Functional determinants and Casimir energies in higher dimensional spherically symmetric background potentials

Klaus Kirsten

Baylor University

Benasque, Sept. 8, 2016

M. Beauregard, M. Bordag and K. Kirsten, J. Phys. A: Math. Theor. **48** (2015) 095401
G. Fucci and K. Kirsten, J. Phys. A: Math. Theor. **49** (2016) 275203.

Outline

Introduction

What are spectral functions?

Eigenvalue problem for a suitable differential operator P:

$$Pu_\ell(x) = \lambda_\ell u_\ell(x), \qquad 0 < \lambda_1 \leq \lambda_2 ..., \quad \lambda_\ell \to \infty \quad \text{as} \quad \ell \to \infty.$$

Introduction

What are spectral functions?

Eigenvalue problem for a suitable differential operator P:

$$P u_\ell(x) = \lambda_\ell u_\ell(x), \qquad 0 < \lambda_1 \le \lambda_2 \ldots, \quad \lambda_\ell \to \infty \quad \text{as} \quad \ell \to \infty.$$

- Functional determinant:

$$\text{"} \det P = \prod_{\ell=1}^{\infty} \lambda_\ell \text{"}$$

Introduction

What are spectral functions?

Eigenvalue problem for a suitable differential operator P:

$$Pu_\ell(x) = \lambda_\ell u_\ell(x), \qquad 0 < \lambda_1 \leq \lambda_2 \ldots, \quad \lambda_\ell \to \infty \quad \text{as} \quad \ell \to \infty.$$

- Functional determinant:

$$\text{" } \det P = \prod_{\ell=1}^{\infty} \lambda_\ell \text{ "}$$

- Casimir energy:

$$\text{" } E_P = \frac{1}{2} \sum_{\ell=1}^{\infty} \lambda_\ell^{1/2} \text{ "}$$

Introduction

- Heat kernel:

$$K_P(\tau) = \sum_{\ell=1}^{\infty} e^{-\tau\lambda_\ell}$$

Introduction

- Heat kernel:

$$K_P(\tau) = \sum_{\ell=1}^{\infty} e^{-\tau\lambda_\ell}$$

$$\overset{\tau\to 0}{\sim} \sum_{\ell=0,1/2,1,\dots}^{\infty} a_\ell(P,\mathcal{B})\ \tau^{\ell-D/2}$$

Introduction

- Heat kernel:

$$K_P(\tau) \quad = \quad \sum_{\ell=1}^{\infty} e^{-\tau\lambda_\ell}$$
$$\overset{\tau\to 0}{\sim} \quad \sum_{\ell=0,1/2,1,\ldots}^{\infty} \quad a_\ell(P,\mathcal{B}) \;\; \tau^{\ell-D/2}$$

- Zeta function:

$$\zeta_P(s) = \sum_{\ell=0}^{\infty} \lambda_\ell^{-s}, \qquad \Re s > \frac{D}{2}$$

Zeta function $\zeta_P(s) = \sum_{\ell=0}^{\infty} \lambda_\ell^{-s}$ as best organization of the spectrum

- Functional determinant:

$$" \quad \ln \det P = \sum_{\ell=0}^{\infty} \ln \lambda_\ell = -\frac{d}{ds} \sum_{\ell=0}^{\infty} \lambda_\ell^{-s} \Big|_{s=0} = -\zeta_P'(0) \; "$$

Zeta function $\zeta_P(s) = \sum_{\ell=0}^{\infty} \lambda_\ell^{-s}$ as best organization of the spectrum

- Functional determinant:

$$" \ln \det P = \sum_{\ell=0}^{\infty} \ln \lambda_\ell = -\frac{d}{ds} \sum_{\ell=0}^{\infty} \lambda_\ell^{-s} \Big|_{s=0} = -\zeta_P'(0) "$$

- Casimir energy:

$$" E_P = \frac{1}{2} \sum_{\ell=0}^{\infty} \lambda_\ell^{1/2} \rightarrow \frac{1}{2} \zeta_P \left(s = -\frac{1}{2} \right) "$$

Basic ideas: Exterior of a sphere

- Let:

$$P = -\Delta + m^2$$

Basic ideas: Exterior of a sphere

- Let:

$$P = -\Delta + m^2$$

- Eigenvalue problem:

$$P\phi_j(\vec{x}) = \lambda_j^2 \phi_j(\vec{x}) \ , \qquad \phi_j(|\vec{x}| = R) = \phi_j(|\vec{x}| = a) = 0$$

