Data Analysis IV
Data Analysis for PTA

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Structure of the lectures

- DA I: Statistic basis for DA: Likelihood, frequentist/Bayesian
- GW Obs I: History, response to GW
- GW Obs II: LISA: LISAPathfinder, noises, ...
- DA II: 3 main classes of signal, parameter estimations, Fisher Matrix
- DA III: LISA DA: Global analaysis, MBHB, stochastic, ...
- GW Obs III: LIGO
- GW Obs IV: PTA
- DA IV: PTA data analysis

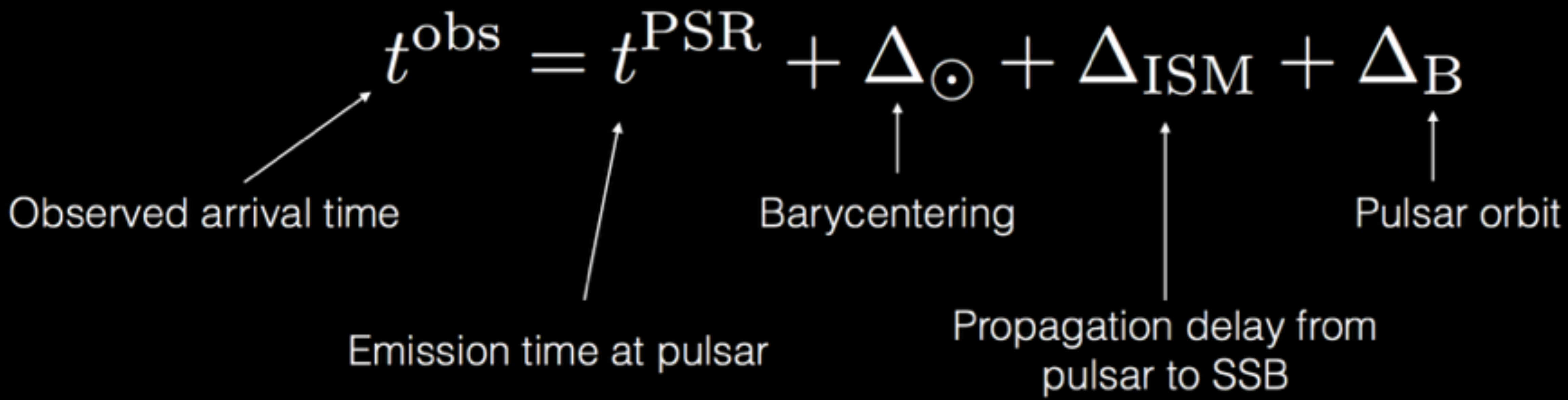
Overview

- PTA data
- Fitting the model of the pulsar
- Continuous Gravitational Wave :
  - Frequentist
  - Bayesian
  - Upper limit
- Stochastic background
- Global analysis

Pulsar Timing Array Data

Pulsar model

- Model of the observed arrival time of the pulsar and radio wave propagation:



$$t^{\mathrm{obs}} = t^{\mathrm{PSR}} + \Delta_{\odot} + \Delta_{\mathrm{ISM}} + \Delta_{\mathrm{B}}$$

PTA data

- **Barycentering:**

$$\Delta_\odot = \underset{\text{Clock corrections}}{\Delta_C} + \underset{\text{Atmospheric delays}}{\Delta_A} + \overset{\text{Solar system Einstein delay}}{\Delta_{\mathrm{E}_\odot}} + \underset{\text{SS Roemer delay}}{\Delta_{\mathrm{R}_\odot}} + \overset{\text{SS Shapiro delay}}{\Delta_{\mathrm{S}_\odot}} + \dots$$

- **Propagation:**

$$\Delta_{\mathrm{ISM}} = \underset{\text{Vacuum propagation}}{\Delta_{\mathrm{VP}}} + \overset{\text{Interstellar dispersion}}{\Delta_{\mathrm{ISD}}} + \underset{\text{Additional Frequency dependent delay}}{\Delta_{\mathrm{FDD}}} + \overset{\text{Einstein delay}}{\Delta_{\mathrm{ES}}} + \dots$$

- **Pulsar orbit:**

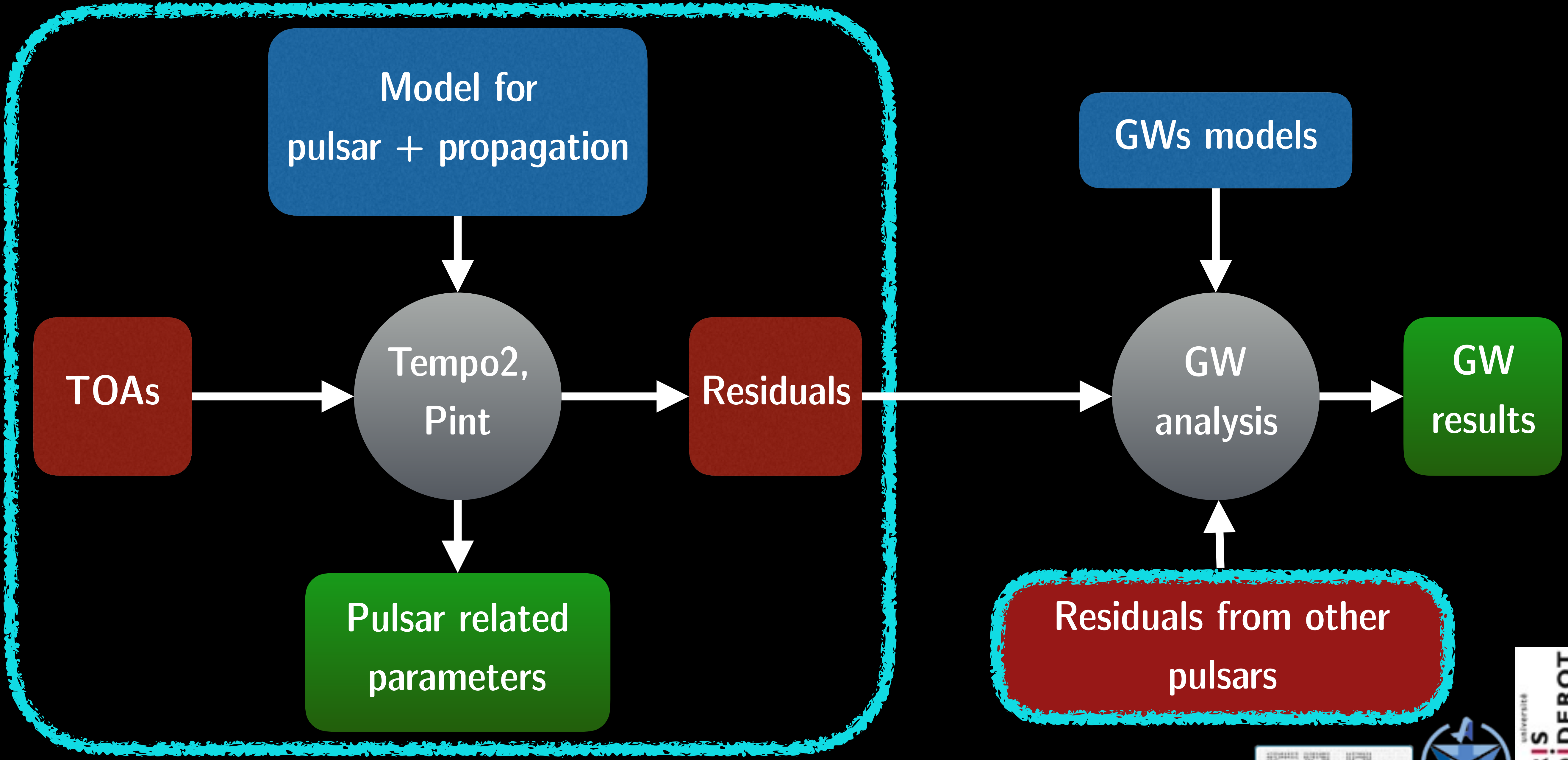
$$\Delta_{\mathrm{B}} = \underset{\text{Roemer delay}}{\Delta_{\mathrm{RB}}} + \overset{\text{Aberration}}{\Delta_{\mathrm{AB}}} + \underset{\text{Einstein delay}}{\Delta_{\mathrm{EB}}} + \overset{\text{Shapiro delay}}{\Delta_{\mathrm{SB}}}$$

- **Spin-down:**

$$P(t) = P_0 + \dot{P}_0\,(t - t_0) + \frac{1}{2}\ddot{P}_0\,(t - t_0)^2 + \dots$$

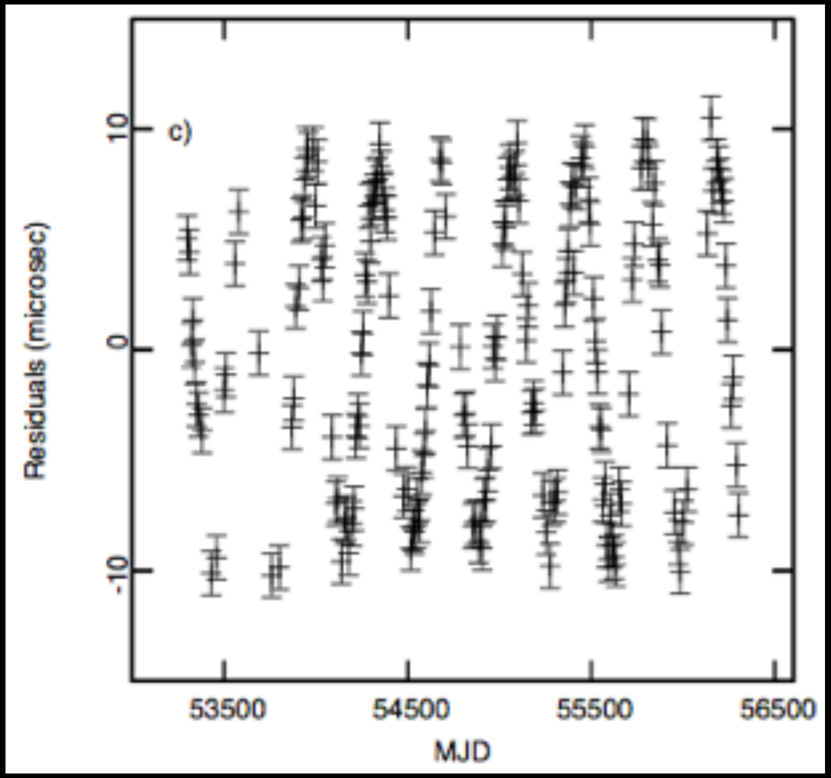
Residuals

- Time Of Arrivals (TOAs) are used to “fit” a model
- TOAs - “best model” = residuals => input to GW analysis

