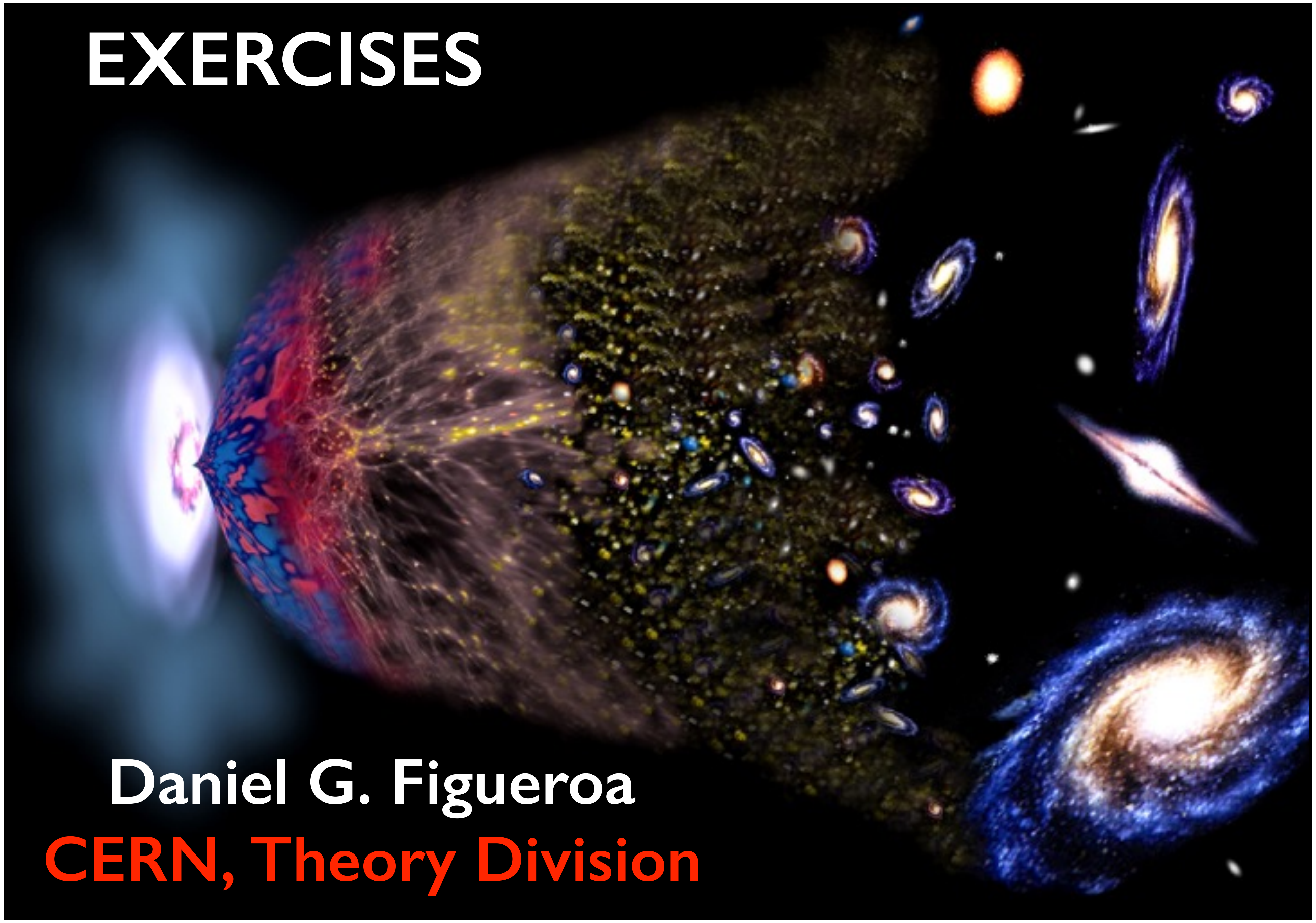
GRAVITATIONAL WAVES
PROBE OF THE EARLY UNIVERSE



School on Gravitational Waves for Cosmology and Astrophysics, Benasque, May 28 - June 10, 2017

Derivation of Inflationary Tensor Spectrum

"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."

