$O(N)$ model in Euclidean de Sitter space: beyond the leading IR approximation

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Outline

- Motivations
- IR effects in de Sitter
- QFT Euclidean de Sitter space
- Resummation of leading IR secular terms
- Concluding remarks

Motivations

QFT in de Sitter spacetime

- Inflationary stage of the early Universe

  Quantum fluctuations give rise to primordial inhomogenities

- Current accelerated expansion

  Quantum effects can afect the time evolution through backreaction

IR effects

- Light and massless fields need special treatment due to the large IR fluctuations

QFT in de Sitter spacetime

- N scalar fields ϕ_a with $O(N)$-symmetry and quartic self-interaction

$$S[\phi_a, g_{\mu\nu}] = -\int d^4 x\sqrt{-g}\left[\frac{1}{2}\phi_a\left(-\Box + m^2 + \xi R\right)\phi_a + \frac{\lambda}{8N}(\phi_a\phi_a)^2\right]$$

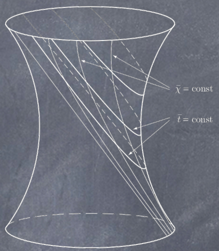
- Cosmological patch of de Sitter

$$ds^2 = -dt^2 + e^{2Ht}d\vec{x}^2$$

or, defining conformal time $\eta = -e^{-Ht}$

$$ds^2 = \frac{1}{\eta^2}\left[-d\eta^2 + d\vec{x}^2\right]$$

with $-\infty < \eta < 0$



- Quantities of interest: Vacuum expectation values

$$\langle\phi^2(t,\vec{x})\rangle = G(x,x)$$

$$\langle T_{\mu\nu}(x)\rangle = D_{\mu\nu}G(x,x')\Big|_{x=x'}$$

$G(x,x')$: field propagator

IR effects

Massive free field propagator

$$G_F^{(m)}(x,x') = \frac{H^2\Gamma(\frac{3}{2}-\nu)\Gamma(\frac{3}{2}+\nu)}{(4\pi)^2}\,{}_2F_1\left(\frac{3}{2}-\nu,\frac{3}{2}+\nu;2;1-\frac{y}{4}-i\epsilon\right)$$

where $\nu = \sqrt{9/4 - m^2/H^2}$ and

$$y(x,x') = \frac{-(\eta-\eta')^2 + |\vec{x}-\vec{x}'|^2}{\eta\eta'}$$

(Invariant distance)

- Large spatial separations or late times ($y \to +\infty$)

$$G_F^{(m)}(y) \sim y^{-\frac{m^2}{3H^2}}$$

- IR limit ($m^2 \ll H^2$)

$$G_F^{(m)} \simeq \frac{3H^4}{8\pi^2 m^2}$$

divergence for $m \to 0$ (not present in flat spacetime)

IR effects

Massless free field

- Field variance

$$\langle \phi(t)^2 \rangle = \frac{H^3 t}{4\pi^2}$$

There is no de Sitter invariant vacuum state for a free massless field.

- Large spatial separations or late times ($y \to +\infty$):

Subtracting the divergent term, the corresponding massless Feynman propagator

$$\hat{G}_F^{(0)}(y) = \lim_{m \to 0} \left[G_F^{(m)}(y) - \frac{3H^4}{8\pi^2 m^2} \right] \sim \log(y)$$

Interacting field (i.e.: $\lambda\phi^4$)

- Loop corrections ($L = \#$ loops)

$$\sim \left(\frac{\lambda H t}{4\pi^2} \right)^L \qquad\qquad \sim \left(\frac{\lambda H^4}{4\pi^2 m^4} \right)^L$$

$\Longrightarrow$ The perturbative expansion is invalid at late times or for small masses!

Burgess et al. (2010)

First non-perturbative solution: Stochastic Inflation

Starobinsky & Yokoyama (1994)

Dinamical mass ($N = 1$)

- Massless field with $\lambda\phi^4/4!$ interaction:

$$\langle\phi^2\rangle = \frac{3}{\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\frac{H^2}{\sqrt{\lambda}} \equiv \frac{3H^4}{8\pi^2 m_{dyn}^2}$$

then

$$\boxed{m_{dyn}^2 = \frac{\sqrt{\lambda}H^2}{8\pi}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}}$$

Result non-analytical in $\lambda \Longrightarrow$ non perturbative!

