The quest for c-functions in three dimensions

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Motivation: Zamolodchikov's c-Theorem in d=2

RG: If the theory, at a given length scale l_0 is characterized by certain dimensionless parameters $g_1(l_0),\ g_2(l_0), \ldots$ (coupling constants,masses,...), then at another scale, l, the physics is the same, but with the dimensionless parameters running according to the FLOW OF THE RG

$$l\,\frac{dg_i(l)}{dl} = -\beta_i(g_1, g_2, \ldots)\, i = 1, 2, \ldots$$

In particular, as $l \to 0$ (UV) and $l \to \infty$ (IR)

$$\{g_1, g_2, \ldots\} \to \{g_1^*, g_2^*, \ldots\} \quad \beta_i(g_1^*, g_2^*, \ldots) = 0$$

Valid for quantum field theories as well as for statistical systems.

RG fixed points correspond to scale-invariant systems; when such theories are, in addition, unitary and Lorentz invariant (rotationally invariant in the statistical case) we have conformal field theories (CFT's). So, all such theories are characterized by their conformal points and the flow of the RG between them.

In d = 2, each CFT is characterised by its conformal anomaly number c (for a free scalar or Dirac fermion, c = 1, but in general c can be non-integer.)

Zamolodchikov's c-Theorem (1986)

There exists a function $C(\{g\})$ on the space of all 2d CFTs such that

- $C(\{g\})$ decreases along RG flows;
- $C(\{g\})$ is stationary at each RG fixed point (CFT). There, its value is the appropriate conformal or trace anomaly c.

In two dimensions there is one anomaly number: the central extension of the Virasoro algebra, which appears in a number of physical quantities,

- The entropy density at finite temperature in flat space, $s = \frac{cT\pi}{3}$,
- The von Neumann entanglement entropy of an interval of length L (PBC's), $S_L = \frac{c}{3} \log L$,
- The trace or conformal anomaly in curved space (R: scalar curvature), $< T^\mu_\mu >= -\frac{cR}{12}$

The last example and the Gauss-Bonnet theorem lead to

$$\int_{\mathcal{M}} d^2x\sqrt{g} < T^\mu_\mu >= -c\frac{\chi}{12}$$

The extension to even dimensions can be sought for by analogy with the last example. Only, in higher even dimensions, there are more anomaly coefficients. Example: a-Theorem in 4-d. Odd dimensions: NO CONFORMAL ANOMALY

The F-theorem

I.Klebanov, S. Pufu and B. Safdi,"F-Theorem without Supersymmetry", JHEP **1110**, 038 (2011)

Conjecture: zeta-regularized Euclidean effective action on S^3 coincides, at the conformal fixed points, with the renormalized entanglement entropy across an S^1 submanifold of $\mathbb{R}^{2,1}$ and, thus, it decreases under the RG flow.

Effective action for free massive scalars:

$$S_{eff}^{S^3} = -\log Z = -\log Det[-\Delta + \frac{3}{4a^2} + m^2]^{-\frac{1}{2}}$$

Zeta regularization:

$$S_{eff}^{S^3} = -\frac{1}{2}\left.\frac{d}{ds}\right|_{s=0} \zeta^{S^3}(s)$$

$$\zeta^{S^3}(s) = \sum_{n=1}^{\infty} n^2 (n^2 - \frac{1}{4} + (ma)^2)^{-s}$$

No need to introduce μ, since $\zeta(s=0) = 0$ in 3-d

a: radius of the sphere. Only dimensionless "coupling constant" is $(ma)^2$

Behavior near the UV fixed point

Binomial expansion, with $\rho^2 = \frac{1}{4} - (ma)^2$ leads to

$$\zeta^{S^3}(s) = \zeta_R(2s-2) + \sum_{j=1}^{\infty} \frac{\Gamma(s+j)}{j!\Gamma(s)} \rho^{2j} \zeta_R(2s+2j-2), \quad 0 \leq \rho^2 < 1$$

$$\begin{aligned} S_{eff}^{S^3} &= -\frac{1}{2}\left\{\rho^2\left[\zeta_H'(0,1-\rho)\zeta_H'(0,1+\rho)\right]\right. \\ &+ 2\rho\left[\zeta_H'(-1,1-\rho) - \zeta_H'(-1,1+\rho)\right] \\ &+ \left.\zeta_H'(-2,1-\rho) + \zeta_H'(-2,1+\rho)\right\} \end{aligned}$$

Positive at $ma = 0$ and $\frac{d}{d(ma)^2} S_{eff}^{S^3} \rfloor_{ma=0} = 0$

UV OK...BUT...

What about the ("trivial") IR fixed point?

Inversion of the heat kernel (inversion formula for Jacobi theta function) gives another expression for the effective action,

$$
\begin{aligned}
S_{eff}^{S^3} &= -\frac{1}{2}\Bigg\{-\frac{\pi^{\frac{1}{2}}}{2}\left[\Gamma(-\frac{3}{2})(\delta^2)^{\frac{3}{2}} + 4\sum_{n=1}^{\infty}\left(\frac{n\pi}{\delta}\right)^{-\frac{3}{2}} K_{\frac{3}{2}}(2\delta n\pi)\right] \\
&- \frac{\pi^{\frac{1}{2}}\delta^2}{2}\left[\Gamma(\frac{1}{2})(\delta^2)^{\frac{1}{2}} + 4\sum_{n=1}^{\infty}\left(\frac{n\pi}{\delta}\right)^{-\frac{1}{2}} K_{\frac{1}{2}}(2\delta n\pi)\right]\Bigg\}
\end{aligned}
$$

$$\delta^2 = (ma)^2 - \frac{1}{4}$$

Gives, in the IR limit, $S_{eff}^{S^3}\rfloor_{ma\to\infty} = -\frac{\pi}{6}((ma)^2 - \frac{1}{4})^{\frac{3}{2}}$

Can't subtract in the whole regime. Becomes imaginary in the UV.