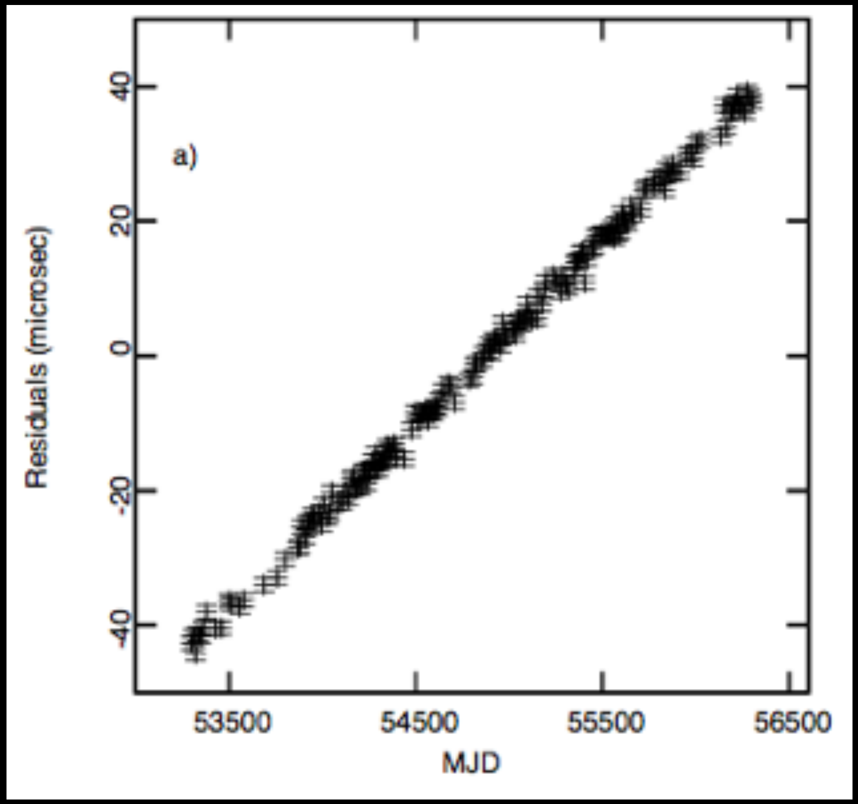


Residuals

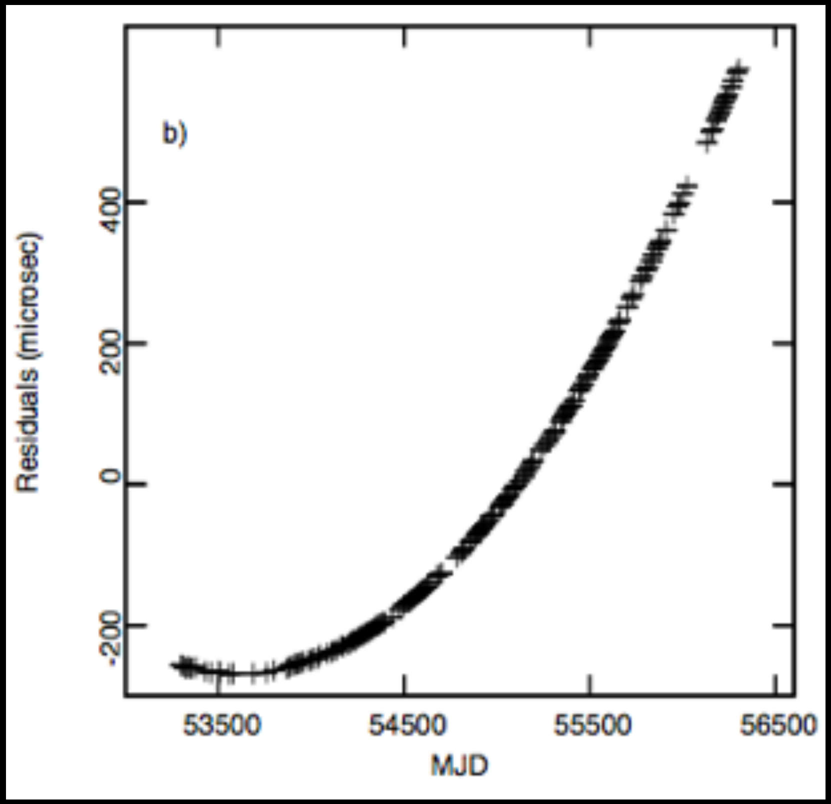
Error in position: annual effect



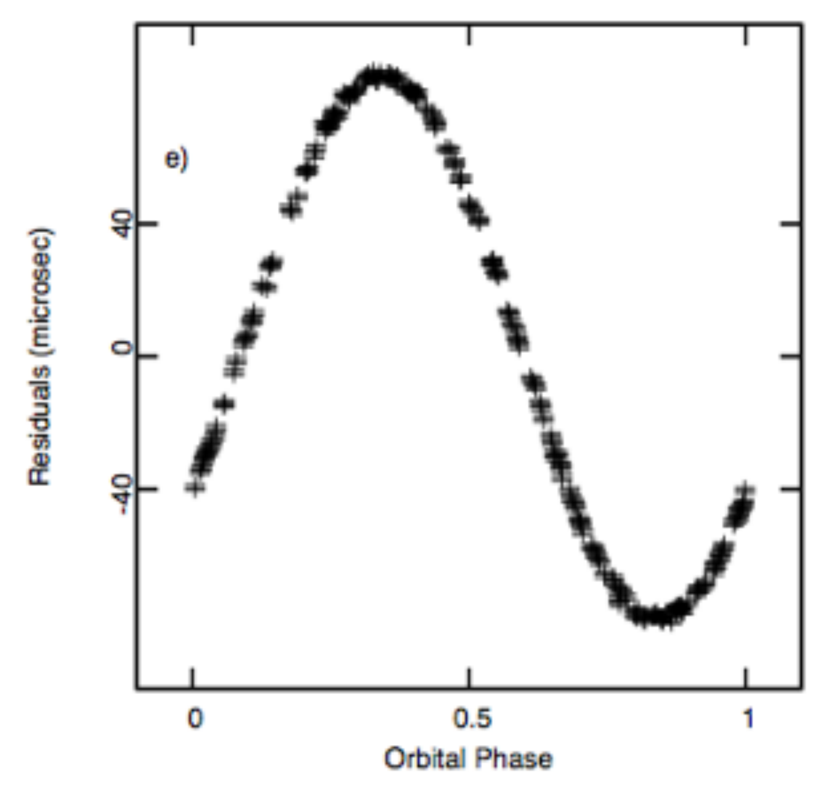
Error in period



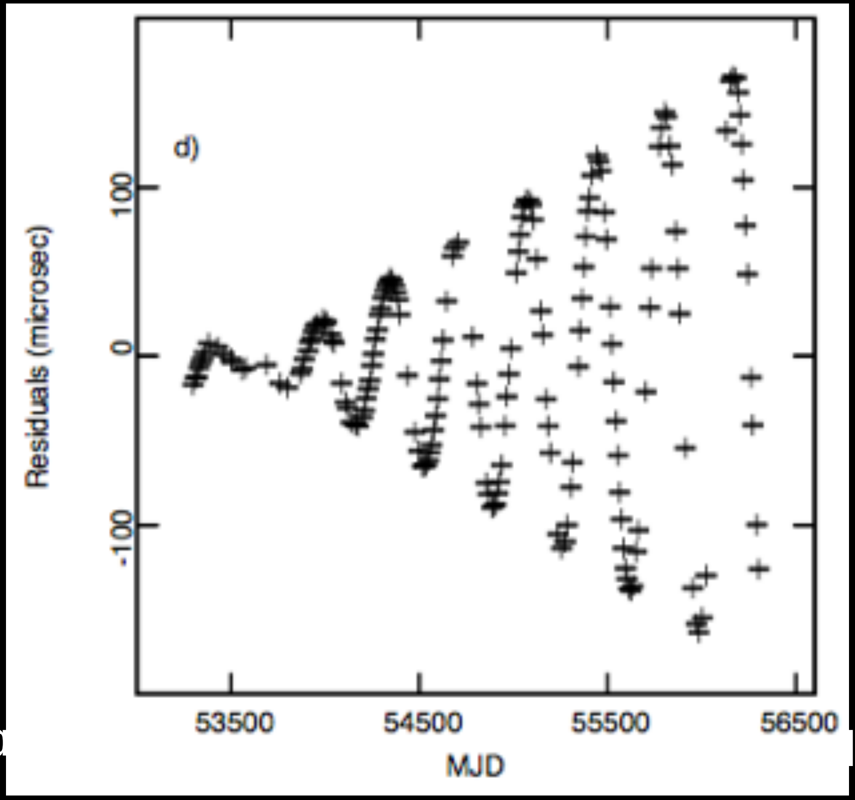
Error in period derivative



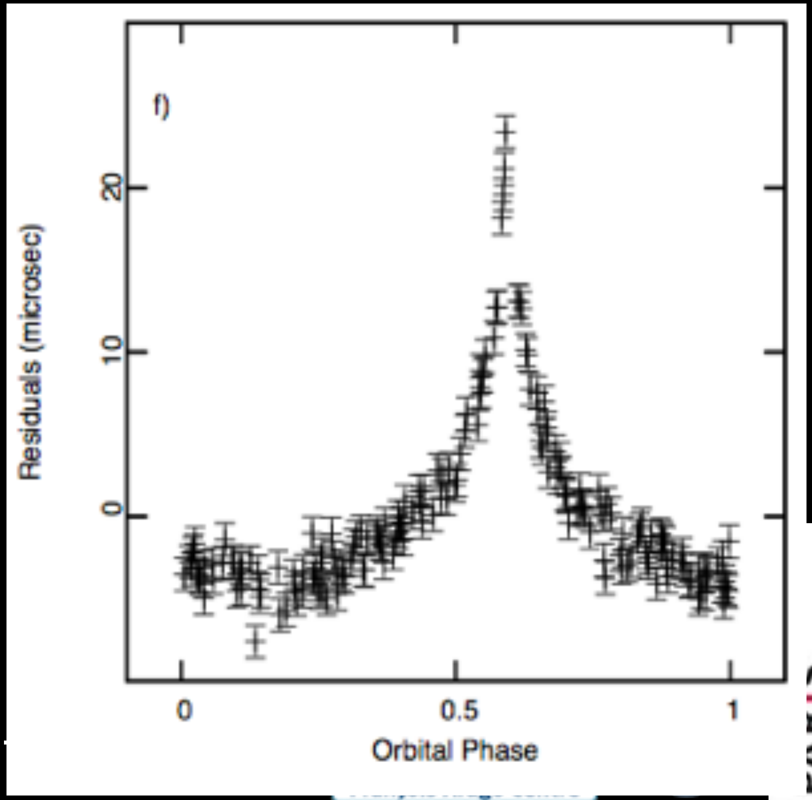
Error in orbital period



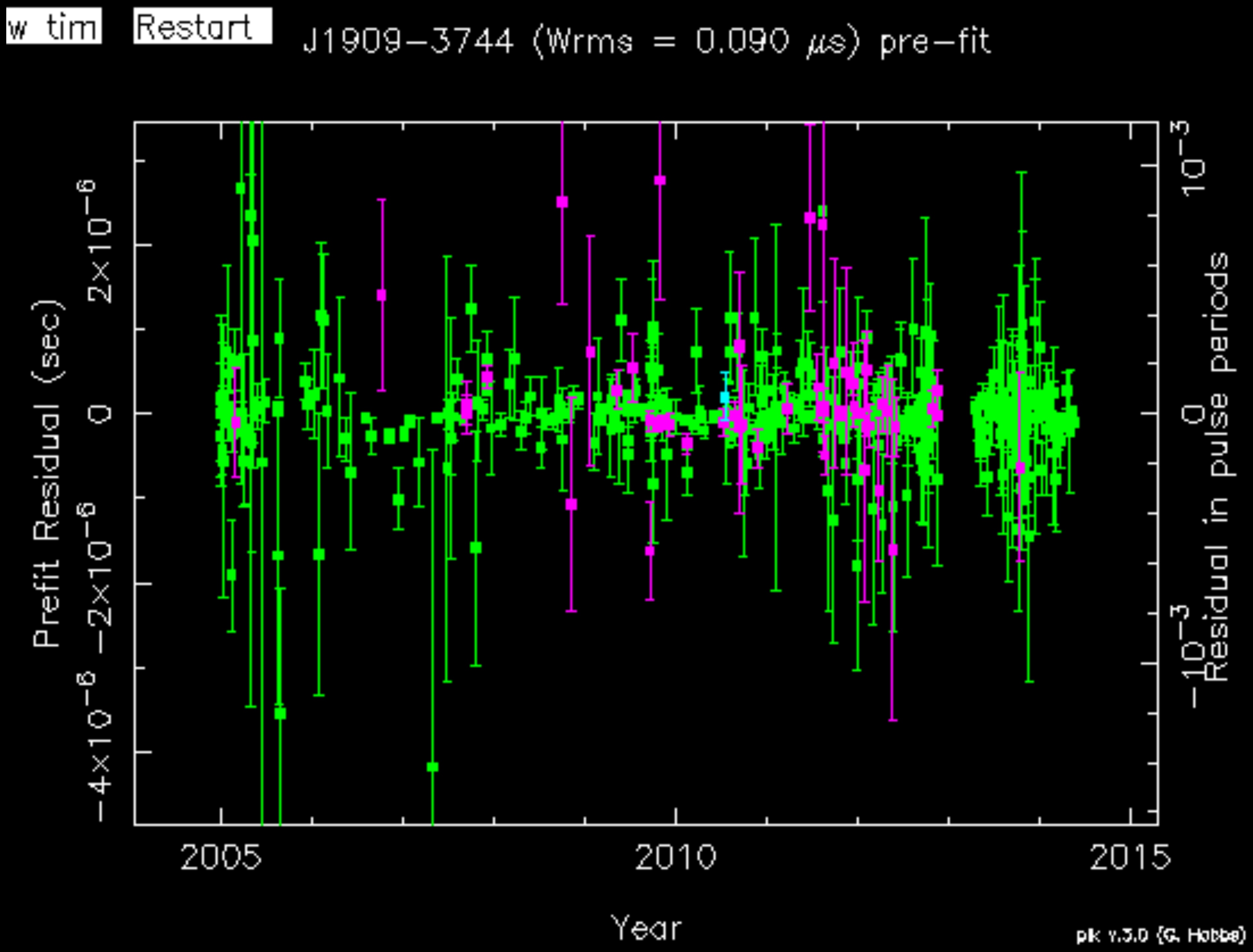
Error in proper motion



Without correction of Shapiro effect



Residuals


w tim
Restart
J1909−3744 (Wrms = 0.090 μs) pre−fit
Prefit Residual (sec)
-4×10^{-6}
-2×10^{-6}
0
2×10^{-6}
Residual in pulse periods
-10^{-3}
0
10^{-3}
2005
2010
2015
Year
plk v.3.0 (G. Hobbs)

Model imperfection

- The fitting of the model is not perfect in particular because we did not consider GWs as the same time as other “pulsars” parameters

- We need to integrate these imprecisions in GWs data analysis.
  - Global analysis ... many parameters
  - Analytical marginalization based on a [van Haasteren et al. 2009, 2012] (see later)

Data analysis

- The general likelihood is:

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^n det(C)}} \exp\left(-\frac{1}{2}\left(\vec{\delta t} - \vec{h}\right)^T C^{-1}\left(\vec{\delta t} - \vec{h}\right)\right)$$

  - n : number of data points
  - C : correlation matrix
  - $\vec{\delta t}$: residual = data points
  - $\vec{\theta}$: parameters

- Unequally sample data

Including pulsar model errors

[van Haasteren et al. 2009, 2012]

- During the fitting of the pulsar model some residual errors
- Assumption:
  - random Gaussian process $\vec{\delta t_i}^G$
  - + some contamination by several systematic signals with known functional forms $f_p(t_i)$ but a-priori unknown amplitudes ξ_p:

$$\vec{\delta t_i} = \vec{\delta t_i}^G + \sum_p \xi_p f_p(t_i)$$

$$= \vec{\delta t_i}^G + M\vec{\xi_p}$$

    - M : “design matrix” : n x m : m number of pulsar fitting parameters

Data analysis

[van Haasteren et al. 2009, 2012]

- Then the likelihood can be rewritten as

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}} \exp\left(-\frac{1}{2}\left(\vec{\delta t} - \vec{h}\right)^T G(G^T C G)^{-1} G^T \left(\vec{\delta t} - \vec{h}\right)\right)$$

- n : number of data points
- C : correlation matrix
- $\vec{\delta t}$: residual = data points
- $\vec{\theta}$: parameters
- G derived from M:
  - $M = U \Sigma V$: (n x n) (n x m) (m x m)
  - $U =$ (F G) : ((n x m) (n x (n-m)))