- How to compute systematic corrections beyond the IR limit?

Lorentzian QFT ($N \to \infty$)

$$m_{dyn}^2 = \frac{\sqrt{3\lambda}H^2}{4\pi}$$

Euclidean de Sitter space

- Global coordinates in Lorenzian d-dimensional de Sitter spacetime

$$ds^2 = \frac{1}{H^2}\left[-dt^2 + \cosh(t)^2 d\Omega^2\right]$$

- Analytical continuation

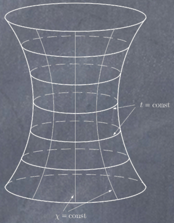
$$t \to -i\left(\tau - \frac{\pi}{2H^2}\right)$$

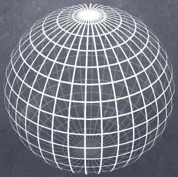
- Compactification

$$\tau = \tau + 2\pi H^{-1}$$

- Euclidean d-sphere

$$ds^2 = \frac{1}{H^2}\left[d\tau^2 + \sin(\tau)^2 d\Omega^2\right]$$





QFT in euclidean de Sitter

- Euclidean Action

$$S_E = \int d^d x \sqrt{g} \left[\frac{1}{2} \phi_a \left(-\Box + m^2 \right) \phi_a + \frac{\lambda}{8N} (\phi_a \phi_a)^2 \right]$$

- We expand in d-dimensional spherical harmonics

$$\phi_a(x) = \sum_{\vec{L}} \tilde{\phi}_{\vec{L},a} Y_{\vec{L}}(x)$$

- Free propagator (symmetric phase)

$$G_{ab}(x,x') = \delta_{ab} \sum_{\vec{L}} H^d \frac{Y_{\vec{L}}(x) Y_{\vec{L}}(x')}{H^2 L(L+d-1) + m^2} \equiv \delta_{ab} G(x,x')$$

- The zero modes ($\vec{L} = 0$) are responsible for this divergence and the breakdown of the perturbative expansion!

$$G(x,x') \longrightarrow \frac{H^d Y_{\vec{0}}^2}{m^2} \qquad \text{when} \quad m^2 \to 0$$

- It can be shown (through the effective potential) that:

$$m^2_{dyn} = \frac{N}{V_d \langle \phi_0^2 \rangle}$$

V_d: surface area of a d-sphere ($V_4 = 8\pi^2 / 3H^4$)

Another nonperturbative solution: Proper treatment of the zero modes

$N = 1$: Rajaraman (2010)

$N > 1$: López Nacir, Mazzitelli, LGT (2016)

- Separating the zero modes

$$\phi_a(x) = \phi_{0a} + \hat{\phi}_a(x) \quad ; \quad G(x,x') = G_0 + \hat{G}(x,x')$$

Generating funcional of the zero modes (LO IR)

- Since ϕ_0 is constant, there is no kinetic term and the path integral turns into an ordinary integral
- Massless field $m = 0$

$$Z_0[J_0] = \mathcal{N}_0 \int d^N\phi_0 \, exp\left[-V_d\left(\frac{\lambda}{8N}|\phi_0|^4 + J_{0a}\phi_{0a}\right)\right]$$

- These expectation values are computed exactly

$$\langle\phi_0^{2p}\rangle_0 = \frac{\int_0^\infty d\phi_0\, \phi_0^{N-1+2p} e^{-\frac{V_d\lambda}{8N}\phi_0^4}}{\int_0^\infty d\phi_0\, \phi_0^{N-1}\, e^{-\frac{V_d\lambda}{8N}\phi_0^4}} = 2^{\frac{3p}{2}}\left(\frac{N}{V_d\lambda}\right)^{\frac{p}{2}} \frac{\Gamma\left[\frac{N+2p}{4}\right]}{\Gamma\left[\frac{N}{4}\right]}$$

Dinamical Mass ($d = 4$)

$$m^2_{dyn,0} = \frac{\sqrt{3\lambda}H^2}{8\pi}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \quad (N=1) \quad ; \quad m^2_{dyn,0} = \frac{\sqrt{3\lambda}H^2}{4\pi} \quad (N\to\infty)$$

Computing corrections from UV (inhomogeneous) modes

$N = 1$ Beneke & Moch (2012)

$N > 1$: López Nacir, Mazzitelli, LGT (2016)

