal{E}_{object1} &= \Delta\mathcal{E}(a) = +\frac{8}{7}\frac{\pi^2}{1440a^3} \\
\Delta\mathcal{E}_{object2} &= \Delta\mathcal{E}(2a) = -\frac{1}{7}\frac{\pi^2}{1440a^3} \\
\Delta\mathcal{E}_{12}(a) &= -\frac{\pi^2}{1440a^3}
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\end{aligned}$$

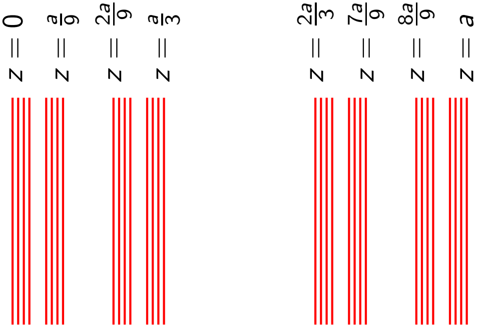
Put together we find,

$$\boxed{\Delta\mathcal{E}_{\text{tot}}(a) = +\frac{8}{7}\frac{\pi^2}{1440a^3} - \frac{1}{7}\frac{\pi^2}{1440a^3} - \frac{\pi^2}{1440a^3} = 0}$$

Both stacks of plates balance each other with opposite tendences so that together they contribute to cero.

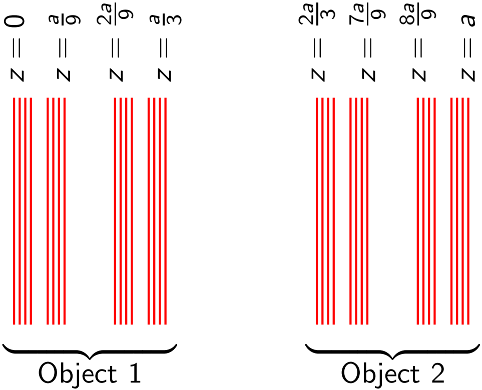
Cantor set of parallel δ-function plates

Divide iteratively a line segment into three parts and delete the central one each time.



Cantor set of parallel δ-function plates

Divide iteratively a line segment into three parts and delete the central one each time.



$$\Delta\mathcal{E}(a) = \Delta\mathcal{E}(a/3) + \Delta\mathcal{E}(a/3) - \frac{\pi^2}{1440(a/3)^3}$$

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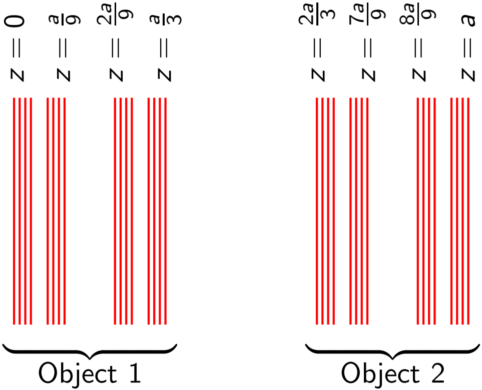

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Then using

$$\Delta\mathcal{E}(a/3) = 3^3\,\Delta\mathcal{E}(a)$$

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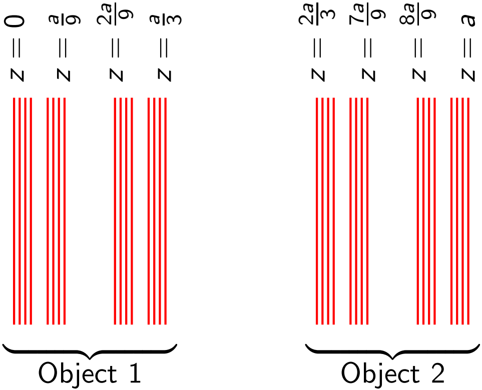

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Then using

$$\Delta\mathcal{E}(a/3) = 3^3\,\Delta\mathcal{E}(a)$$

we have the Casimir interaction for the configuration given by

$$\boxed{\Delta\mathcal{E}(a) = +\frac{27}{53}\frac{\pi^2}{1440a^3}}$$

Positive $\rightarrow$ the vacuum pressure tends to inflate the set of plates.

Green's functions

Let's consider N parallel δ-function plates located at points a_i,

$$\left[-\frac{d^2}{dz^2}+\kappa^2+\sum_{i=1}^{N}\lambda_i\delta(z-a_i)\right]g_{1\dots N}(z,z')=\delta(z-z'),$$

where

- $\kappa^2=k_\perp^2-\omega^2$,
- $g_{1\dots N}(z,z')$ is the reduced Green function
- there is symmetry in the transverse components.