Likelihood

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}}$$
$$\times \exp\left(-\frac{1}{2}\left(\vec{\delta t} - \vec{h}\right)^T G(G^T C G)^{-1} G^T \left(\vec{\delta t} - \vec{h}\right)\right)$$

Likelihood

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}}$$

$$\times \exp\left(-\frac{1}{2} \left(\vec{\delta t} - \vec{h} \right)^T G (G^T C G)^{-1} G^T \left(\vec{\delta t} - \vec{h} \right) \right)$$

- **GW in**

Likelihood

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}} \times \exp\left(-\frac{1}{2} \left(\vec{\delta t} \boxed{- \vec{h}} \right)^T G(G^T C G)^{-1} G^T \left(\vec{\delta t} \boxed{- \vec{h}} \right) \right)$$

- **GW in**
  - h : **deterministic GW like binaries**

Likelihood

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}}$$

$$\times \exp\left(-\frac{1}{2} \left(\vec{\delta t} - \vec{h} \right)^T G (G^T C G)^{-1} G^T \left(\vec{\delta t} - \vec{h} \right) \right)$$

- **GW in**
  - h : **deterministic GW like binaries**
  - C : **stochastic background as correlation between pulsars**

Likelihood

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}}$$

$$\times \exp\left(-\frac{1}{2} \left(\vec{\delta t} - \vec{h} \right)^T G (G^T C G)^{-1} G^T \left(\vec{\delta t} - \vec{h} \right) \right)$$

- GW in
  - h : deterministic GW like binaries
  - C : stochastic background as correlation between pulsars
- Noises in C

Gravitational wave signal

- GW signal in the pulsar residual:

$$r(t) = \int_0^t \frac{\delta\nu}{\nu}(t')dt'$$

$$\frac{\delta\nu}{\nu}(t) = \frac{1}{2}\frac{\hat{n}^i\hat{n}^j}{1+\hat{n}.\hat{k}}(\underbrace{h_{ij}(t - L(1+\hat{n}.\hat{k}))}_{\boxed{\text{Strain of GW at the pulsar}}} - \underbrace{h_{ij}(t)}_{\boxed{\text{Strain of GW at Earth}}})$$

- n : direction of the pulsar
- L : distance Earth – pulsar
- k : direction of the GW propagation
- h_{ij} : GW strain

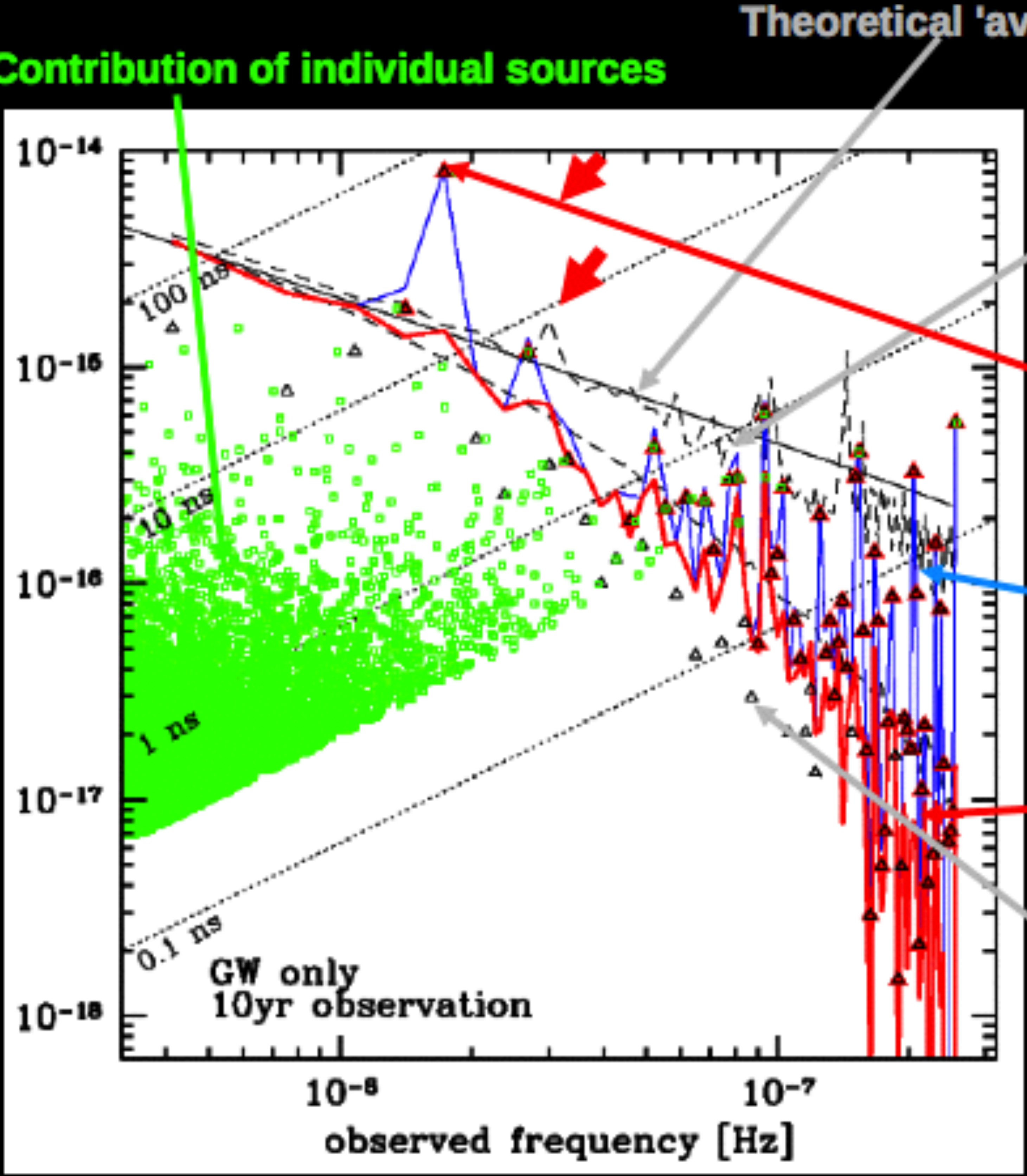
Continous GWs

Sources

- SuperMassive Black Hole Binaries (SMBHB)
  - 10^7 to 10^9 solar masses
  - far from the merger =>
    - quasi-monochromatic sources
    - spins can be neglected
- Continuous waves: sources that can be resolved individually

Distribution of sources

From A. Sesana



Contribution of individual sources

Theoretical 'average' spectrum

Spectrum averaged over 1000 Monte Carlo realizations

Resolvable systems: i.e. *systems whose signal is larger than the sum of all the other signals falling in their frequency bin*

Total signal

Unresolved background

Brightest sources in each frequency bin

Model of the signal

- GW waveform:

$$h_+(t) = \mathcal{A}\left(1+\cos^2 i\right)\cos\left(\phi(t)+\phi_0\right)$$

$$h_\times(t) = -2\mathcal{A}\cos i \sin\left(\phi(t)+\phi_0\right)$$

- with $\mathcal{A} = 2\frac{\mathcal{M}_c^{5/3}}{D_L}(\pi f)^{2/3}$
- Parameters:
  - $\mathcal{M}_c$: chirp mass
  - D_L : luminosity distance
  - $f = 2\pi\omega$: frequency of GW
  - i : inclination
  - $\phi(t)$: phase
  - ϕ_0 : initial phase

Model of the signal

- GW signal in the residual: 2 terms pulsar term & Earth term

$$r_a(t) = r_a^p(t) - r_a^e(t)$$

$$r_a^e(t) = \frac{\mathcal{A}}{\omega}\left\{(1+\cos^2\iota)F_a^+\left[\sin(\omega t+\Phi_0) - \sin\Phi_0\right] + 2\cos\iota F_a^\times\left[\cos(\omega t+\Phi_0) - \cos\Phi_0\right]\right\},$$

$$r_a^p(t) = \frac{\mathcal{A}_a}{\omega_a}\left\{(1+\cos^2\iota)F_a^+\left[\sin(\omega_a t+\Phi_a+\Phi_0) - \sin(\Phi_a+\Phi_0)\right] + 2\cos\iota F_a^\times\left[\cos(\omega_a t+\Phi_a+\Phi_0) - \cos(\Phi_a+\Phi_0)\right]\right\}.$$

$$F_a^+ = \frac{1}{2}\frac{(\hat{n}^a.\hat{p})^2 - (\hat{n}^a.\hat{q})^2}{1+\hat{n}^a.\hat{k}}$$

$$F_a^\times = \frac{(\hat{n}^a.\hat{p})(\hat{n}^a.\hat{q})}{1+\hat{n}^a.\hat{k}}$$

- Parameters
  - ϕ_0 : initial phase at Earth
  - ϕ_a : initial phase at pulsar
  - $F_a{}^+$, $F_a{}^\times$: beam patterns depending on direction & polarisation

Model of the signal

- Beam patterns $F_a{}^+$, $F_a{}^\times$ are:

$$F_a^+ = \frac{1}{2}\frac{(\hat{n}^a.\hat{p})^2 - (\hat{n}^a.\hat{q})^2}{1+\hat{n}^a.\hat{k}}$$

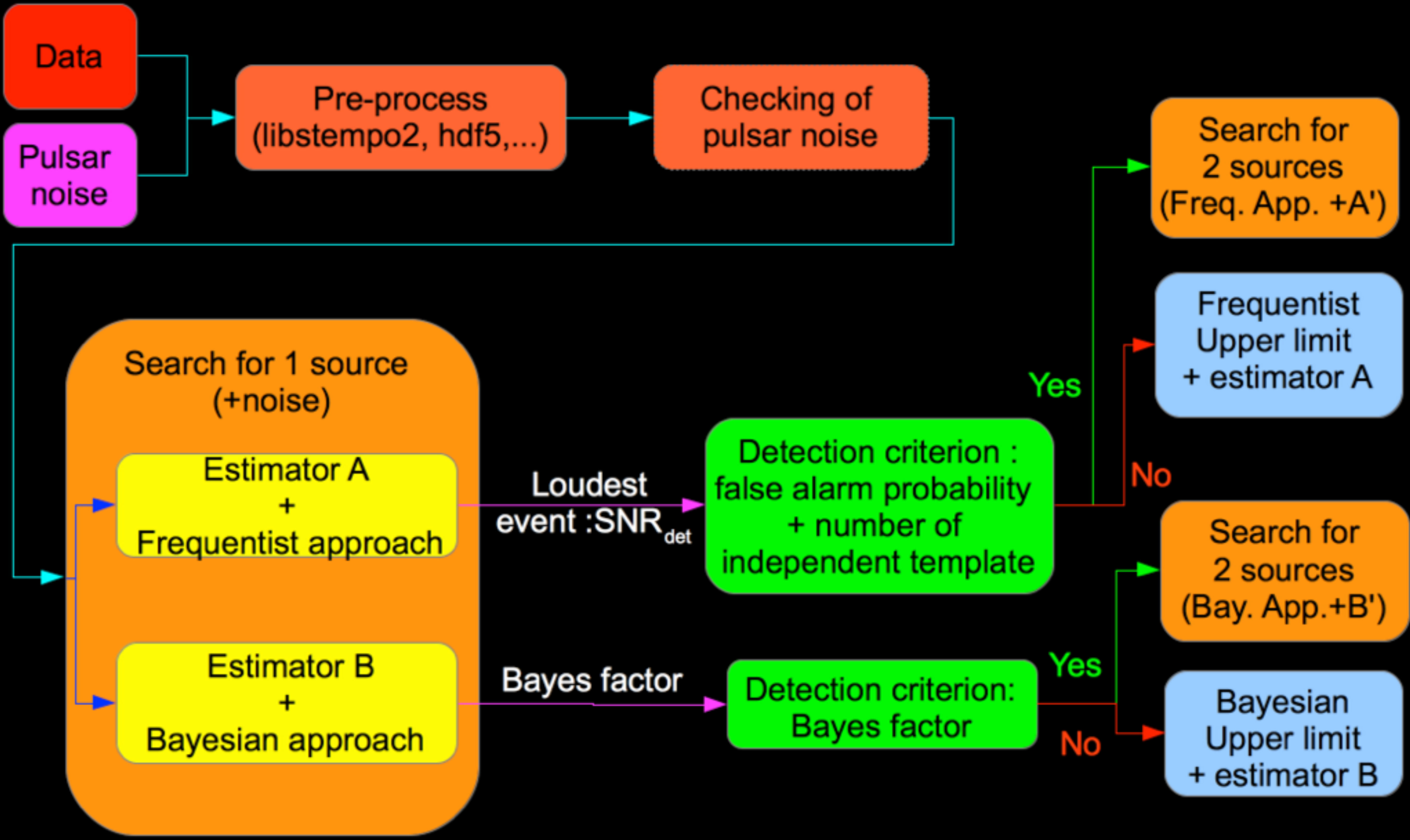
$$F_a^\times = \frac{(\hat{n}^a.\hat{p})(\hat{n}^a.\hat{q})}{1+\hat{n}^a.\hat{k}}$$

- with

$$\begin{aligned}
\hat{k} &= -\{\sin\theta_S\cos\phi_S, \sin\theta_S\sin\phi_S, \cos\theta_S\} \\
\hat{p} &= \hat{u}\cos\psi + \hat{v}\sin\psi \\
\hat{q} &= \hat{u}\cos\psi + \hat{v}\sin\psi \\
\hat{u} &= \{\cos\theta_S\cos\phi_S, \cos\theta_S\sin\phi_S, -\sin\theta_S\} \\
\hat{v} &= \{\sin\phi_S, -\cos\phi_S, 0\}
\end{aligned}$$