- Perturbative treatment of the UV modes

$$S_{int} = \frac{\lambda}{8N}\int d^dx\sqrt{g}\left[|\phi_0|^4 + 2|\phi_0|^2|\hat{\phi}|^2 + 4(\phi_0\cdot\hat{\phi})^2 + 4(\phi_0\cdot\hat{\phi})|\hat{\phi}|^2 + |\hat{\phi}|^4\right]$$

$$= \boxed{\frac{V_d\lambda}{8N}|\phi_0|^4} + \tilde{S}_{int}[\phi_0,\hat{\phi}]$$

$$Z[J_0,\hat{J}] = \mathcal{N}\int_{-\infty}^{\infty} d\phi_0 \int \mathcal{D}\hat{\phi}\, exp\left[-S_E - \int d^dx\,\sqrt{g}\,(J_0\phi_0 + \hat{J}\hat{\phi})\right]$$

$$= exp\left(-\tilde{S}_{int}\left[\frac{\delta}{\delta J_0},\frac{\delta}{\delta\hat{J}}\right]\right) Z_0[J_0]\hat{Z}_{free}[\hat{J}]$$

- We expand on $\tilde{S}_{int}$, while treating $\frac{V_d\lambda}{4!}\phi_0^4$ exactly in $Z_0[J_0]$

<u>Example for $m = 0$:</u>

Each ϕ_0 scales as $\lambda^{-1/4}$

Each $\hat{\phi}$ scales as 1 $\quad\Longrightarrow\quad \lambda|\phi_0|^2|\hat{\phi}|^2$ scales as $\sqrt{\lambda}$

2-point functions in a double expansion in $\sqrt{\lambda}$ and $1/N$ $(m=0)$

IR part

$$\langle\phi_{0a}\phi_{0b}\rangle = \delta_{ab}\left[\sqrt{\frac{2}{V_d\lambda}} - \frac{1}{2}[\hat{G}^{(0)}]_{ren} + \frac{1}{8}\sqrt{\frac{V_d\lambda}{2}}[\hat{G}^{(0)}]^2_{ren} - \frac{1}{2}\sqrt{\frac{\lambda}{2V_d}}\left(\frac{\partial[\hat{G}^{(m)}]}{\partial m^2}\right)_{0,fin}\right] + \mathcal{O}(N^{-1})$$

UV part

$$\langle\hat{\phi}_a(x)\hat{\phi}_b(x')\rangle = \delta_{ab}\left\{\hat{G}^{(0)}(x,x') + \left(\frac{\partial\hat{G}^{(m)}(x,x')}{\partial m^2}\right)_0\left[\sqrt{\frac{\lambda}{2V_d}} + \frac{\lambda}{4}[\hat{G}^{(0)}]_{ren}\right] + \frac{1}{2}\left(\frac{\partial^2\hat{G}^{(m)}(x,x')}{\partial(m^2)^2}\right)_0\left[\sqrt{\frac{\lambda}{2V_d}}\right]^2\right\} + \mathcal{O}(N^{-1})$$

where we used that

$$\int\ldots\int_{x_2,\ldots,x_{k-1}}\hat{G}^{(m)}(x_1,x_2)\ldots\hat{G}^{(m)}(x_{k-1},x_k) = \frac{(-1)^k}{(k-2)!}\frac{\partial^{k-2}\hat{G}^{(m)}(x_1,x_k)}{\partial(m^2)^{k-2}}$$

- Agreement with Lorentzian calculations at the given order in $\sqrt{\lambda}$ and $1/N$.

Back to Lorentzian spacetime

- Analytical continuation:

$$\begin{aligned} y(x,x') &\longrightarrow \frac{-\left(|\eta-\eta'|-i\epsilon\right)^2+|\vec{x}-\vec{x'}|^2}{\eta\eta'} \\ \hat{G}(x,x') &\longrightarrow \hat{G}_F(x,x') \end{aligned}$$

- Points x and x' can now be infinitelly far apart (unlike on the sphere)

- Since the results are built up from massless free propagators $\hat{G}^{(0)}(x,x')$ (and its derivatives) $\Longrightarrow$ The are still IR issues at large separations and/or late times

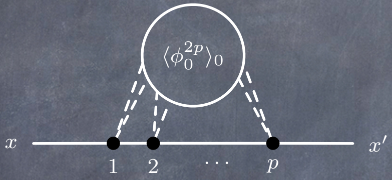
- Is there a way of further resumming contributions that afect the behavior in that regime?