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- $g_{1\ldots N}(z,z')$ is the reduced Green function
- there is symmetry in the transverse components.

In the absence of plates, the free Green function satisfies

$$\left(-\frac{d^2}{dz^2}+\kappa^2\right)g_0(z-z')=\delta(z-z'),$$

and has the solution

$$g_0(z-z')=\frac{1}{2\kappa}e^{-\kappa|z-z'|}.$$

Green's function

The solution $g_{1\dots N}(z, z')$ to our problem is given in terms of the free Green's functions and can be written using the ansatz,

$$\boxed{g_{1\dots N}(z, z') = g_0(z, z') - \mathbf{r}(z) \cdot \mathbf{t}_{1\dots N} \cdot \mathbf{r}(z')}$$

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where

$$\mathbf{r}(z) \cdot \mathbf{t}_{1\ldots N} \cdot \mathbf{r}(z') = r_i(z)\, t^{ij}_{1\ldots N}\, r_j(z'),$$

- Vector $\mathbf{r}(z)$ with components $r_i(z) = g_0(z - a_i)$.
- Tensor $\mathbf{t}_{1\ldots N}$ obeying the Faddeev equation

$$\mathbf{t}_{1\ldots N} = (\mathbf{1} + \boldsymbol{\lambda} \cdot \mathbf{R})^{-1} \cdot \boldsymbol{\lambda}$$

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$$\boxed{g_{1\dots N}(z,z') = g_0(z,z') - \mathbf{r}(z)\cdot \mathbf{t}_{1\dots N}\cdot \mathbf{r}(z')}$$

where

$$\mathbf{r}(z)\cdot \mathbf{t}_{1\dots N}\cdot \mathbf{r}(z') = r_i(z)\, t^{ij}_{1\dots N}\, r_j(z'),$$

- Vector $\mathbf{r}(z)$ with components $r_i(z) = g_0(z-a_i)$.
- Tensor $\mathbf{t}_{1\dots N}$ obeying the Faddeev equation

$$\mathbf{t}_{1\dots N} = (\mathbf{1} + \boldsymbol{\lambda}\cdot\mathbf{R})^{-1}\cdot\boldsymbol{\lambda}$$

where $\mathbf{1}$ is the identity matrix, and $\boldsymbol{\lambda}$ and $\mathbf{R}$ are given by

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & \lambda_N \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} g_0(0) & g_0(a_1-a_2) & \dots & g_0(a_1-a_N) \\ g_0(a_2-a_1) & g_0(0) & \dots & g_0(a_2-a_N) \\ \vdots & \vdots & \ddots & \vdots \\ g_0(a_N-a_1) & g_0(a_N-a_2) & \dots & g_0(0) \end{bmatrix}$$

Nature of the resulting GF

Let's look at the behavior of the resulting Green's functions,

$$g_{1\ldots N}(z,z') = g_0(z,z') - \mathbf{r}(z)\cdot\mathbf{t}_{1\ldots N}\cdot\mathbf{r}(z') = g_0(z,z') - \text{Tr}\Big[\,\mathbf{t}_{1\ldots N}\cdot\mathbf{r}(z')\,\mathbf{r}(z)\Big]$$

$$\left(\tilde{\boldsymbol{\lambda}} = \tfrac{\boldsymbol{\lambda}}{2\kappa}, \quad \tilde{\mathbf{t}}_{1\ldots N} = \tfrac{\mathbf{t}_{1\ldots N}}{2\kappa}, \quad \text{and} \quad \tilde{\mathbf{R}} = 2\kappa\mathbf{R}\right)$$

$\boxed{N=1}$ A single plate

$$\tilde{t}_1 = \frac{\tilde{\lambda}_1}{1+\tilde{\lambda}_1} \rightarrow g_1(z,z') = \frac{1}{2\kappa}e^{-\kappa|z-z'|} - \frac{t_1}{2\kappa}e^{-\kappa|z-a_1|}e^{-\kappa|z'-a_1|}.$$