Basic ideas: Exterior of a sphere

- Let:

$$P = -\Delta + m^2$$

- Eigenvalue problem:

$$P\phi_j(\vec{x}) = \lambda_j^2 \phi_j(\vec{x}) \ , \qquad \phi_j(|\vec{x}| = R) = \phi_j(|\vec{x}| = a) = 0$$

- Spherical coordinates:

$$\begin{aligned} \phi_{n\ell m}(\vec{x}) &= \frac{1}{r}\psi_{n\ell}(r) Y_{\ell m}(\theta, \varphi), \\ 0 &= \left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + \lambda_{n\ell}^2 \right] \psi_{n\ell}(r). \end{aligned}$$

Basic ideas: Exterior of a sphere

- Let:

$$P = -\Delta + m^2$$

- Eigenvalue problem:

$$P\phi_j(\vec{x}) = \lambda_j^2 \phi_j(\vec{x}) \ , \qquad \phi_j(|\vec{x}| = R) = \phi_j(|\vec{x}| = a) = 0$$

- Spherical coordinates:

$$\begin{aligned} \phi_{n\ell m}(\vec{x}) &= \frac{1}{r}\psi_{n\ell}(r) Y_{\ell m}(\theta, \varphi), \\ 0 &= \left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + \lambda_{n\ell}^2\right]\psi_{n\ell}(r). \end{aligned}$$

- Solutions typically written as $(\nu = \ell + \frac{1}{2})$:

$$\psi_{n\ell}(r) = a_1\sqrt{r}J_\nu(\lambda_{n\ell}r) + a_2\sqrt{r}N_\nu(\lambda_{n\ell}r).$$

Basic ideas: Exterior of a sphere

- Within scattering theory common:

$$\psi_{n\ell}(r) = b_1 \hat{h}_\ell^+(\lambda_{n\ell}\, r) + b_2 \hat{h}_\ell^-(\lambda_{n\ell}\, r),$$

Basic ideas: Exterior of a sphere

- Within scattering theory common:

$$
\begin{aligned}
\psi_{n\ell}(r) &= b_1 \hat{h}_\ell^+(\lambda_{n\ell}\, r) + b_2 \hat{h}_\ell^-(\lambda_{n\ell}\, r),\\
\hat{h}_\ell^+(z) &= i\sqrt{\frac{\pi z}{2}} H_\nu^{(1)}(z), \qquad \hat{h}_\ell^-(z) = -i\sqrt{\frac{\pi z}{2}} H_\nu^{(2)}(z),\\
H_\nu^{(1)}(z) &= J_\nu(z) + iN_\nu(z),\\
H_\nu^{(2)}(z) &= J_\nu(z) - iN_\nu(z).
\end{aligned}
$$

Basic ideas: Exterior of a sphere

- Within scattering theory common:

$$\psi_{n\ell}(r) = b_1 \hat{h}_\ell^+(\lambda_{n\ell}\; r) + b_2 \hat{h}_\ell^-(\lambda_{n\ell}\; r),$$

$$\hat{h}_\ell^+(z) = i\sqrt{\frac{\pi z}{2}} H_\nu^{(1)}(z), \qquad \hat{h}_\ell^-(z) = -i\sqrt{\frac{\pi z}{2}} H_\nu^{(2)}(z),$$

$$H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z),$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z).$$

- Impose boundary conditions:

$$F_\nu(\lambda_{n\ell}) \equiv H_\nu^{(1)}(\lambda_{n\ell} a) H_\nu^{(2)}(\lambda_{n\ell} R) - H_\nu^{(1)}(\lambda_{n\ell} R) H_\nu^{(2)}(\lambda_{n\ell} a) = 0.$$

Basic ideas: Exterior of a sphere

- Within scattering theory common:

$$\psi_{n\ell}(r) = b_1 \hat{h}_\ell^+(\lambda_{n\ell}\ r) + b_2 \hat{h}_\ell^-(\lambda_{n\ell}\ r),$$

$$\hat{h}_\ell^+(z) = i\sqrt{\frac{\pi z}{2}} H_\nu^{(1)}(z), \qquad \hat{h}_\ell^-(z) = -i\sqrt{\frac{\pi z}{2}} H_\nu^{(2)}(z),$$

$$H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z),$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z).$$

- Impose boundary conditions:

$$F_\nu(\lambda_{n\ell}) \equiv H_\nu^{(1)}(\lambda_{n\ell} a) H_\nu^{(2)}(\lambda_{n\ell} R) - H_\nu^{(1)}(\lambda_{n\ell} R) H_\nu^{(2)}(\lambda_{n\ell} a) = 0.$$