Resumming the leading IR secular terms to the two-point functions

- We focus on the bi-quadratic interaction terms:

$$\frac{\lambda}{4N}\int d^d\sqrt{g}\,\left[|\phi_0|^2|\hat{\phi}|^2+2(\phi_0\cdot\hat{\phi})\right]$$

- Consider a generic correction to the 2-point functions with p-insertions of ϕ_0^2:



$$\sim \lambda^p\langle\phi_0^{2p}\rangle_0\,\frac{\partial^p\hat{G}^{(m)}(x,x')}{\partial(m^2)^p}\bigg|_0$$

- This leads to:

$$\langle\hat{\phi}_a(x)\hat{\phi}_b(x')\rangle^{(0)} \quad = \quad \delta_{ab}\left\{\sum_{p=0}^{\infty}\frac{1}{p!}\frac{\partial^p\hat{G}^{(m)}(x,x')}{\partial(m^2)^p}\bigg|_0\left(\frac{\lambda}{2N}\right)^p\left[1+\frac{(3^p-1)}{N}\right]\langle\phi_0^{2p}\rangle_0\right\}$$

- Which can be resummed order by order in $1/N$ (expanding de factor $\langle\phi_0^{2p}\rangle_0$). For example at leading order:

$$\langle\hat{\phi}_a(x)\hat{\phi}_b(x')\rangle^{(0)}=\hat{G}^{(m_{dyn,0})}(x,x')+\mathcal{O}(N^{-1},\sqrt{\lambda})$$

with $m^2_{dyn,0}=\sqrt{\lambda/2V_d}$

Resumming the leading IR secular terms to the two-point functions

Redefinition of perturbation theory (once again)

- ϕ_0-dependent mass term for $\hat{\phi}$:

$$\frac{\lambda}{4N}\int d^d\sqrt{g}\,m^2_{ab}(\phi_0)\hat{\phi}_a\hat{\phi}_d$$

- Generating functional

$$\begin{aligned} Z[J_0,\hat{J}] &= \mathcal{N}e^{-\tilde{\hat{S}}_{int}\left[\frac{\delta}{\delta J_0},\frac{\delta}{\delta\hat{J}}\right]}\int d^N\phi_0\,e^{-\left[\frac{\lambda V_d}{8N}|\phi_0|^4+V_dJ_{0a}\phi_{0a}\right]} \\ &\times\int\mathcal{D}\hat{\phi}\,\exp\left(-\frac{1}{2}\iint_{x,y}\hat{\phi}_a\hat{G}^{-1}_{ab}(\phi_0)\hat{\phi}_b+\int_x\hat{J}_a\hat{\phi}_a\right) \end{aligned}$$

with

$$\hat{G}^{-1}_{ab}(\phi_0)(x,x')=\left[-\Box\delta_{ab}+m^2_{ab}(\phi_0)\right]\frac{\delta^{(d)}(x-x')}{\sqrt{g}}$$

- The remaining interaction terms $\phi_0\hat{\phi}^3$ and $\hat{\phi}^4$ are contained in $\tilde{\hat{S}}_{int}$ and are treated pertubatively

Example:

Up to order λ we only need the first perturbative correction coming from the term $\frac{\lambda}{8N}|\hat{\phi}|^4$

$$\langle \hat{\phi}_a(x)\hat{\phi}_b(x')\rangle^{(1)} = \frac{1}{Z^{(1)}[0,0]} \left. \frac{\delta^2 Z^{(1)}[J_0,\hat{J}]}{\delta \hat{J}_a(x)\delta \hat{J}_b(x')} \right|_{J_0,\hat{J}=0}$$

$$\begin{aligned}\langle \hat{\phi}_a(x)\hat{\phi}_b(x')\rangle^{(0)} = \; & \delta_{ab}\Bigg\{ \sum_{p=0}^{\infty} \frac{1}{p!} \left. \frac{\partial^p \hat{G}^{(m)}(x,x')}{\partial (m^2)^p}\right|_0 \left(\frac{\lambda}{2N}\right)^p \left[1 + \frac{(3^p - 1)}{N}\right] \langle \phi_0^{2p}\rangle_0 \\ & - \frac{\lambda^2 (N+2)}{4N} V_d [\hat{G}^{(0)}]_{ren} \times \\ & \sum_{p=0}^{\infty} \frac{1}{p!} \left. \frac{\partial^p \hat{G}^{(m)}(x,x')}{\partial (m^2)^p}\right|_0 \left(\frac{\lambda}{2N}\right)^p \left[1 + \frac{(3^p - 1)}{N}\right] \left[\langle \phi_0^{2(p+1)}\rangle_0 - \langle \phi_0^2\rangle_0 \langle \phi_0^{2p}\rangle_0\right] \Bigg\}\end{aligned}$$