- Parameters:
  - θ_S, ϕ_S : equatorial sky position of GW source
  - ψ : polarisation
  - p^a : direction of the pulsar

DA pipeline (EPTA)



Continous GWs
Frequentist analysis

F-statistic Earth term only

- We consider only the Earth term which is the coherent term between all pulsars
- To apply F-statistic we want to rewrite the signal in the form

$$r_a^E(t) = \sum_j a_{(j)} h^a_{(j)}$$

Model of the signal

- Beam patterns $F_a{}^{+}$, $F_a{}^{\times}$ can be rewritten in the form:

$$F_a^{+} = F_c^a \cos(2\psi) + F_s^a \sin(2\psi)$$
$$F_a^{\times} = -F_s^a \cos(2\psi) + F_c^a \sin(2\psi)$$

- with

$$F_c^a = \left\{\frac{1}{4}(\sin^2(\theta_a) - 2\cos^2(\theta_a))\sin^2(\theta_S) - \right.$$
$$\frac{1}{2}\cos(\theta_a)\sin(\theta_a)\sin(2\theta_S)\cos(\phi_S - \phi_a) +$$
$$\left.\frac{1}{4}(1+\cos^2(\theta_S))\sin^2(\theta_a)\cos(2\phi_S - 2\phi_a)\right\}\frac{1}{1+\hat{n}^a.\hat{k}}$$
$$F_s^a = \{\cos(\theta_a)\sin(\theta_a)\sin(\theta_S)\sin(\phi_S - \phi_a) +$$
$$\left.\frac{1}{2}\sin^2(\theta_a)\cos(\theta_S)\sin(2\phi_a - 2\phi_S)\right\}\frac{1}{1+\hat{n}^a.\hat{k}}.$$

F-statistic Earth term only

- Thus, we have $r_a^E(t) = \sum_j a_{(j)} h_{(j)}^a$
- with

$$h_{(1)} = F_c^a \sin(\Phi(t)), \quad h_{(2)} = F_s^a \sin(\Phi(t)),$$
$$h_{(3)} = F_c^a \cos(\Phi(t)), \quad h_{(4)} = F_s^a \cos(\Phi(t)),$$

- and

$$a_{(1)} = \frac{\mathcal{A}}{2\pi f}[(1+\cos^2 \iota)\cos(2\psi)\cos(\Phi_0) - 2\cos\iota\sin(2\psi)\sin(\Phi_0)],$$
$$a_{(2)} = \frac{\mathcal{A}}{2\pi f}[(1+\cos^2 \iota)\sin(2\psi)\cos(\Phi_0) + 2\cos\iota\cos(2\psi)\sin(\Phi_0)],$$
$$a_{(3)} = \frac{\mathcal{A}}{2\pi f}[(1+\cos^2 \iota)\cos(2\psi)\sin(\Phi_0) + 2\cos\iota\sin(2\psi)\cos(\Phi_0)],$$
$$a_{(4)} = \frac{\mathcal{A}}{2\pi f}[(1+\cos^2 \iota)\sin(2\psi)\sin(\Phi_0) - 2\cos\iota\cos(2\psi)\cos(\Phi_0)].$$

F-statistic Earth term only

- We parametrize only the deterministic signal h, not the noises and the stochastic background

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}} \exp\left(-\frac{1}{2}\left(\vec{\delta t} - \vec{r}\right)^T G(G^T CG)^{-1} G^T \left(\vec{\delta t} - \vec{r}\right)\right)$$

=> the log likelihood is:

$$\log \mathcal{L} = < \delta\vec{t_a} | \vec{r_a} > - \frac{1}{2} < \vec{r_a} | \vec{r_a} >$$

with inner product: $< \vec{x} | \vec{y} > = \vec{x}\ G(G^T CG)^{-1} G^T\ \vec{y}$

F-statistic Earth term only

- Analytic maximisation of the likelihood over $a_{(j)}$: $\dfrac{\partial \log(\mathcal{L})}{\partial a_{(j)}} = 0$
- Solution: $a_{(j)} = M^{-1}_{kj} \; X_j$

with $X_j \equiv \sum_{\alpha=1}^{P} < x_\alpha | h^{\alpha}_{(j)} >$ and $M_{jk} \equiv \sum_{\alpha=1}^{P} < h^{\alpha}_{(j)} | h^{\alpha}_{(k)} >$

- New statistic: $\boxed{\mathcal{F}_e = \{\log(\mathcal{L})\}_{max} = \frac{1}{2} X_k M^{-1}_{jk} X_j}$
- To be maximized over the 3 remaining parameters (intrinsic):
  - Equatorial latitude: θ_S,
  - Equatorial latitude: ϕ_S
  - Frequency: f

Search algorithm MS-GA: Genetic Algorithm

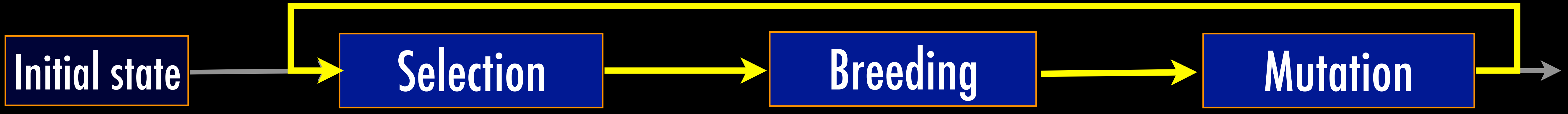
Description :

Evolution :

gene ⟷ parameter : binary representation (binary or Grey code):

parameter value ⟷ 0 1 1 1 0 0 1 0 1 0 0 1

organism ⟷ template described by a set of genes



Selection : Selection of parents for the breeding

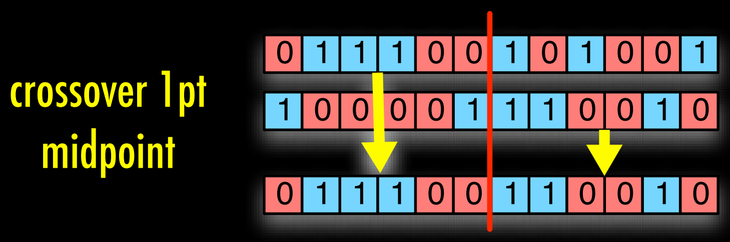
Probability of selecting one organism depend on Quality.

1. Quality $Q_i = Maximized\ Likelihood$,
2. Sort organisms by decreasing normalized quality
3. Roulette selection : Select one organism with probability equal to $Q_{Ni} / \Sigma_j^N Q_{Nj}$

Breeding : Making 1 child from the 2 selected parents

Mixing parts of corresponding parental genes. Several types of breeding :

- Crossover one point randomly chosen. Example :



- Others possibilities ...

Mutation : Change few bits in gene

Probability of change described by the 'Probability Mutation Rate' (PMR) $\in[0,1]$.

Several types of mutation :

- Mutate all the gene : If a random value α $\alpha<PMR$, mutate the gene. Several types :
  - Choose randomly N bits and flip them.
  - Complete random value
- Mutate bits independently : for each bit compare PMR to a random value α. If $\alpha<PMR$, flip bit (0→1 or 1→0).

Search algorithm MS-GA: Genetic Algorithm

Description : gene ⟷ parameter : binary representation (binary or Grey code):

parameter value ⟷ 0 1 1 1 0 0 1 0 1 0 0 1

organism ⟷ template described by a set of genes

Evolution :

Initial state

- GA can be seen as a method for clustering templates around the best one.
- GA has several possibilities of tuning which allows local or global searches.

Selection : Sele

the breeding

Probability

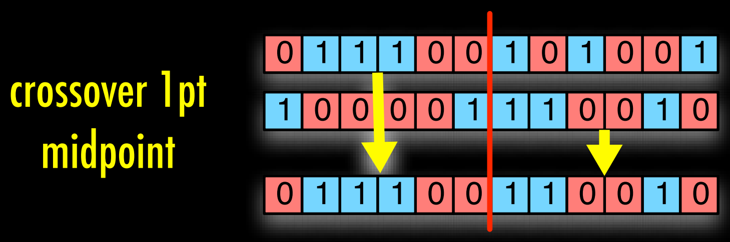
organism depe

1. Quality $Q_i =$

   $Likelihood,$
2. Sort organisms by decreasing normalized quality
3. Roulette selection : Select one organism with probability equal to $Q_{Ni} / \Sigma_j^N Q_{Nj}$

Dat

chosen. Example :



- Others possibilities ...

asq

gene

the

∈[0,1].

n value α

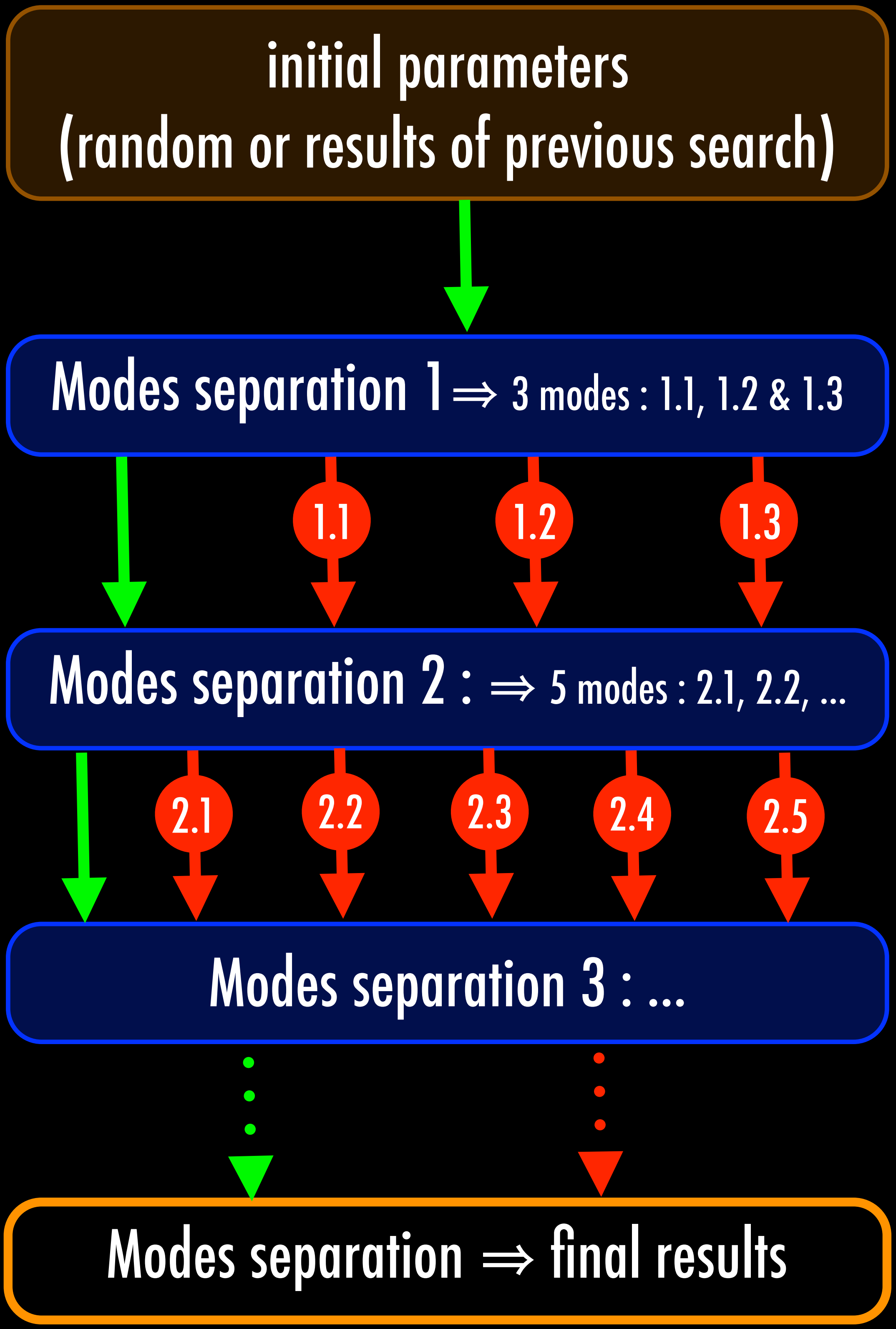
al types :

- Choose randomly N bits and flip them.
- Complete random value
- Mutate bits independently : for each bit compare PMR to a random value α. If $\alpha < PMR$, flip bit (0→1 or 1→0).

François Arago Centre

Search algorithm MS-GA: Multi Search

- Framework to run in parallel several dedicated search methods :
  - → "Global searches" looks for new good candidates avoiding the ones already found.
  - → "Local searches" explores in details the best candidates found at the previous step.
- "Modes separation" : the results are combined to find a new set of best candidates using some criterions (high SNR and not to close to the others).
- Each search is done by a GA with a special tuning or a parallel MCMC (EMCEE)



initial parameters
(random or results of previous search)

Modes separation 1 ⇒ 3 modes : 1.1, 1.2 & 1.3

Modes separation 2 : ⇒ 5 modes : 2.1, 2.2, ...

Modes separation 3 : ...