- Zeta function representation:

$$\frac{d}{dp} \ln F_\nu(p).$$

Basic ideas: Exterior of a sphere

- Within scattering theory common:

$$\begin{aligned}\psi_{n\ell}(r) &= b_1 \hat{h}_\ell^+(\lambda_{n\ell}\ r) + b_2 \hat{h}_\ell^-(\lambda_{n\ell}\ r),\\ \hat{h}_\ell^+(z) &= i\sqrt{\frac{\pi z}{2}} H_\nu^{(1)}(z), \qquad \hat{h}_\ell^-(z) = -i\sqrt{\frac{\pi z}{2}} H_\nu^{(2)}(z),\\ H_\nu^{(1)}(z) &= J_\nu(z) + iN_\nu(z),\\ H_\nu^{(2)}(z) &= J_\nu(z) - iN_\nu(z).\end{aligned}$$

- Impose boundary conditions:

$$F_\nu(\lambda_{n\ell}) \equiv H_\nu^{(1)}(\lambda_{n\ell} a) H_\nu^{(2)}(\lambda_{n\ell} R) - H_\nu^{(1)}(\lambda_{n\ell} R) H_\nu^{(2)}(\lambda_{n\ell} a) = 0.$$

- Zeta function representation:

$$\zeta_{ext}^{(\ell)}(s) = \frac{1}{2\pi i} \int_\gamma dp (p^2 + m^2)^{-s} \frac{d}{dp} \ln F_\nu(p).$$

Basic ideas: Exterior of a sphere

- Subtract Minkowski space contribution:

$$J_\nu(\mu_{n\ell} a) = 0$$

Basic ideas: Exterior of a sphere

- Subtract Minkowski space contribution:

$$J_\nu(\mu_{n\ell}a) = 0$$

- Better for current purposes:

$$H_\nu^{(1)}(\mu_{n\ell}a) - H_\nu^{(2)}(\mu_{n\ell}a) = 0.$$

Basic ideas: Exterior of a sphere

- Subtract Minkowski space contribution:

$$J_\nu(\mu_{n\ell}a) = 0$$

- Better for current purposes:

$$H_\nu^{(1)}(\mu_{n\ell}a) - H_\nu^{(2)}(\mu_{n\ell}a) = 0.$$

- Zeta function representation:

$$\zeta_{rel}^{(\ell)}(s) = \frac{1}{2\pi i}\int_\gamma dp(p^2+m^2)^{-s}\frac{d}{dp}\ln\frac{H_\nu^{(1)}(pa)H_\nu^{(2)}(pR) - H_\nu^{(1)}(pR)H_\nu^{(2)}(pa)}{H_\nu^{(1)}(pa) - H_\nu^{(2)}(pa)}.$$

Basic ideas: Exterior of a sphere

- Full exterior sphere contribution ($a \to \infty$):

$$\zeta_{rel}^{(\ell)}(s) = \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \frac{d}{dk} \ln H_\nu^{(1)}(ikR)$$

$$= \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \frac{d}{dk} \ln K_\nu(kR).$$

Background potentials

- Eigenvalue equation with background potentials:

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - V(r) + p^2\right] \psi_{\ell,p}(r) = 0.$$

Background potentials

- Eigenvalue equation with background potentials:

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - V(r) + p^2\right]\psi_{\ell,p}(r) = 0.$$

- Eigenfunctions used in scattering theory:

$$\phi_{\ell,p}(r) \quad \sim \quad \hat{j}_\ell(pr) = \frac{i}{2}\left[\hat{h}_\ell^-(pr) - \hat{h}_\ell^+(pr)\right] \qquad \text{as } r \to 0$$

$$\phi_{\ell,p}(r) \quad \sim \quad \frac{i}{2}\left[f_\ell(p)\hat{h}_\ell^-(pr) - f_\ell^*(p)\hat{h}_\ell^+(pr)\right] \qquad \text{as } r \to \infty$$

with the Jost function $f_\ell(p)$.

Background potentials

- Impose boundary conditions at $r = a$ (a sphere containing the compact support of $V(r)$):

$$f_\ell(p)\hat{h}_\ell^-(pa) - f_\ell^*(p)\hat{h}^+(pa) = 0.$$

Background potentials

- Impose boundary conditions at $r = a$ (a sphere containing the compact support of $V(r)$):

$$f_\ell(p)\hat{h}_\ell^-(pa) - f_\ell^*(p)\hat{h}^+(pa) = 0.$$

- Subtract Minkowski space contribution:

$$\zeta_{rel}^{(\ell)}(s) \quad = \quad \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \frac{d}{dk} \ln f_\ell(ik)$$