and

$$\begin{aligned}\Delta \langle \hat{\phi}_a(x)\hat{\phi}_b(x')\rangle = \; & \delta_{ab} \frac{\lambda (N+2)}{2N} [\hat{G}^{(0)}]_{ren} \\ & \times \sum_{p=0}^{\infty} \frac{1}{p!} \left. \frac{\partial^{p+1} \hat{G}^{(m)}(x,x')}{\partial (m^2)^{p+1}}\right|_0 \left(\frac{\lambda}{2N}\right)^p \left[1 + \frac{(3^p - 1)}{N}\right] \langle \phi_0^{2p}\rangle .\end{aligned}$$

Result

$$\langle \hat{\phi}_a(x)\hat{\phi}_b(x')\rangle^{(1)} = \delta_{ab} \quad \Bigg\{ \quad \hat{G}^{(m)}(x,x') + \frac{\lambda}{4}[\hat{G}^{(0)}]_{ren}\frac{\partial \hat{G}^{(m)}(x,x')}{\partial m^2}$$
$$+\frac{1}{2N}\Bigg[2\hat{G}^{(\sqrt{3}m)}(x,x') - 2\hat{G}^{(m)}(x,x')$$
$$-\sqrt{\frac{\lambda}{2V_d}}\frac{\partial \hat{G}^{(m)}(x,x')}{\partial m^2} + \frac{\lambda}{2V_d}\frac{\partial^2 \hat{G}^{(m)}(x,x')}{\partial (m^2)^2}$$
$$+\frac{\lambda}{4}[\hat{G}^{(0)}]_{ren}\left(7\frac{\partial \hat{G}^{(m)}(x,x')}{\partial m^2} - 6\frac{\partial \hat{G}^{(\sqrt{3}m)}(x,x')}{\partial m^2}\right)\Bigg]\Bigg\}_{m_{dyn,0}}$$

Remarks

- All free propagators that build up the expression now have masses $\sim \sqrt{\lambda}$
  $\Longrightarrow$ $\boxed{\text{Decay at large distances and/or late times!!}}$
- Other recent results in Lorentzian QFT (Gautier & Serreau (2015)) obtain a decay in the massless case in the IR limit. But their approximations are not systematic.
- Our results are consistent with, and improve on, the know Lorentzian calculations.
- It is possible to go to higher orders in both $\sqrt{\lambda}$ and $1/N$ in a systematic way

Summary

- The QFT in euclidean de Sitter space with a proper treatment of the zero mode, allows to recover known non-perturbative results in the IR limit (dynamical mass generation).

- Furthermore, the computation of corrections from the UV modes is systematic order by order in $\sqrt{\lambda}$ and $1/N$, unlike other methods.

- However, the nonperturbative treatment of the zero mode alone does not solve the IR problems associated with large separations and/or late times.

- As we have shown, it is possible to also resum the leading secular terms in order to obtain a proper decay in that regime

Outlook

- Computation of higher correlation functions

  $$\langle \phi(x_1) \ldots \phi(x_n) \rangle$$

  How to do the analytical continuation?
- The vacumm expectation value of the stress-energy tensor

  $$\langle T_{\mu\nu} \rangle$$

  as a source of the Einstein semiclassical equations.
  Stress-tensor fluctuations

  $$\langle T_{\mu\nu}(x) T_{\mu'\nu'}(x') \rangle$$

  to study the validity of the semiclassical treatment
- Generalization to $m^2 < 0$ to study spontaneous symmetry breaking and symmetry restoration

  ϕ_0 scales as $\lambda^{-1/2} \Longrightarrow \lambda |\phi_0|^2 |\hat{\phi}|^2 \sim \lambda^0$
- In the Lorenztian QFT, there is a continuum of modes. How can we define the analogue of the zero mode? Is it posible to do a systematic nonperturbative treatment as in the Euclidean case?

THANK YOU FOR YOUR ATTENTION!!