Modes separation ⇒ final results

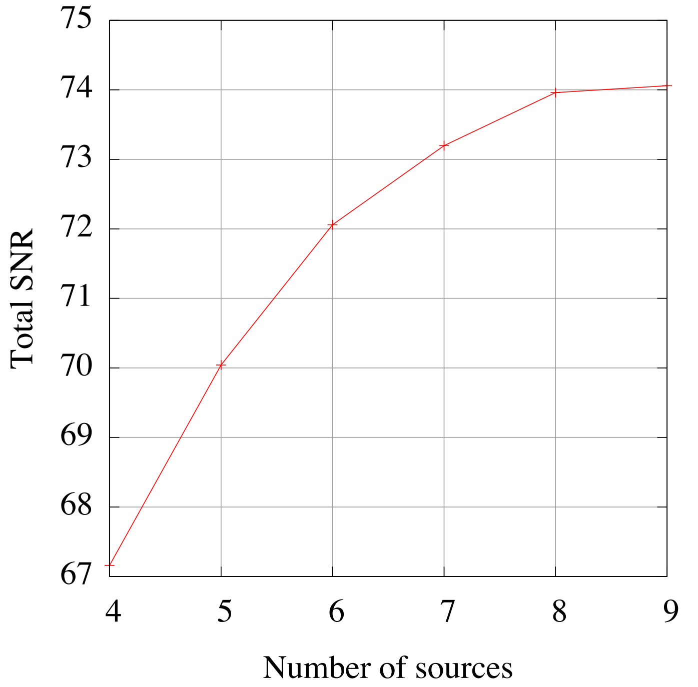
Individual sources and data

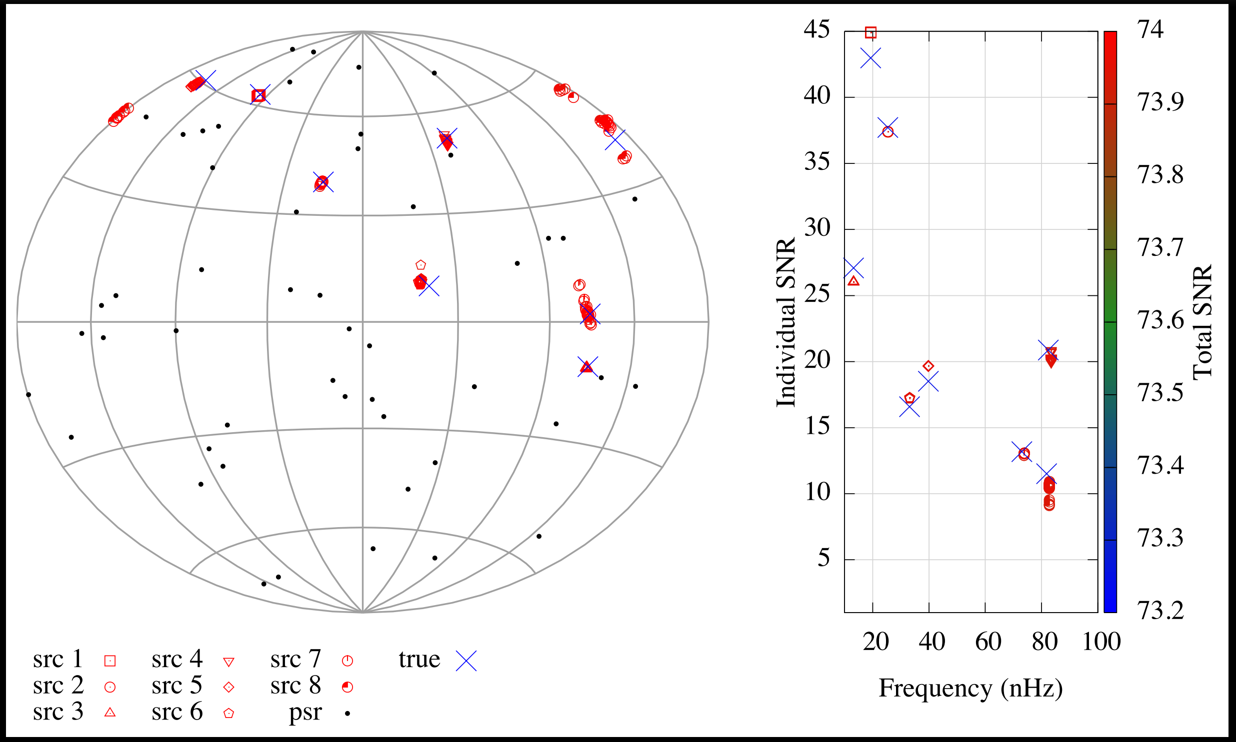
- Template :
  - Several individual sources : Earth term only, non-eccentric, fixed frequencies.
  - Fstatistic : analytic maximisation over 4 parameters (Petiteau & al. 2012 , Babak & Sesana 2012, Ellis & al. 2012) ⇒ search for Nsrc x 3 parameters (sky positions & frequencies).
  - GW background (GWB) and red-noise on pulsars (RN) can be taken into account in likelihood computation (use of design matrix based on Van Hasterren & Levin 2012) ...
    MS-GA can search for GWB + RN + individual source parameters (ex. in Stas Babak's talk).
- Search :
  - MS-GA coupled with several technics for removing «ghost detections» (checking of correlation between pulsars, high-pass filtering, cyclic removing of pulsars).
  - Characterisation of errors and likelihood distribution via MCMC (Antoine Lassus's talk).
- Data : simplified dataset or par/tim dataset (⇒ MS-GA can work with real dataset).

Results on simulated data

Npulsars	noise	dataset	Nsrc	ind. src SNR	signal	Results
30-50	noiseless or white 50-100 ns	simplified	1-5	> 10	earth term, same frequency	Pilot study Babak & Sesana 2012
30-50	white 30-200 ns	simplified	3-8	> 10	earth term, ≠ freq.	OK Petiteau & al. 2012

- MS-GA successfully identified all the injected sources in all datasets
- MS-GA found all source parameters : sky position offset by less than few degrees and frequencies found with precision better than 0.1 nHz (errors characterisation : Antoine Lassus' talk)





Statistics of Fe

- $2\,\mathcal{F}e$ is distributed as $\mathcal{X}^2$ with n degrees of freedom [Janaroski & Krolak LRR, 15 (2012)]

$$p_0'(\mathcal{F}) = p_{\chi^2}(2\mathcal{F}) = \frac{(2\mathcal{F})^{n/2-1} e^{-(2\mathcal{F})/2}}{2^{n/2}\ \Gamma(n/2)} = \frac{(\mathcal{F})^{n/2-1} e^{-\mathcal{F}}}{2\ \Gamma(n/2)}$$

- After normalization and approximations:

$$p_0(\mathcal{F}) = e^{-\mathcal{F}} \frac{\mathcal{F}^{n/2-1}}{(n/2-1)!} \left(\mathrm{exact} : \mathrm{e}^{-\mathcal{F}} \frac{\mathcal{F}^{\mathrm{n}/2-1}}{\Gamma(\mathrm{n}/2)} \right)$$

Statistics of Fe

- The false alarm probability, P_F that $\mathcal{F}_e$ exceed $\mathcal{F}_{th}$ when there is no signal is

$$P_F(\mathcal{F}_{th}) = \int_{\mathcal{F}_{th}}^{\infty} p_0(\mathcal{F}) d\mathcal{F}$$

$$= e^{-\mathcal{F}_{th}} \sum_{k=0}^{n/2-1} \frac{\mathcal{F}_{th}^k}{k!} \left(\mathrm{exact} : \frac{\Gamma(\mathrm{n}/2, \mathcal{F}_{\mathrm{th}})}{\Gamma(\mathrm{n}/2)} \right)$$

- Probability that $\mathcal{F} < \mathcal{F}_{th}$ for one template is: $1 - P_F(\mathcal{F}_{th})$
- Probability that $\mathcal{F} < \mathcal{F}_{th}$ for one template is: $[1 - P_F(\mathcal{F}_{th})]^{N_{cell}}$
- Finally, probability that $\mathcal{F} > \mathcal{F}_{\mathrm{th}}$ for at least one template is the total false alarm probability:

$$P_F^T(\mathcal{F}_{th}) = 1 - [1 - P_F(\mathcal{F}_{th})]^{N_{cell}}$$

Statistics of Fe

- In the case of Fe, we marginalize over 4 parameters per source so $N = 4\ N_{src}$ then :

$$P_F(\mathcal{F}_{e,th}) = e^{-\mathcal{F}_{e,th}} \sum_{k=0}^{2N_{src}-1} \frac{\mathcal{F}_{e,th}^k}{k!}$$

- The total false alarm probability is than

$$P_F^T = 1 - \left[1 - e^{-\mathcal{F}_{e,th}} \sum_{k=0}^{2N_{src}-1} \frac{\mathcal{F}_{e,th}^k}{k!}\right]^{N_{cell}}$$

- We need N_{cell} the total number of templates ?

Statistics of Fe

- We need N_{cell} the total number of templates:
  - criterion for considering that 2 templates defined by parameters ξ and ξ' are independent is that the autocovariance function is

$$C(\xi, \xi') \leq \rho C(\xi, \xi) = \rho \frac{n}{2}$$

  - with $\rho = 0.5$
  - Stochastic template bank
- 1 source: False Alarm Probabilty

$$P_F^T = 1 - \left[1 - e^{-\mathcal{F}_{e,th}} (1 + \mathcal{F}_{e,th})\right]^{N_{cell,1}}$$

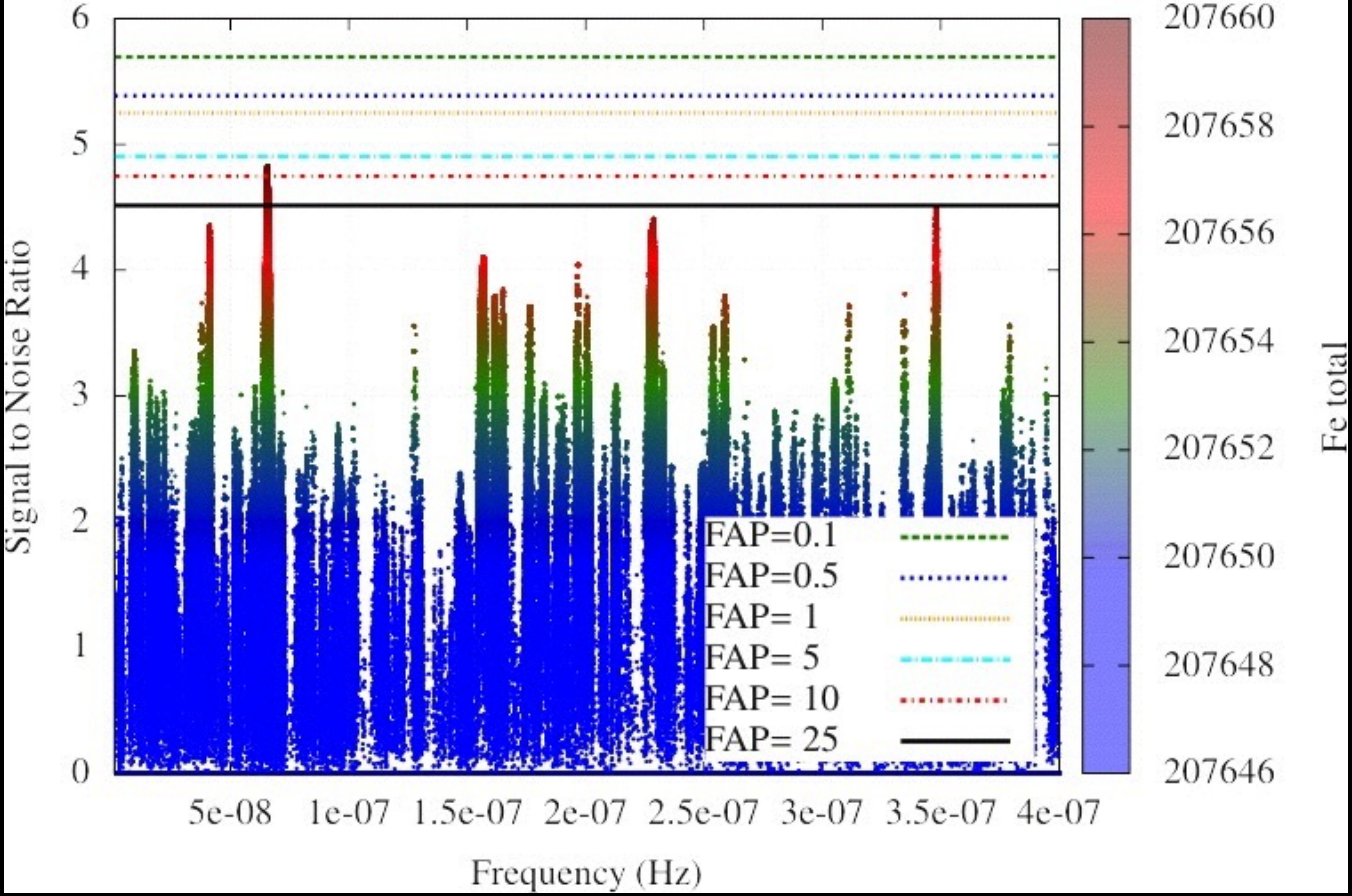
- By inverting, we got the threshold corresponding to the false alarm probability