Expand? $\frac{\pi}{6}[(ma)^3 - \frac{3}{8}ma]$ would spoil the UV stationary character (divergent derivative w.r.to $(ma)^2$)

The Rényi entanglement entropy

Entanglement entropy evaluated through the “replica trick”: qS^3 is an orbifolded q-covering of the sphere

$$S = \lim_{q \to 1} S_q = \lim_{q \to 1} \left(\frac{q S_{eff}^{S^3} - S_{eff}^{qS^3}}{q-1} \right)$$

I. Klebanov, T. Nishioka, S. Pufu and B. Safdi,“Is Renormalized Entanglement Entropy Stationary at RG Fixed Points?”, JHEP **1210** (2012) 058

J.S. Dowker,“Sphere Rényi entropies”, J. Phys. A **46**, 225401 (2013).

UV fixed point

As before, using the binomial expansion for small values of ma,

$$S = S_{eff}^{S^3} - \frac{\pi}{6}[\rho^2 - 1] \cot \pi\rho, \quad \rho = \sqrt{\frac{1}{4} - (ma)^2}$$

From this expression, one readily sees it reduces to $S_{eff}^{S^3}$ at $ma = 0$

But $\frac{d}{d(ma)^2}\Big\rfloor_{ma=0} S = \frac{\pi^2}{16} \neq 0$. S is non-stationary

Even worse:

IR limit can be studied though Jacobi inversion or analytical extension

$$\lim_{ma\to\infty} S = \frac{\pi}{6}((ma)^2 - \frac{1}{4})^{\frac{1}{2}}$$

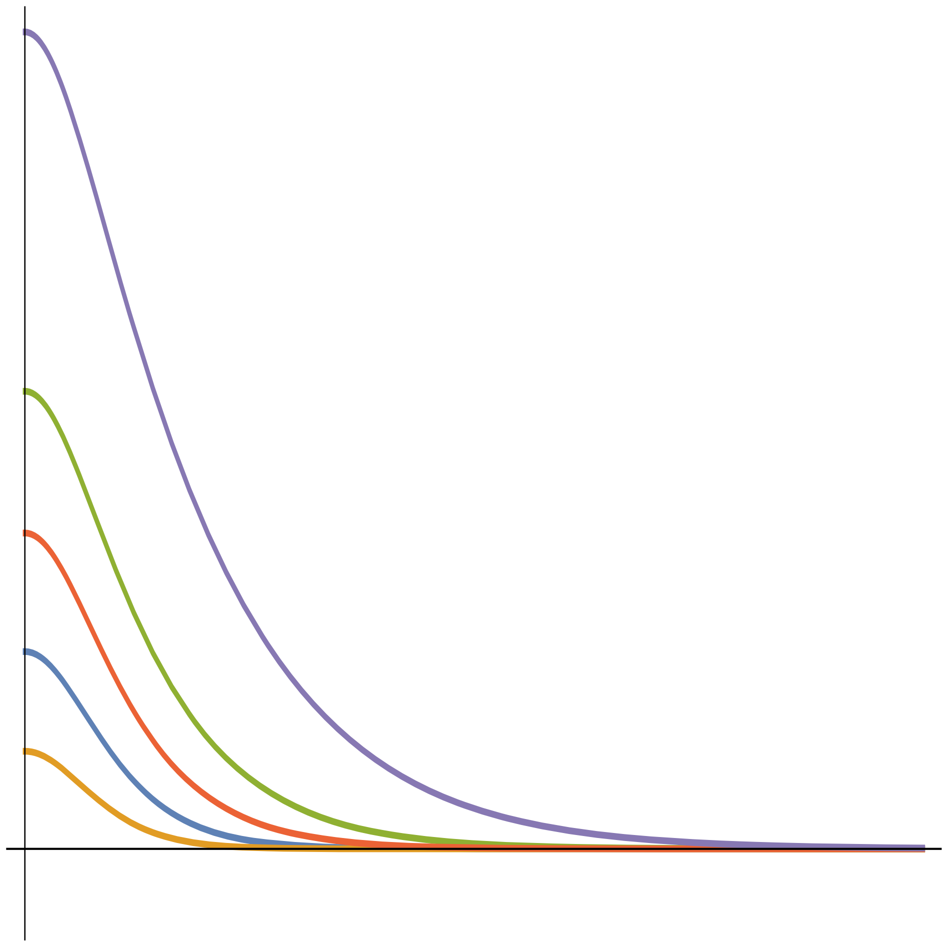
Can't be subtracted without getting a divergent derivative in the UV regime. ORIGIN: singularity

Our proposal: The holonomy entropy

M. Asorey, C.G. Beneventano, I. Cavero-Peláez, D. D'Ascanio and E.M. Santangelo,"Topological Entropy and Renormalization Group flow in 3-dimensional spherical spaces", JHEP **1501**, 078 (2015)

Study free quotients S^3/Γ instead of orbifolds. In particular, for lens spaces S^3/Z_p

$$S_{\rm hol}(p, ma) = S_{eff}^{S^3/Z_p}(ma) - \frac{1}{p} S_{eff}^{S^3}(ma)$$



p grows upwards

Always positive in the UV and zero in the IR: nice candidate for a c-function

Further developments of the idea and pending tasks

We have (more in Asorey's talk):

- generalized to free massive scalars in other dimensions restricting to $RP^d = S^d/Z_2$
- studied massive free fermions in arbitrary number of dimensions.

Need to:

- Study interacting theories