Sidney Coleman

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1) Review of QM Harmonic Oscillator

2) Tensor modes in (quasi-) de Sitter

1) Review of QM Harmonic Oscillator

$$S = \int \mathrm{d}t \, \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2(t) x^2 \right) \equiv \int \mathrm{d}t \, L$$

$$\frac{\delta S}{\delta x} = 0 \quad \Rightarrow \quad \boxed{\ddot{x} + \omega^2(t)\, x = 0}$$

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$$p \equiv \frac{dL}{d\dot{x}} = \dot{x} \quad \Longrightarrow \quad \text{Q.M.:} \; \boxed{[\hat{x}, \hat{p}] = i\hbar}$$ **(quantization)**

$$\left([x(t), \dot{x}(t)] = i\hbar \right)$$

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$$\boxed{[\hat{a}, \hat{a}^\dagger] = \mathbf{C}} \quad \Longleftrightarrow \quad \langle v, v\rangle \equiv \frac{1}{\mathbf{C}}$$

C > 0

(const.)

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choice of vacuum

(not completely fixed)

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$$\boxed{\hat{a}|0\rangle = 0} \longrightarrow \quad |n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \quad , \quad \hat{N}|n\rangle = n|n\rangle$$

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Hamiltonian:

$$\begin{aligned} \hat{H} &= \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2 \\ &= \frac{1}{2}\left[(\dot{v}^2 + \omega^2 v^2)\hat{a}\hat{a} + (\dot{v}^2 + \omega^2 v^2)^*\, \hat{a}^\dagger\hat{a}^\dagger + (|\dot{v}|^2 + \omega^2|v|^2)(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\right] \end{aligned}$$

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2) compatible with 1) $\Longleftrightarrow \boxed{\omega(t) = \omega = const.}$

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IF $\boxed{\omega(t) = \omega = const.}$ $\xrightarrow[2)]{}$ $\dot{v} = \pm i\omega v$ $\Rightarrow$

$$\Rightarrow \quad \langle v, v \rangle = \mp \frac{2\omega}{\hbar} |v|^2$$

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$\Rightarrow$ $\langle v, v\rangle = \cancel{\mp}\frac{2\omega}{\hbar}|v|^2$ $\Rightarrow$ $\left\{ \begin{array}{l} \dot{v} = -i\omega v \\ |v| = \sqrt{\frac{\hbar}{2\omega}} \end{array} \right\}$

$(\langle v, v\rangle = 1)$

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$$\underset{(\langle v,v\rangle = 1)}{\Longrightarrow} \langle v, v\rangle = \cancel{\mp}\frac{2\omega}{\hbar}|v|^2 \Longrightarrow \left\{ \begin{array}{l} \dot{v} = -i\omega v \\ |v| = \sqrt{\dfrac{\hbar}{2\omega}} \end{array} \right\} \Rightarrow$$

$$\Longrightarrow \boxed{v(t) = \sqrt{\frac{\hbar}{2\omega}}\, e^{-i\omega t}}$$

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$$\boxed{v(t) = \sqrt{\frac{\hbar}{2\omega}}\, e^{-i\omega t}} \longrightarrow \boxed{\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)}$$

vacuum $|0\rangle$ is the state of
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$$\Downarrow$$

$$\boxed{\langle|\hat{x}|^2\rangle} \equiv \langle 0|\hat{x}^\dagger\hat{x}|0\rangle = |v(\omega,t)|^2 \boxed{= \frac{\hbar}{2\omega}}$$