$\boxed{N=2}$ Two plates

$$\begin{aligned} g_{12}(z,z') \quad &= \quad \frac{1}{2\kappa}e^{-\kappa|z-z'|} - \frac{1}{2\kappa}\frac{1}{\Delta_{12}} \times \\ &\text{Tr}\begin{bmatrix} \tilde{t}_1 & -\tilde{t}_1\tilde{R}_{12}\tilde{t}_2 \\ -\tilde{t}_2\tilde{R}_{21}\tilde{t}_1 & \tilde{t}_2 \end{bmatrix}\begin{bmatrix} e^{-\kappa|z-a_1|}e^{-\kappa|z'-a_1|} & e^{-\kappa|z-a_1|}e^{-\kappa|z'-a_2|} \\ e^{-\kappa|z-a_2|}e^{-\kappa|z'-a_1|} & e^{-\kappa|z-a_2|}e^{-\kappa|z'-a_2|} \end{bmatrix}\end{aligned}$$

where $\Delta_{12} = 1 - \tilde{t}_1\tilde{R}_{12}\tilde{t}_2\tilde{R}_{21}$.

Recursion relation

N plates

$$\tilde{\mathbf{t}}_{12\ldots N} = \frac{1}{\Delta_{12\ldots N}} \times$$

$$\begin{bmatrix} \tilde{t}_1\Delta_{23\ldots N} & -\tilde{t}_1\tilde{R}_{1[34\ldots N]2}\tilde{t}_2\Delta_{34\ldots N} & \cdots & -\tilde{t}_1\tilde{R}_{1[23\ldots N-1]N}\tilde{t}_N\Delta_{23\ldots N-1} \\ -\tilde{t}_2\tilde{R}_{2[34\ldots N]1}\tilde{t}_1\Delta_{34\ldots N} & \tilde{t}_2\Delta_{134\ldots N} & \cdots & -\tilde{t}_2\tilde{R}_{2[13\ldots N-1]N}\tilde{t}_N\Delta_{13\ldots N-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{t}_N\tilde{R}_{N[23\ldots N-1]1}\tilde{t}_1\Delta_{23\ldots N-1} & -\tilde{t}_N\tilde{R}_{N[13\ldots N-1]2}\tilde{t}_2\Delta_{13\ldots N-1} & \cdots & \tilde{t}_N\Delta_{12\ldots N-1} \end{bmatrix},$$

We have used the notation $R_{i[k]j} = g_k(a_i, a_j)$, $R_{i[mn]j} = g_{mn}(a_i, a_j)$ and so on.

- This can then be extended for the $N \to \infty$ case.
- The Green's function for N plates is given in terms of all possible Green's function for $(N-2)$ plates, obtained by deleting two plates.
- In this sense we have a recursion relation for the Green's function.

Casimir energy

In terms of the Greens function, the Casimir energy per unit area for N parallel δ-function plates is

$$\mathcal{E}_{1\dots N} = -\frac{1}{6\pi^2}\int_0^\infty \kappa^4 d\kappa \int_{-\infty}^\infty dz\, g_{1\dots N}(z,z).$$

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$$\mathcal{E}_{1\dots N} = -\frac{1}{6\pi^2}\int_0^\infty \kappa^4 d\kappa \int_{-\infty}^{\infty} dz\, g_0(z,z) + \frac{1}{6\pi^2}\int_0^\infty \kappa^4 d\kappa \,\mathrm{Tr}\, \mathbf{t}_{1\dots N} \cdot \int_{-\infty}^{\infty} dz\, \mathbf{r}(z)\mathbf{r}(z)^T$$

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We remind ourselves that we are looking for

$$\Delta\mathcal{E}_{1\ldots N} = \mathcal{E}_{1\ldots N} - \mathcal{E}_0 - \sum_{i=1}^{N} \Delta\mathcal{E}_i,$$

Greens function. Dirichlet b. c.

The δ-function plates satisfying Dirichlet b. c. are described by $\boxed{\lambda_i \to \infty}$

Then, the transition matrix $\mathbf{t}_{1\ldots N} = (\mathbf{1} + \boldsymbol{\lambda} \cdot \mathbf{R})^{-1} \cdot \boldsymbol{\lambda}$ in this limit becomes

$$\mathbf{t}_{1\ldots N} = \mathbf{R}^{-1},$$

$$\mathbf{t}_{1\ldots N} = \kappa \begin{bmatrix} D_{11} & S_{12} & & & & & \\ S_{21} & D_{22} & S_{23} & & & & \\ & S_{32} & D_{33} & S_{34} & & \mathbf{0} & \\ & & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & & \ddots & D_{N-1,N-1} & S_{N-1,N} \\ & & & & & S_{N,N-1} & D_{N,N} \end{bmatrix},$$

where

$$D_{ii} = \frac{e^{\kappa a_{i-1,i}}}{\sinh \kappa a_{i-1,i}} - 2 + \frac{e^{\kappa a_{i,i+1}}}{\sinh \kappa a_{i,i+1}}, \qquad \text{if} \qquad i \neq 1, i \neq N,$$

and

$$D_{11} = \frac{e^{\kappa a_{12}}}{\sinh \kappa a_{12}}, \quad D_{N,N} = \frac{e^{\kappa a_{N-1,N}}}{\sinh \kappa a_{N-1,N}}, \quad S_{i,i+1} = S_{i+1,i} = -\frac{1}{\sinh \kappa a_{i,i+1}}$$