Background potentials

- Impose boundary conditions at $r = a$ (a sphere containing the compact support of $V(r)$):

$$f_\ell(p)\hat{h}_\ell^-(pa) - f_\ell^*(p)\hat{h}^+(pa) = 0.$$

- Subtract Minkowski space contribution:

$$\zeta_{rel}^{(\ell)}(s) = \frac{\sin \pi s}{\pi} \int_m^\infty dk (k^2 - m^2)^{-s} \frac{d}{dk} \ln f_\ell(ik)$$

- Summation $\sum_{\ell=0}^\infty (2\ell+1)$ and integration cannot be interchanged:

$$\ln f_\ell^{asym}(ik) \sim \frac{1}{2\nu} \int_0^\infty dr \frac{rV(r)}{\left(1 + \left(\frac{kr}{\nu}\right)^2\right)^{1/2}} + \ldots$$

Phase shifts

- Eigenfunctions using phase shift $\delta_\ell(p)$:

$$\psi_{\ell,p}(r) \sim c_\ell \sin\left(pr - \frac{\pi\ell}{2} + \delta_\ell(p)\right) \qquad \text{as } r \to \infty.$$

Phase shifts

- Eigenfunctions using phase shift $\delta_\ell(p)$:

$$\psi_{\ell,p}(r) \sim c_\ell \sin\left(pr - \frac{\pi\ell}{2} + \delta_\ell(p)\right) \qquad \text{as } r \to \infty.$$

- Compare with Jost function representation:

$$c_\ell \sin\left(pr - \frac{\pi\ell}{2} + \delta_\ell(p)\right) \sim \frac{i}{2}\left[f_\ell(p)\hat{h}_\ell^-(pr) - f_\ell^* \hat{h}_\ell^+(pr)\right]$$

Phase shifts

- Eigenfunctions using phase shift $\delta_\ell(p)$:

$$\psi_{\ell,p}(r) \sim c_\ell \sin\left(pr - \frac{\pi\ell}{2} + \delta_\ell(p)\right) \qquad \text{as } r \to \infty.$$

- Compare with Jost function representation:

$$c_\ell \sin\left(pr - \frac{\pi\ell}{2} + \delta_\ell(p)\right) \sim \frac{i}{2}\left[f_\ell(p)\hat{h}_\ell^-(pr) - f_\ell^* \hat{h}_\ell^+(pr)\right]$$

$$\sim \frac{i}{2}\left[f_\ell^*(p)e^{i(pr-\ell\pi/2)} - f_\ell(p)e^{-i(pr-\ell\pi/2)}\right].$$

Phase shifts

- Eigenfunctions using phase shift $\delta_\ell(p)$:

$$\psi_{\ell,p}(r) \sim c_\ell \sin\left(pr - \frac{\pi\ell}{2} + \delta_\ell(p)\right) \qquad \text{as } r \to \infty.$$

- Compare with Jost function representation:

$$c_\ell \sin\left(pr - \frac{\pi\ell}{2} + \delta_\ell(p)\right) \sim \frac{i}{2}\left[f_\ell(p)\hat{h}_\ell^-(pr) - f_\ell^* \hat{h}_\ell^+(pr)\right]$$

$$\sim \frac{i}{2}\left[f_\ell^*(p)e^{i(pr-\ell\pi/2)} - f_\ell(p)e^{-i(pr-\ell\pi/2)}\right].$$

- This shows the relation:

$$f_\ell(p) = |f_\ell(p)|e^{-i\delta_\ell(p)}.$$

Phase shifts

- Dispersion relation (in absence of bound states):

$$f_\ell(ik) = \exp\left\{ -\frac{2}{\pi} \int_0^\infty \frac{q}{q^2+k^2}\, \delta_\ell(q)\, dq \right\}.$$

Phase shifts

- Dispersion relation (in absence of bound states):

$$f_\ell(ik) = \exp\left\{ -\frac{2}{\pi} \int_0^\infty \frac{q}{q^2+k^2}\ \delta_\ell(q)\ dq \right\}.$$

- Zeta function using phase shifts:

$$\zeta(s) \quad = \quad \frac{2s}{\pi} \sum_{\ell=0}^\infty (2\ell+1) \int_0^\infty \frac{q}{(q^2+m^2)^{s+1}} \delta_\ell(q)\ dq$$

Phase shifts

- Dispersion relation (in absence of bound states):

$$f_\ell(ik) = \exp\left\{ -\frac{2}{\pi} \int_0^\infty \frac{q}{q^2+k^2}\ \delta_\ell(q)\ dq \right\}.$$