Apply on real-data

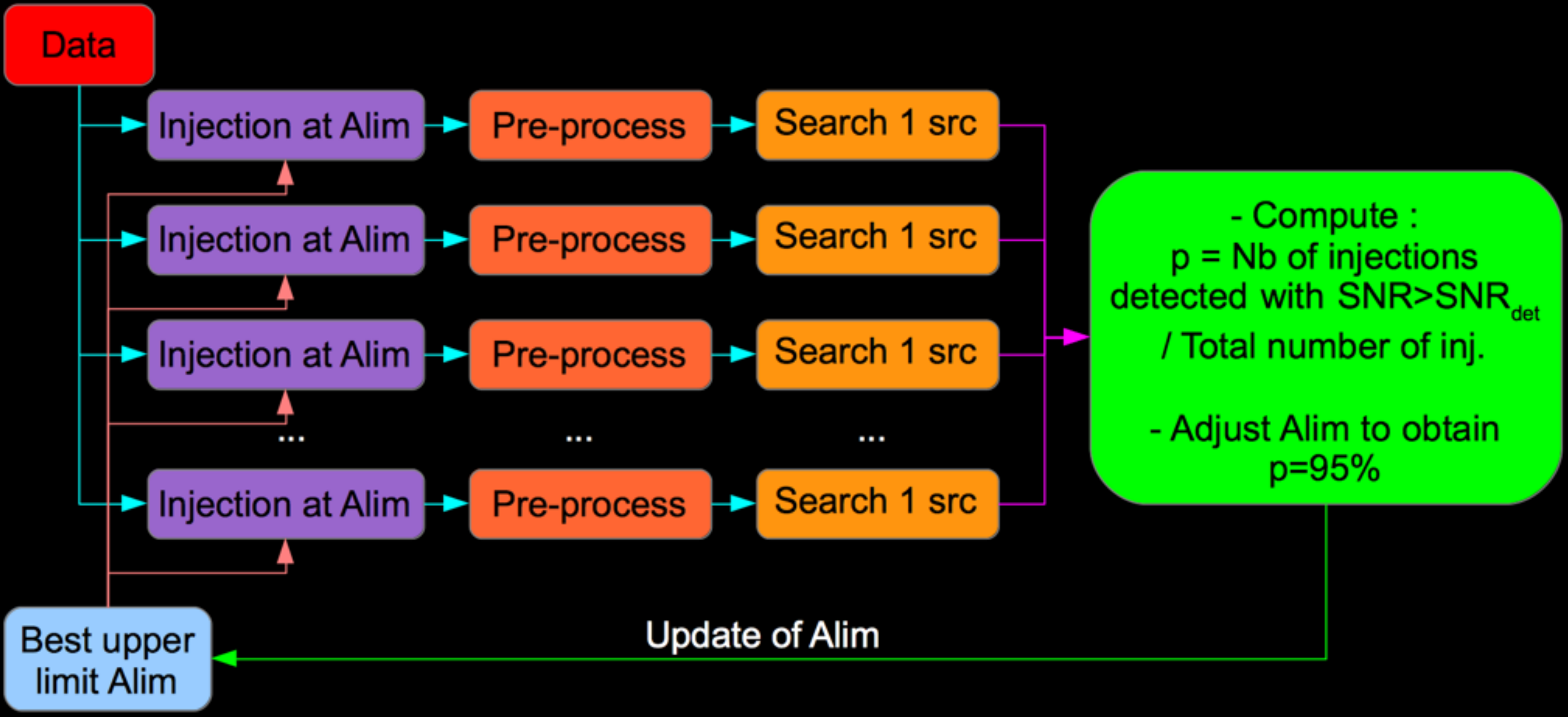
- Data: EPTA data release 2
  - 10- of 41 millisecond pulsars
- Noises analysis:
  - Hardest part: estimate C
  - Estimation of the noise contribution for each pulsar using various technics:
    - Red noise
    - Dispersion measurements
    - White noise component per back-end per pulsar
      - EFAC
      - EQUAD

Apply on real-data

- Detection results of Fp on real data



Upper limit

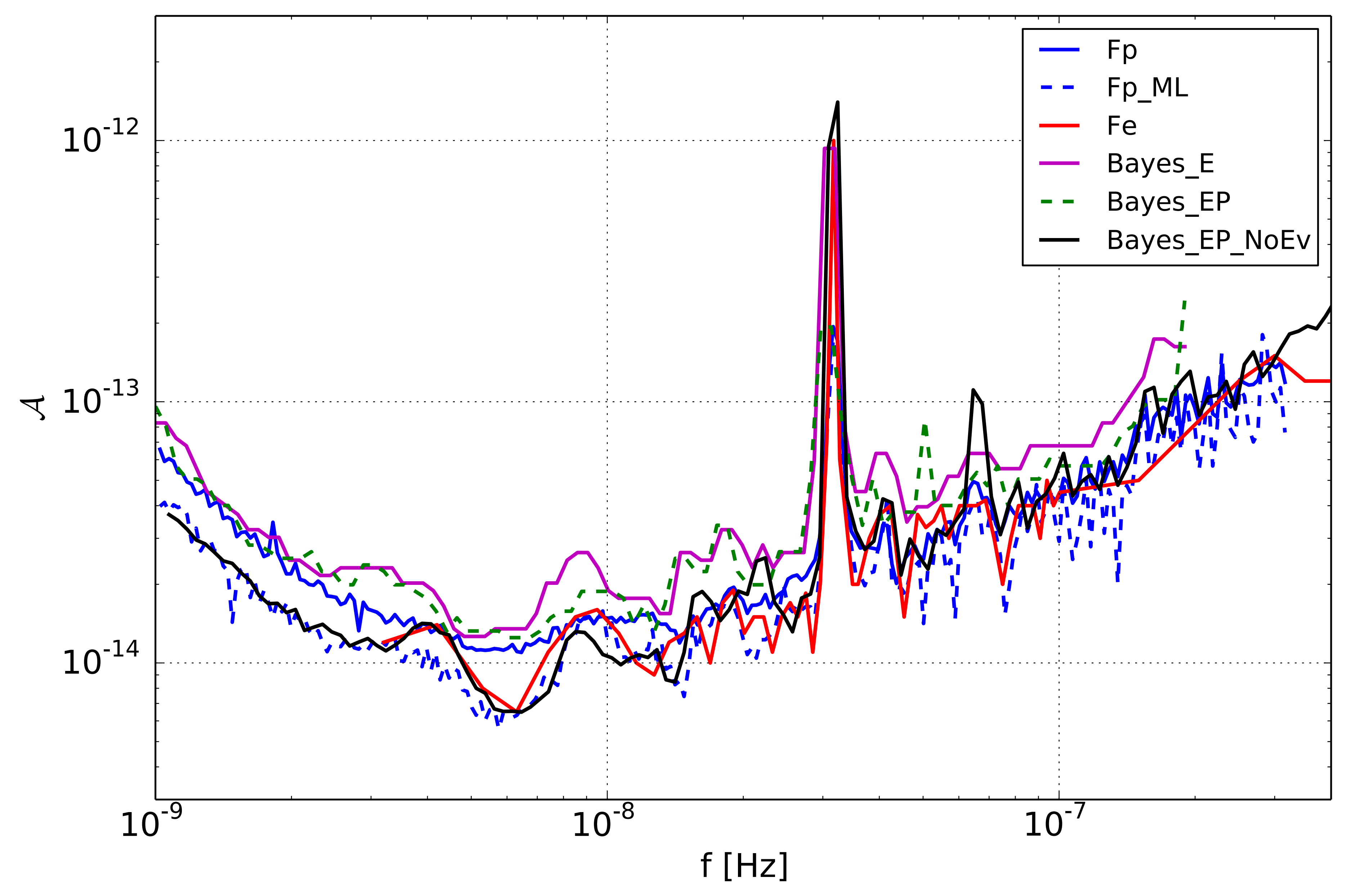


- Compute :
p = Nb of injections detected with SNR>SNR_{det} / Total number of inj.

- Adjust Alim to obtain p=95%

Results

Babak et al. EPTA MNRAS 455.2 (2016)



Fp statistic

Ellis et al. (2012)

- Excess power in which we basically search for extra power at a given frequency in each pulsar data
- Maximisation over all parameters except frequency f

$$r_a(t) = \sum_{j=1}^{2} b_{(j,a)}(\mathcal{A}, \theta_S, \phi_S, \Psi, \iota, \Phi_0, \Phi_a)\kappa_{(j)}(\omega, t)$$

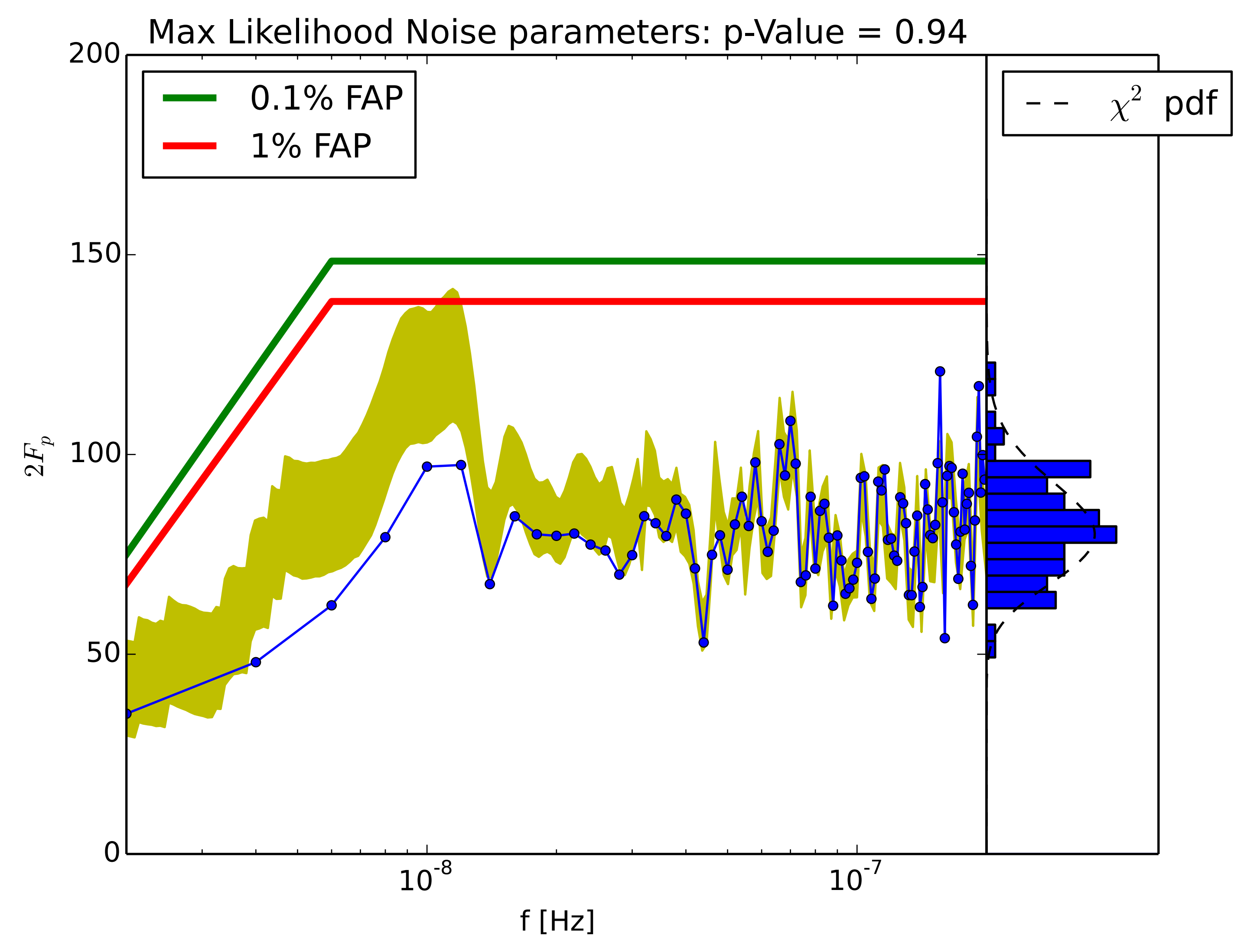
- Distribution of Fp statistic:
  - Gaussian noise: central : $p_0(\mathcal{F}_p) = \frac{\mathcal{F}_p^{n/2-1}}{(n/2-1)!}\exp(-\mathcal{F}_p)$
  - Signal: non-central with optimal SNR ρ :

$$p_1(\mathcal{F}_p, \rho) = \frac{(2\mathcal{F}_p)^{(n/2-1)/2}}{\rho^{n/2-1}} I_{n/2-1}(\rho\sqrt{2\mathcal{F}_p})e^{-\mathcal{F}_p-\frac{1}{2}\rho}$$

Fe detection results

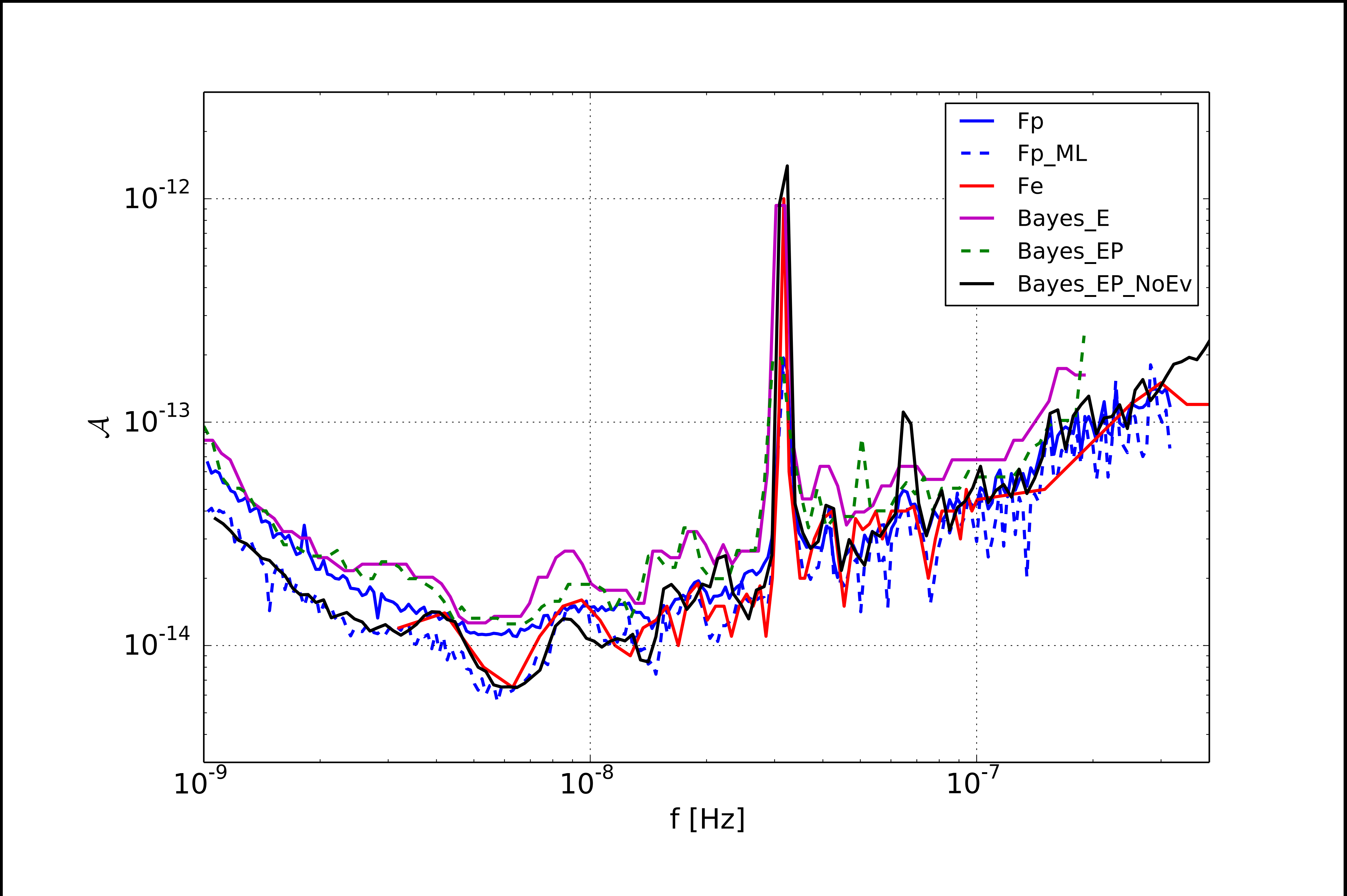
Babak et al. EPTA MNRAS 455.2 (2016)