Casimir Energy for a sequence of N plates

We had found

$$\mathcal{E}_{1\ldots N} = -\frac{1}{6\pi^2}\int_0^\infty \kappa^4 d\kappa \int_{-\infty}^\infty dz\, g_0(z,z) + \frac{1}{6\pi^2}\int_0^\infty \kappa^4 d\kappa \, \mathrm{Tr}\, \mathbf{t}_{1\ldots N} \cdot \int_{-\infty}^\infty dz\, \mathbf{r}(z)\mathbf{r}(z)^T,$$

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Using the $\mathbf{t}_{1\ldots N}$ just calculated and integrating the dyadic,

$$\mathcal{E}_{1\ldots N} = \mathcal{E}_0 + \frac{1}{12\pi^2}\int_0^\infty \kappa^2 d\kappa \left[-(N-2) + \sum_{i=1}^{N-1} \frac{[e^{\kappa a_{i,i+1}} - (1+\kappa a_{i,i+1})e^{-\kappa a_{i,i+1}}]}{\sinh a_{i,i+1}}\right]$$

$(N-2)$ comes from summing the -2's in the diagonal terms of the transition matrix.

Casimir Energy for a sequence of N plates

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$(N-2)$ comes from summing the -2's in the diagonal terms of the transition matrix. This and the a independent terms can be combined as

$$\frac{1}{12\pi^2}\int_0^\infty \kappa^2 d\kappa \Big[-(N-2) + 2(N-1)\Big] = \frac{N}{12\pi^2}\int_0^\infty \kappa^2 d\kappa,$$

which is identified as the sum of single-body (divergent) contributions to the Casimir energy from the N individual plates,

$$\sum_{i=1}^{N} \Delta\mathcal{E}_i = N\Delta\mathcal{E}_1.$$

Interaction Casimir Energy for a sequence of N plates

The remaining term is distance dependent and it is identified with the interaction vacuum energy,

$$\Delta\mathcal{E}_{1\dots N} = -\frac{1}{12\pi^2}\sum_{i=1}^{N-1}\int_0^\infty \kappa^2 d\kappa \frac{\kappa a_{i,i+1} e^{-\kappa a_{i,i+1}}}{\sinh \kappa a_{i,i+1}},$$

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after integration we find the not surprising result,

$$\boxed{\Delta\mathcal{E}_{1\dots N} = -\frac{\pi^2}{1440}\sum_{i=1}^{N-1}\frac{1}{a_{i,i+1}^3}}$$

since a Dirichlet plate physically disconnects the two half-spaces across it.

Interaction Casimir Energy for a sequence of infinity plates

For an infinite sequence of Dirichlet plates we have

$$\boxed{\Delta\mathcal{E}_{1\ldots N} = -\frac{\pi^2}{1440}\sum_{i=1}^{\infty}\frac{1}{a_{i,i+1}^3}}$$

Example 1. Positions of the plates at $a_{i+1} = \frac{a}{2^i}$
We substitute above and find,

$$\Delta\mathcal{E}_{12\ldots} = -\frac{\pi^2}{1440a^3}(8+8^2+8^3+\ldots).$$

Using $x = 8+8x$, with $x = 8+8^2+8^3+\ldots$. We make the formal assignment $x = -\frac{8}{7}$ and get

$$\boxed{\Delta\mathcal{E}_{12\ldots} = \frac{8}{7}\frac{\pi^2}{1440a^3}}$$

exactly as we derived earlier.