Vacuum unique thanks to $\boxed{\omega(t) = \omega = const.}$

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$$S_g^{(2)} = -\frac{m_{\mathrm{Pl}}^2}{8} \int d\eta \, d^3\mathbf{x} \, a^2(\eta) \, \eta^{\mu\nu} \, \partial_\mu h_{ij} \, \partial_\nu h_{ij}$$

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Let's quantize ($\hbar = 1$): $\pi_r(\mathbf{x},\eta) \equiv v_r'(\mathbf{x},\eta)$

$$\boxed{\begin{aligned} [\hat{v}_r(\mathbf{x},\eta)\,,\, \hat{\pi}_{r'}(\mathbf{x}',\eta)] &= i\,\delta_{rr'}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}') \\ [\hat{v}_r(\mathbf{x},\eta)\,,\, \hat{v}_{r'}(\mathbf{x}',\eta)] &= [\hat{\pi}_r(\mathbf{x},\eta)\,,\, \hat{\pi}_{r'}(\mathbf{x}',\eta)] = 0 \end{aligned}}$$

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- -

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C > 0 (const.)

$$\boxed{\left[\hat{a}_{\mathbf{k}r}\ ,\ \hat{a}^+_{\mathbf{k}'r'}\right] = \mathbf{C}\, \delta_{rr'}\, \delta^{(3)}(\mathbf{k} - \mathbf{k}')} \Longleftrightarrow \boxed{v_k\, v_k'^* - v_k^*\, v_k' = i\, \frac{1}{\mathbf{C}}}$$

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time-dependent harmonic-oscillator !

so is there a unique vacuum ?

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$$\hat{a}_{\mathbf{k}r}\, |0\rangle = 0, \implies \hat{H} v_k^{(+)} = +k v_k^{(+)}$$

unique vacuum !

(positive freq. modes)

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$$v_k \simeq \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad \text{for} \qquad k \gg aH \qquad \text{sub-Hubble modes}$$

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$$v_k \simeq e^{i\frac{\pi}{2}\left(\nu-\frac{1}{2}\right)} 2^{\left(\nu-\frac{3}{2}\right)} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\eta)^{\frac{1}{2}-\nu} \;, \qquad \text{for} \; -k\eta \ll 1\,,$$

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$$\hat{h}_{ij}(\mathbf{x},\eta) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(h_k(\eta)\, e^{i\mathbf{kx}}\, \hat{a}_{\mathbf{k}r} + h_k^*(\eta)\, e^{-i\mathbf{kx}}\, \hat{a}_{\mathbf{k}r}^{+} \right) e_{ij}^r(\hat{\mathbf{k}})$$

$$v_r(\mathbf{k},\eta) = \frac{m_{\rm Pl}}{\sqrt{2}}\, a(\eta)\, h_r(\mathbf{k},\eta) \;\Rightarrow\; \boxed{|h_k(\eta)| \simeq \frac{H}{m_{\rm Pl}\, k^{3/2}}\, f(\epsilon) \left(\frac{k}{aH}\right)^{-\epsilon} ,} \quad \boxed{k \ll aH}$$

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$$\langle 0|\hat{h}_{ij}(\mathbf{x},\eta)\, \hat{h}_{ij}(\mathbf{x},\eta)|0\rangle = \int \frac{dk}{k}\, \Delta_h^2 \;\Rightarrow\; \boxed{\Delta_h^2 \simeq \frac{2}{\pi^2} \frac{H^2}{m_{\rm Pl}^2} f^2(\epsilon) \left(\frac{k}{aH}\right)^{-2\epsilon}}$$

super-Hubble spectrum

$$\left(f(\epsilon) \equiv 2^\epsilon (1-\epsilon)^{1+\epsilon} \frac{\Gamma\left(\frac{3}{2}+\epsilon\right)}{\Gamma\left(\frac{3}{2}\right)} \simeq 1 - \left(1 - \ln(2) - \psi_0\left(\tfrac{3}{2}\right)\right)\epsilon \simeq 1 - 0.27\epsilon \right)$$