- Fp evaluated at 99 independent frequencies
- p-value = 0.93
- Take into account the uncertainty in the noise parameters by sampling from their posterior distribution derived from the single pulsar analysis



Results

Babak et al. EPTA MNRAS 455.2 (2016)



Continous GWs
Bayesian analysis

Bayesian analysis 1

- No evolving sources: frequency at the pulsar is the same as the earth frequency.
  - we should sample on 7+N_{pulsar} parameters

$$(\mathcal{A}, \theta_S, \phi_S, \Psi, \iota, \omega, \Phi_0, \Phi_a)$$

  - numerical marginalization over the pulsar phase ϕ_a [Taylor et al., 2014]
  - MultiNest
  - Analysis:
    - 41 pulsars with fixed noise
    - 6 pulsars with varying noise

Bayesian analysis 2

- Full response:
  - 7 + 2 N_{pulsar} parameter space
  - Parallel tempering MCMC
  - Analysis:
    - 41 pulsars with Earth term only
    - 6 pulsars with pulsar and Earth terms

Results

- Bayes factor

$$\mathcal{B} = \frac{\int \mathcal{L}(\vec{\theta}, \vec{\lambda} | \vec{\delta t}) \pi(\vec{\theta}, \vec{\lambda}) d\vec{\theta} \, d\vec{\lambda}}{\int \mathcal{L}(\vec{\theta} | \vec{\delta t}) \pi(\vec{\theta}) d\vec{\theta}}.$$

- Non evolving: log($\mathcal{B}$) = -0.27
- Earth term only: log($\mathcal{B}$) = -0.31

=> no detection

=> upper limit

Upper limit

$$0.95 = \int_0^{\tilde{A}} d\mathcal{A} \int d\vec{\lambda}' \mathcal{L}(\mathcal{A}, \vec{\lambda}' | \vec{\delta t}) \pi(\mathcal{A}) \pi(\vec{\lambda}')$$

Data

Pre-process

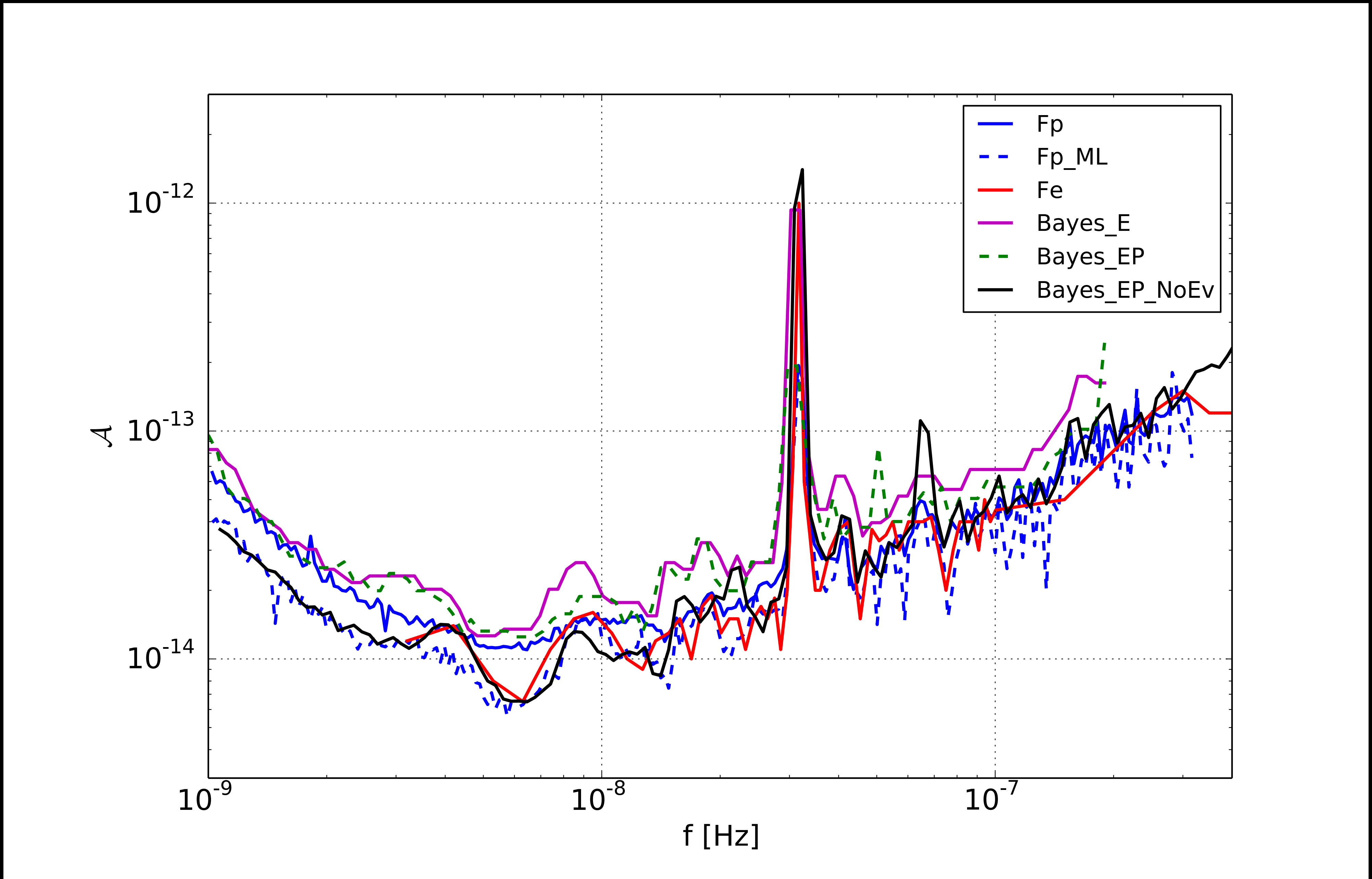
Full exploration of all parameters within the box in frequency (+sky)

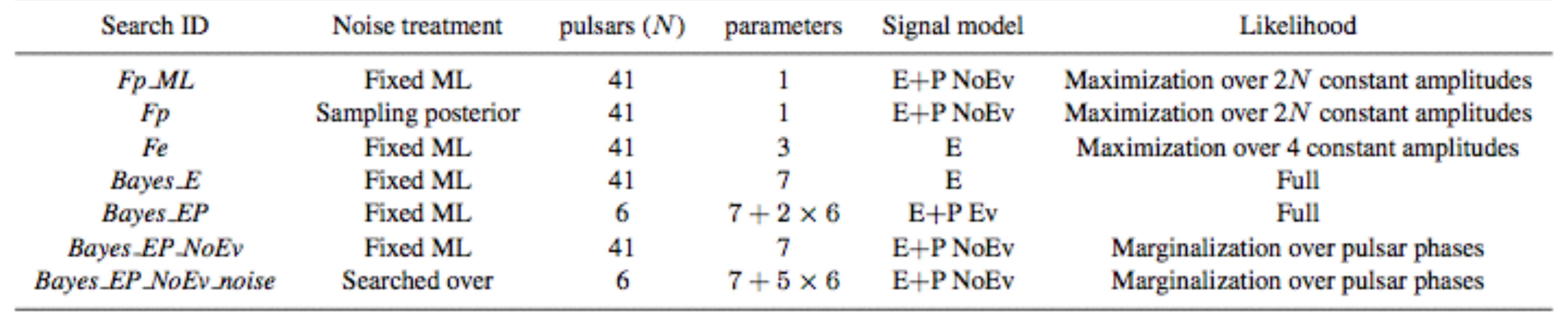
On amplitude distribution, take the value of A including 95% of the distribution

Upper limit Alim

Results EPTA

Babak et al. EPTA MNRAS 455.2 (2016)



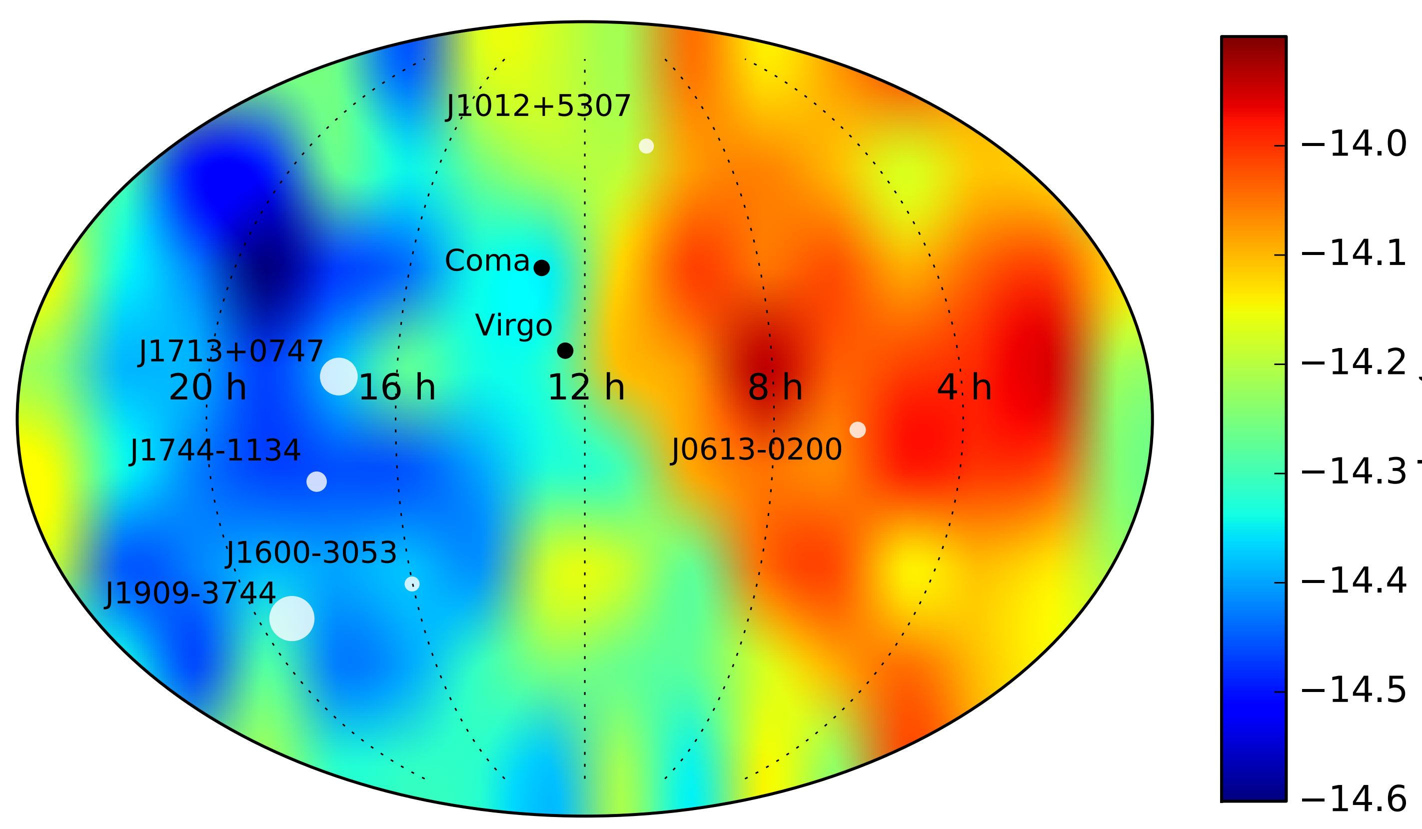


Search ID	Noise treatment	pulsars (N)	parameters	Signal model	Likelihood
Fp_ML	Fixed ML	41	1	E+P NoEv	Maximization over $2N$ constant amplitudes
Fp	Sampling posterior	41	1	E+P NoEv	Maximization over $2N$ constant amplitudes
Fe	Fixed ML	41	3	E	Maximization over 4 constant amplitudes
Bayes_E	Fixed ML	41	7	E	Full
Bayes_EP	Fixed ML	6	$7 + 2 \times 6$	E+P Ev	Full
Bayes_EP_NoEv	Fixed ML	41	7	E+P NoEv	Marginalization over pulsar phases
Bayes_EP_NoEv_noise	Searched over	6	$7 + 5 \times 6$	E+P NoEv	Marginalization over pulsar phases