- Zeta function using phase shifts:

$$\begin{aligned}\zeta(s) &= \frac{2s}{\pi} \sum_{\ell=0}^\infty (2\ell+1) \int_0^\infty \frac{q}{(q^2+m^2)^{s+1}} \delta_\ell(q)\ dq \\ &= \sum_{\ell=0}^\infty (2\ell+1) \int_0^\infty \frac{1}{(q^2+m^2)^s}\ \frac{1}{\pi}\frac{d}{dq}\delta_\ell(q)\end{aligned}$$

Phase shifts

- Dispersion relation (in absence of bound states):

$$f_\ell(ik) = \exp\left\{ -\frac{2}{\pi} \int_0^\infty \frac{q}{q^2+k^2}\ \delta_\ell(q)\ dq \right\}.$$

- Zeta function using phase shifts:

$$\zeta(s) = \frac{2s}{\pi} \sum_{\ell=0}^\infty (2\ell+1) \int_0^\infty \frac{q}{(q^2+m^2)^{s+1}} \delta_\ell(q)\ dq.$$

Phase shifts

- Dispersion relation (in absence of bound states):

$$f_\ell(ik) = \exp\left\{ -\frac{2}{\pi} \int_0^\infty \frac{q}{q^2+k^2}\ \delta_\ell(q)\ dq \right\}.$$

- Zeta function using phase shifts:

$$\zeta(s) = \frac{2s}{\pi} \sum_{\ell=0}^\infty (2\ell+1) \int_0^\infty \frac{q}{(q^2+m^2)^{s+1}} \delta_\ell(q)\ dq.$$

- Interchanging summation and integration now?

$$\delta_\ell(q) \sim \frac{m\ell V\left(\frac{\ell}{q}\right)}{q^2}, \quad \text{so yes for} \quad V(r) \sim \frac{1}{r^{3+\epsilon}}.$$

Phase shifts

- Interchange summation and integration:

$$\delta(q) = \sum_{\ell=0}^{\infty}(2\ell+1)\delta_\ell(q),$$

$$\zeta(s) = \frac{2s}{\pi}\int_0^\infty \frac{q}{(q^2+m^2)^{s+1}}\,\delta(q)\,dq.$$

Phase shifts

- Interchange summation and integration:

$$\delta(q) = \sum_{\ell=0}^{\infty}(2\ell+1)\delta_\ell(q),$$

$$\zeta(s) = \frac{2s}{\pi}\int_0^\infty \frac{q}{(q^2+m^2)^{s+1}}\ \delta(q)\ dq.$$

- The pole structure of the zeta function shows for $q \to \infty$:

$$\delta(q) \sim \pi\left(\frac{4q^3}{3\sqrt{\pi}}a_0 + q^2 a_{1/2} + \frac{2q}{\sqrt{\pi}}a_1 + a_{3/2} + \frac{1}{q\sqrt{\pi}}a_2 + \dots\right).$$

Phase shifts

- Renormalized Casimir energy defined by normalizing:

$$\lim_{m\to\infty} E_{Cas}^{(ren)}(s) = 0.$$

Phase shifts

- Renormalized Casimir energy defined by normalizing:

$$\lim_{m\to\infty} E_{Cas}^{(ren)}(s) = 0.$$

- Phase shift asymptotics equals terms to be subtracted, so:

$$E_{Cas}^{(ren)} = -\frac{1}{2\pi}\int_0^\infty dq\, \frac{q}{\sqrt{q^2+m^2}}\delta_{subtr}(q),$$

$$\delta_{subtr}(q) = \delta(q) - \frac{4\sqrt{\pi}}{3}a_0 q^3 - \pi a_{1/2} q^2 - 2\sqrt{\pi} a_1 q - \pi a_{3/2} - \sqrt{\pi} a_2 \frac{1}{q}.$$

Examples

- Phase shift determined from the Jost function:

$$\frac{f_\ell(p)}{f_\ell^*(p)} = e^{-2i\delta_\ell(p)} \Longrightarrow \delta_\ell(p) = -\arctan\frac{\Im f_\ell(p)}{\Re f_\ell(p)}.$$

Examples

- Phase shift determined from the Jost function:

$$\frac{f_\ell(p)}{f_\ell^*(p)} = e^{-2i\delta_\ell(p)} \Longrightarrow \delta_\ell(p) = -\arctan \frac{\Im f_\ell(p)}{\Re f_\ell(p)}.$$

- Jost function determined as follows: $V(r) = 0$ for $r \geq R$

$$\psi_{\ell,p}(r) = u_{\ell,p}(r)\Theta(R-r) + \frac{i}{2}\left[f_\ell(p)\hat{h}_\ell^-(pr) - f_\ell^*(p)\hat{h}_\ell^+(pr)\right]\Theta(r-R).$$