Interaction Casimir Energy for a sequence of infinity plates

Example 2. Positions of the plates at $a_i = 2^i a$

$$\Delta\mathcal{E}_{12\ldots} = -\frac{\pi^2}{1440a^3}\left(\frac{1}{8}+\frac{1}{8^2}+\frac{1}{8^3}+\ldots\right).$$

This involves the convergent series $\frac{1}{8}+\frac{1}{8^2}+\frac{1}{8^3}+\cdots=\frac{1}{7}$, which implies that the Casimir interaction energy for this configuration is

$$\boxed{\Delta\mathcal{E}_{12\ldots} = -\frac{1}{7}\frac{\pi^2}{1440a^3}}$$

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Example 3. Positions of the plates at $a_i = i\, a$

$$\Delta\mathcal{E}_{12\ldots} = -\frac{\pi^2}{1440a^3}(1+1+1+\ldots).$$

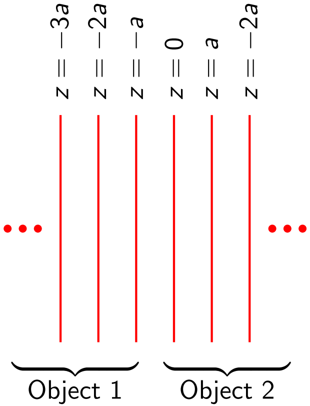
If we now make the formal assignment $1+1+1+\cdots=\zeta(0)=-\frac{1}{2}$, the Casimir interaction energy of this configuration becomes

$$\boxed{\Delta\mathcal{E}_{12\ldots} = \frac{1}{2}\frac{\pi^2}{1440a^3}}$$

The tendency for the plates is to inflate under the pressure of vacuum.

Piecewise uniform string

Let's concentrate on the last example, plates located at points ia but let's make i going from $-\infty$ to ∞.



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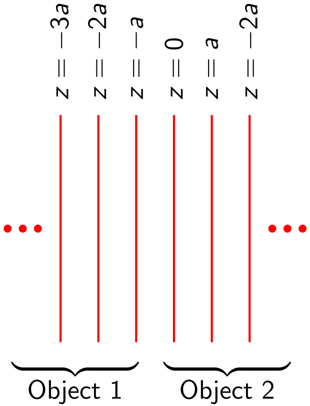

$$\Delta\mathcal{E}_{total} = \Delta\mathcal{E}_{object1} + \Delta\mathcal{E}_{object2} + \Delta\mathcal{E}_{12} = 0$$

since $\Delta\mathcal{E}_{object_i} = \frac{1}{2}\frac{\pi^2}{1440a^3}$.
It extends to the whole space, being equivalent to periodic boundary conditions.

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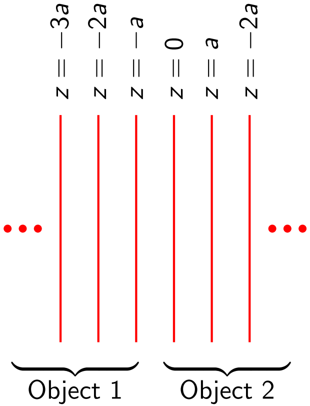

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since $\Delta\mathcal{E}_{object_i} = \frac{1}{2}\frac{\pi^2}{1440a^3}$.
It extends to the whole space, being equivalent to periodic boundary conditions.
There is a formal analogy with the case of a closed cosmic string of length L divided into N identical pieces.
By a completly independent method (and using different regularizations)it is being calculated that for $N = 2$

$$E = -\frac{\pi}{24L}\left(\frac{L_2}{L_1} + \frac{L_1}{L_2} - 2\right) = 0 \qquad (\text{if } L_1 = L_2)$$

Brevik, Elizalde and others have worked on this. The result has been generalized to a string formed of $2N$ pieces.

Conclusions and Outlook

Conclusions

- We have derived the Casimir energies of simple self-similar configurations using the idea of self-similarity alone and corroborated our results for Casimir energies using the completely independent Green's functions formalism.
- We have thus shown that an infinite stack of parallel plates can have positive, negative, or zero Casimir energy.
- In particular, we have successfully derived the Casimir energy of a stack of plates positioned at the points of the Cantor set, thus computing the Casimir energy of a simple fractal for the first time.

Outlook

- Even though the plates arrangement treated are trivial, they open a way to deal with more elaborate fractal configurations. In particular, some fractal structure have fractional dimension. By calculating the vacuum energy we could say something about the dimension of the object checking in this manner Berrys conjecture that the Weyl formula for the asymptotic Casimir energy extends to fractal regions.
- To our knowledge it is the first time a calculation for an infinite stack of plates is done (maybe except for equidistant plates which could be equivalent to periodic bc). Then, we probably could calculate de Casimir energy of a quasi-cristal.

Appreciation to Martin Schaden

We dedicate this work to Martin since the motivation to work on this topic comes from his idea of the "many-body Casimir energies"

K.V. Shajesh worked with him on this subject.

Even though for personal reasons he refused the invitation to collaborate on the presented work, he was completely aware of it.

Unfortunately Martin left us while this work was being reviewed.