Examples

- Phase shift determined from the Jost function:

$$\frac{f_\ell(p)}{f_\ell^*(p)} = e^{-2i\delta_\ell(p)} \Longrightarrow \delta_\ell(p) = -\arctan\frac{\Im f_\ell(p)}{\Re f_\ell(p)}.$$

- Jost function determined as follows: $V(r) = 0$ for $r \geq R$

$$\psi_{\ell,p}(r) = u_{\ell,p}(r)\Theta(R-r) + \frac{i}{2}\left[f_\ell(p)\hat{h}_\ell^-(pr) - f_\ell^*(p)\hat{h}_\ell^+(pr)\right]\Theta(r-R).$$

- Continuity of $\psi_{\ell,p}$ and its derivative:

$$f_\ell(p) = -\frac{1}{p}\left(pu_{\ell,p}(R)\left(\hat{h}_\ell^+\right)'(pR) - u'_{\ell,p}(R)\hat{h}_\ell^+(pR)\right).$$

Examples

- Step potential $V_0 > 0$:

$$V(r) = \begin{cases} V_0 & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases}$$

Examples

- Step potential $V_0 > 0$:

$$V(r) = \begin{cases} V_0 & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases}$$

- Phase shift: $q = \sqrt{p^2 - V_0}$

$$\delta_\ell(p) = -\arctan \frac{qJ_\nu(pR)J'_\nu(qR) - pJ_\nu(qR)J'_\nu(pR)}{pJ_\nu(qR)N'_\nu(pR) - qN_\nu(pR)J'_\nu(qR)}.$$

Examples

- Step potential $V_0 > 0$:

$$V(r) = \begin{cases} V_0 & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases}$$

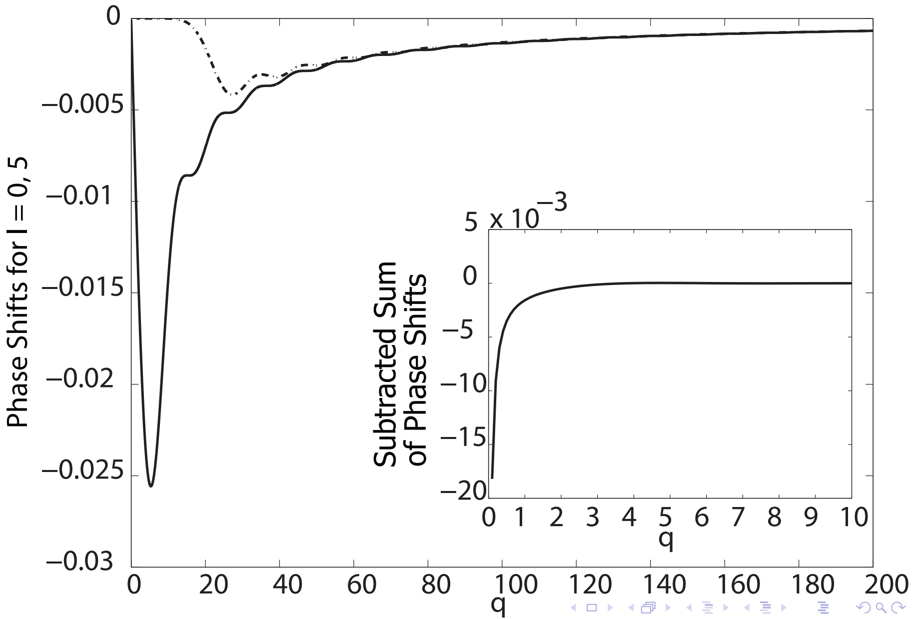
- Phase shift: $q = \sqrt{p^2 - V_0}$

$$\delta_\ell(p) = -\arctan \frac{qJ_\nu(pR)J'_\nu(qR) - pJ_\nu(qR)J'_\nu(pR)}{pJ_\nu(qR)N'_\nu(pR) - qN_\nu(pR)J'_\nu(qR)}.$$

- Subtracted terms:

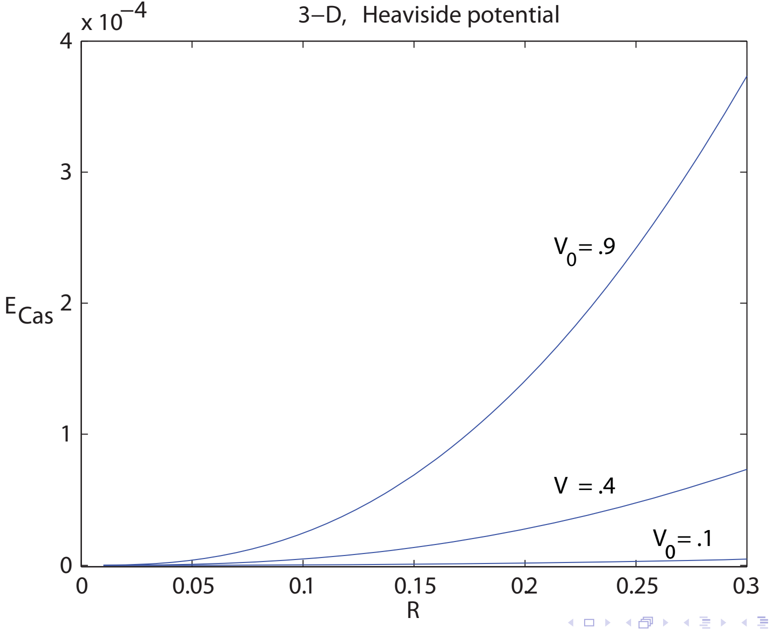
$$a_0 = a_{1/2} = a_{3/2} = 0, \qquad a_1 = -\frac{R^3 V_0}{6\sqrt{\pi}}, \qquad a_2 = \frac{R^3 V_0^2}{12\sqrt{\pi}}.$$

- $V_0 = 0.9, \; R = 1$:



Phase Shifts for l = 0, 5
0
−0.005
−0.01
−0.015
−0.02
−0.025
−0.03
0 20 40 60 80 100 120 140 160 180 200
q
Subtracted Sum of Phase Shifts
5 x 10−3
0
−5
−10
−15
−20
0 1 2 3 4 5 6 7 8 9 10
q

- $m = 1$:



x 10−4
3−D, Heaviside potential
4
3
2
1
0
E Cas
V0= .9
V = .4
V0= .1
0
0.05
0.1
0.15
0.2
0.25
0.3
R

Examples

- Delta potential:

$$V(r) = \frac{\alpha}{R}\delta(r - R).$$

Examples

- Delta potential:

$$V(r) = \frac{\alpha}{R}\delta(r - R).$$

- Phase shift:

$$\delta_\ell(p) = -\arctan \frac{\frac{\pi\alpha}{2} J_\nu^2(pR)}{1 - \frac{\pi\alpha}{2} J_\nu(pR) N_\nu(pR)}.$$

Examples

- Delta potential:

$$V(r) = \frac{\alpha}{R}\delta(r-R).$$

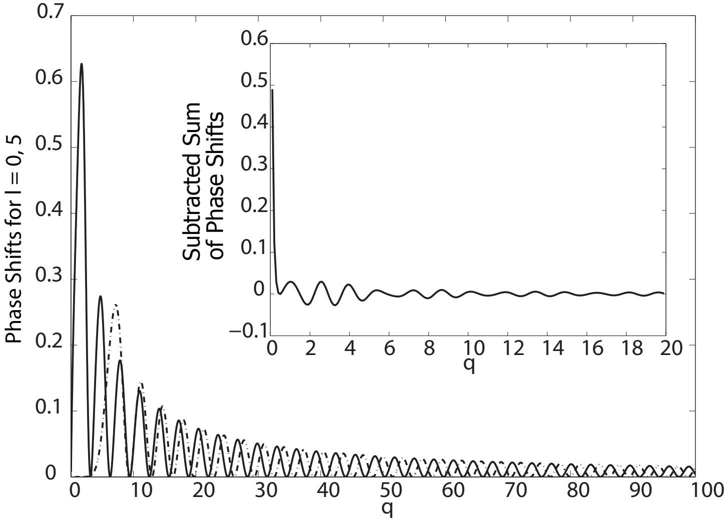
- Phase shift:

$$\delta_\ell(p) = -\arctan \frac{\frac{\pi\alpha}{2} J_\nu^2(pR)}{1-\frac{\pi\alpha}{2}J_\nu(pR)N_\nu(pR)}.$$

- Subtracted terms:

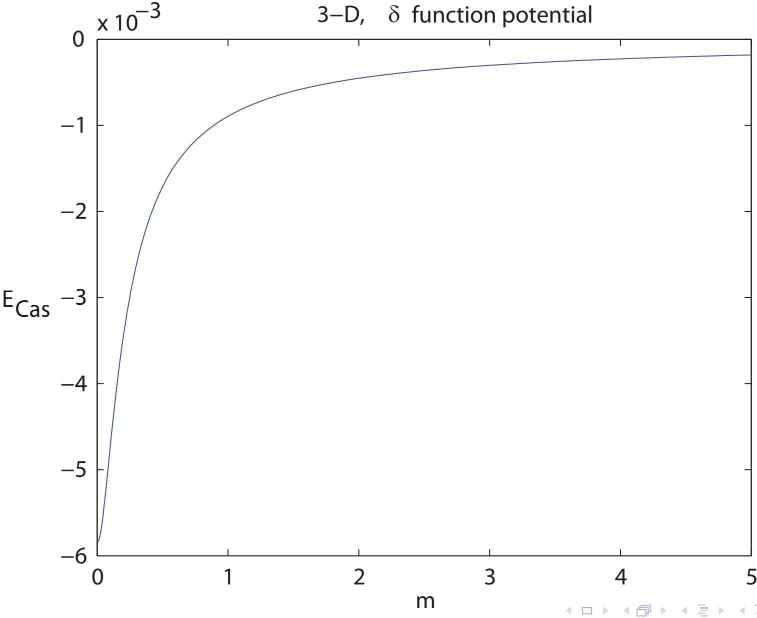
$$a_0 = a_{1/2} = 0, \qquad a_1 = -\frac{\alpha R}{2\sqrt{\pi}}, \qquad a_{3/2} = \frac{\alpha^2}{8}, \qquad a_2 = -\frac{\alpha^3}{12\sqrt{\pi}R}.$$

- $\alpha = 1, \ R = 1$:



Phase Shifts for l = 0, 5
q
0
10
20
30
40
50
60
70
80
90
100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Subtracted Sum
of Phase Shifts
q
0
2
4
6
8
10
12
14
16
18
20
−0.1
0
0.1
0.2
0.3
0.4
0.5
0.6

- $\alpha = 1,\ R = 1$:



Conclusions

- Casimir energy in arbitrary dimension D:

$$E_{Cas}^{(ren)} = -\frac{1}{2\pi} \int_0^\infty \frac{q}{\sqrt{q^2+m^2}} \left\{ \delta(q) - \pi \sum_{k=0}^{D+1} \frac{q^{D-k}}{\Gamma\left(\frac{D-k}{2}+1\right)} \, a_{k/2} \right\} dq.$$

Conclusions

- Casimir energy in arbitrary dimension D:

$$E_{Cas}^{(ren)} = -\frac{1}{2\pi}\int_0^\infty \frac{q}{\sqrt{q^2+m^2}}\left\{\delta(q) - \pi\sum_{k=0}^{D+1}\frac{q^{D-k}}{\Gamma\left(\frac{D-k}{2}+1\right)}\ a_{k/2}\right\}dq.$$

- Functional determinants in arbitrary even dimension $D = 2M$:

$$\begin{aligned}\zeta'(0) &= \frac{2}{\pi}\int_0^\infty \frac{q}{q^2+m^2}\left\{\delta(q) - \pi\sum_{k=0}^{2M}\frac{q^{2M-k}}{\Gamma\left(\frac{2M-k}{2}+1\right)}\ a_{k/2}\right\}dq \\ &+ m^{2M}\sum_{j=0}^{M}\frac{(-1)^{M-j}}{(M-j)!}m^{-2j}a_j(-\log m^2 + H_{M-j}) \\ &+ m^{2M}\sum_{j=0}^{M-1}\Gamma\left(-M+j+\frac{1}{2}\right)m^{-2j-1}a_{j+1/2}.\end{aligned}$$

Moss I.G. and Naylor W., Nucl. Phys. B **632** (2002) 173

Conclusions

- Higher spin particles: very complicated uniform asymptotic expansions of Jost functions replaced by a simple computation of heat kernel coefficients
  (soliton, instanton, magnetic flux tube, color magnetic vortex, cosmic strings, Nielsen-Olesen vortex, dielectric backgrounds...)

Conclusions

- Higher spin particles: very complicated uniform asymptotic expansions of Jost functions replaced by a simple computation of heat kernel coefficients
  (soliton, instanton, magnetic flux tube, color magnetic vortex, cosmic strings, Nielsen-Olesen vortex, dielectric backgrounds...)
- Question: Can something like this be done for cases with boundary like the sphere? What corresponds to the phase shift?
  Exterior yes:

$$\sqrt{\frac{2p}{\pi}} e^p K_{\ell+1/2}(p) = \exp\left\{ -\frac{2}{\pi} \int_0^\infty dz \frac{z}{z^2+p^2} \delta_\ell(z) \right\}$$

$$\delta_\ell(z) = -\arctan \frac{\cos z \; J_{2n+1/2}(z) + \sin z \; N_{2n+1/2}(z)}{\sin z \; J_{2n+1/2}(z) - \cos p \; N_{2n+1/2}(z)}$$