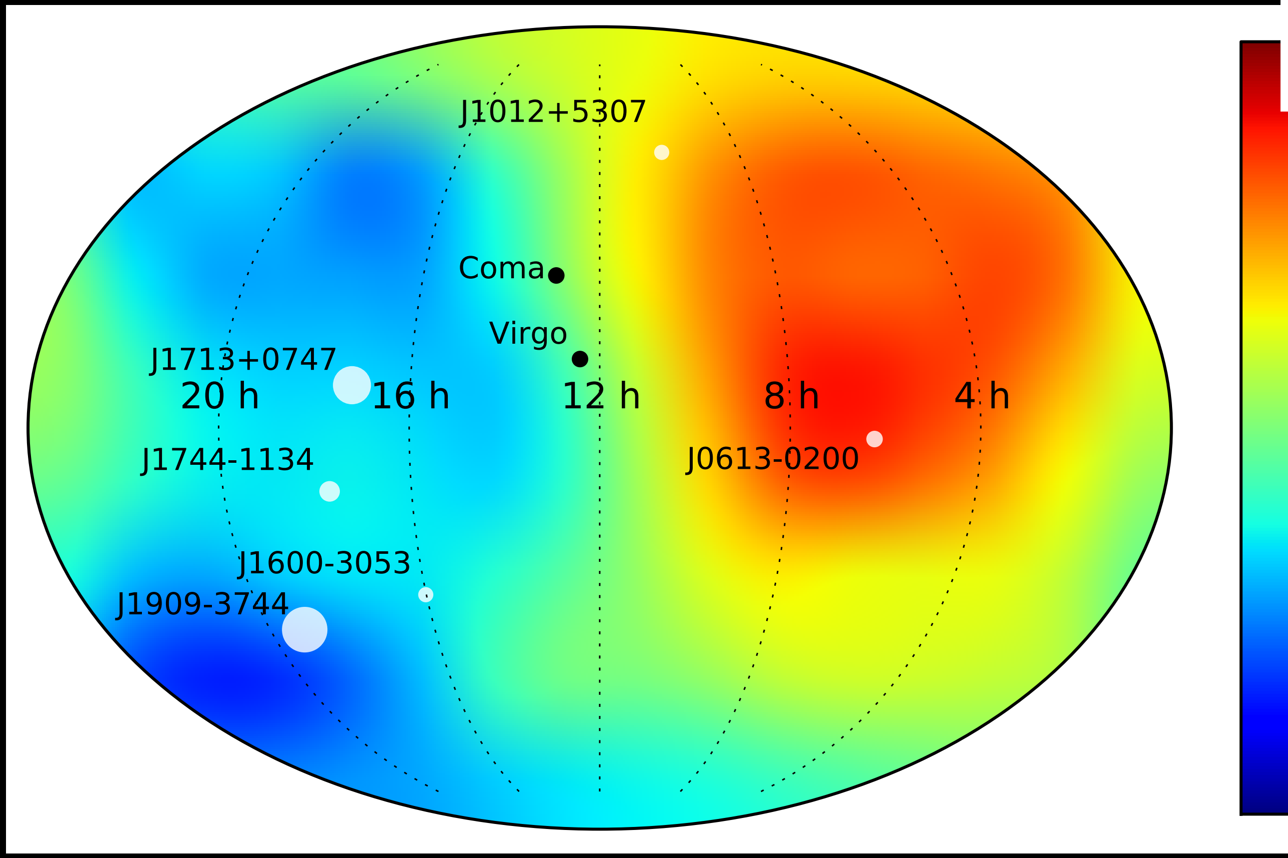
Upper limit

- Sky map upper limit at 7nHz

Bayesian



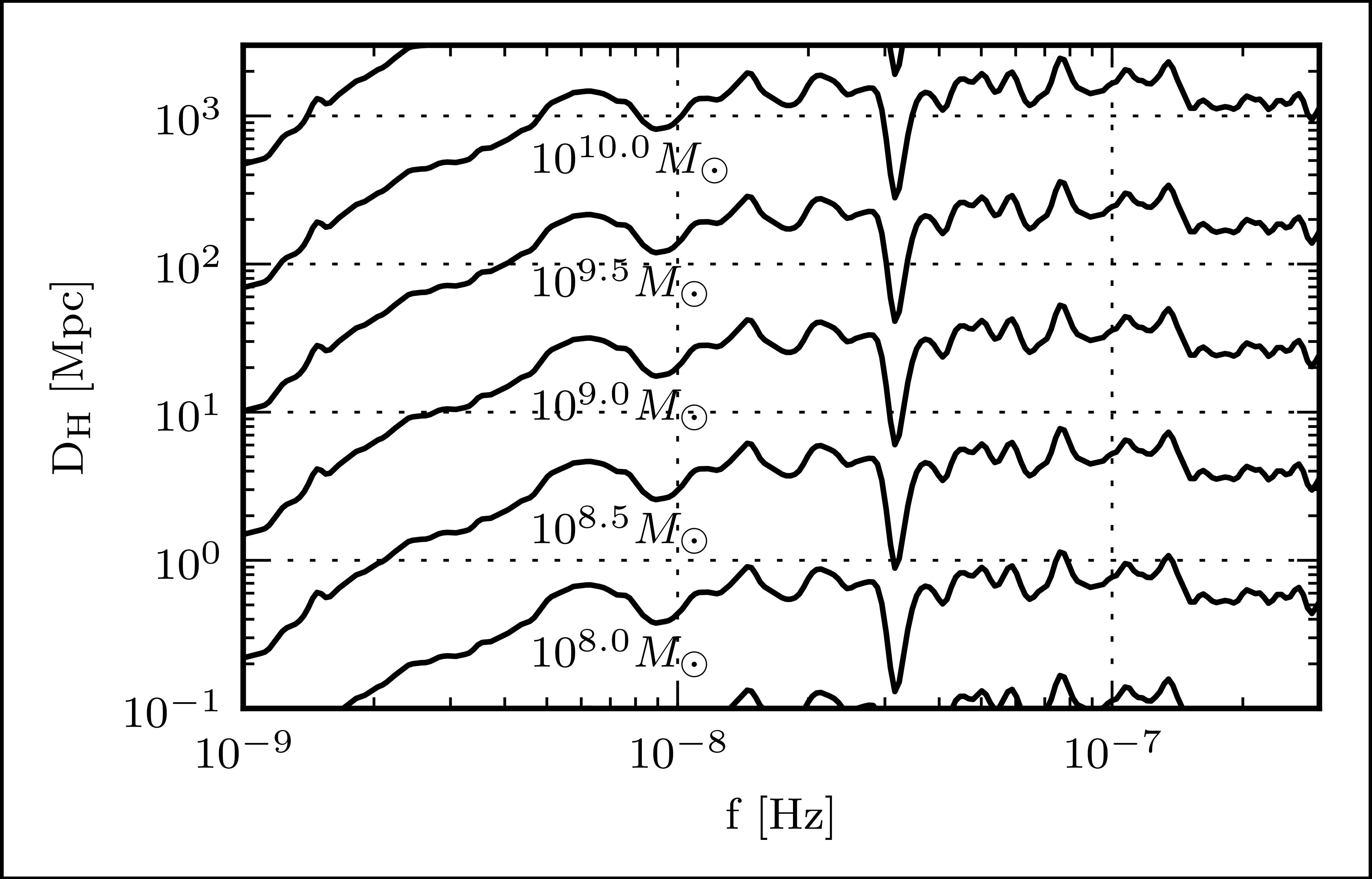
Frequentist



Horizon

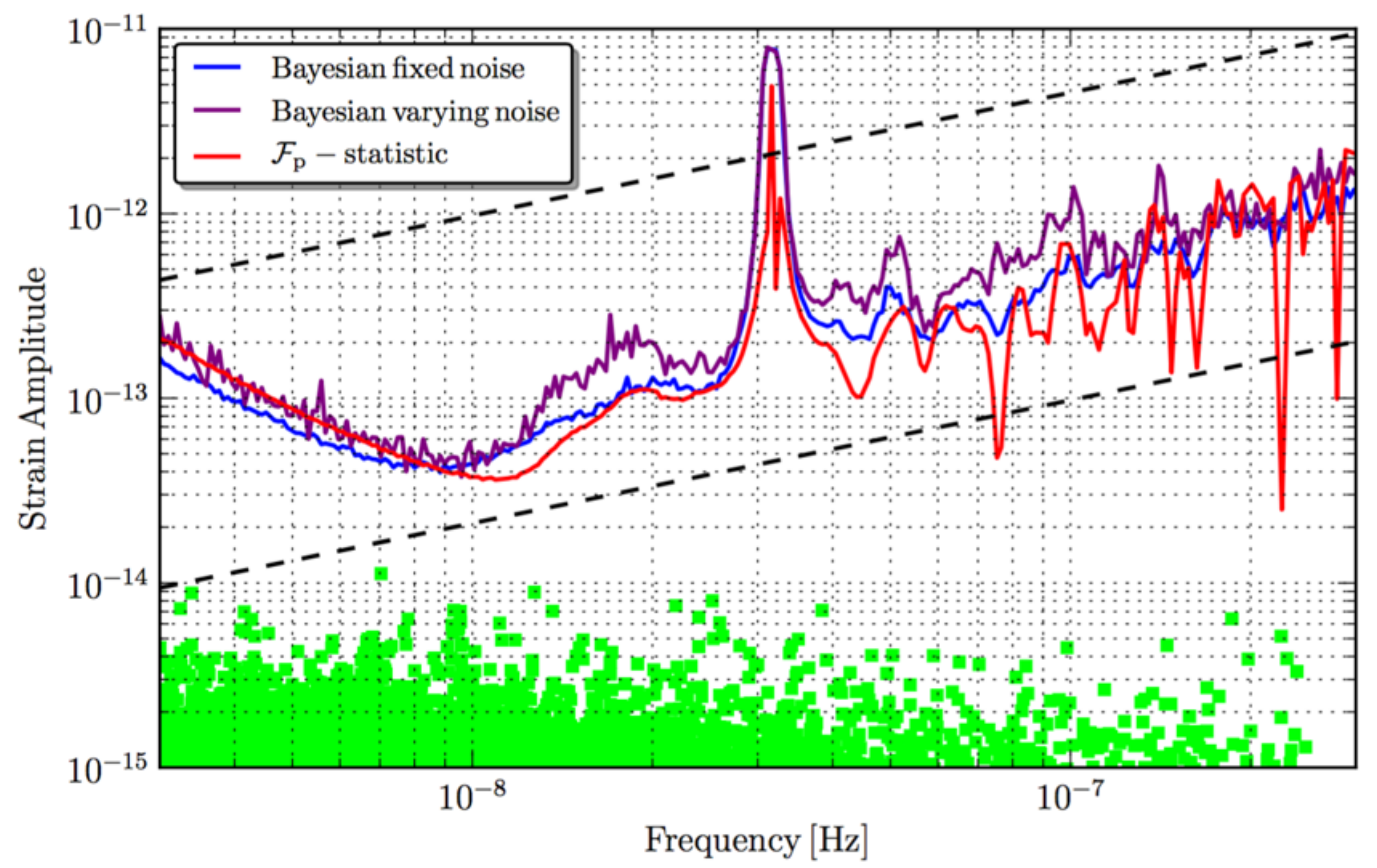
- Invert amplitude

$$\mathcal{A} = 2\frac{\mathcal{M}_c^{5/3}}{D_L}(\pi f)^{2/3}$$



Results from NANOGrav

Arzoumanian et al. NANOGrav (2014)



Stochastic background

DA stochastic background

- Likelihood:

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}} \times \exp\left(-\frac{1}{2}\left(\vec{\delta t} - \vec{h}\right)^T G(G^T C G)^{-1} G^T \left(\vec{\delta t} - \vec{h}\right)\right)$$

- Stochastic background shape: power-law

- Parametrization of the correlation matrix:

$$C_{GWB} = \zeta_{\alpha\beta} A^2 \left(\frac{1yr^{-1}}{f_L}\right)^{\gamma-1} \left[\Gamma(1-\gamma)\sin\frac{\pi\gamma}{2}(f_L\tau_{ij})^{\gamma-1} - \sum_{n=0}^{\infty}\frac{(f_L\tau_{ij})^{2n}}{(2n)!(2n+1-\gamma)}\right]$$

$$\zeta_{\alpha\beta} = \frac{3}{2} y \ln y - \frac{1}{4} y + \frac{1}{2} + \frac{1}{2}\delta_{\alpha\beta} \, , \quad y = \frac{1-\cos\theta_{\alpha\beta}}{2} \, , \quad \tau_{ij} = 2\pi |t_i - t_j|$$

DA stochastic background

- **Likelihood:**

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}}$$

$$\times \exp\left(-\frac{1}{2}\left(\vec{\delta t} - \vec{h}\right)^T G(G^T C G)^{-1} G^T \left(\vec{\delta t} - \vec{h}\right)\right)$$

- **Stochastic background shape: power-law**

- **Parametrization of the correlation matrix:**

$$C_{GWB} = \zeta_{\alpha\beta} A^2 \left(\frac{1yr^{-1}}{f_L}\right)^{\gamma-1} \left[\Gamma(1-\gamma)\sin\frac{\pi\gamma}{2}(f_L\tau_{ij})^{\gamma-1} - \sum_{n=0}^{\infty}\frac{(f_L\tau_{ij})^{2n}}{(2n)!(2n+1-\gamma)}\right]$$

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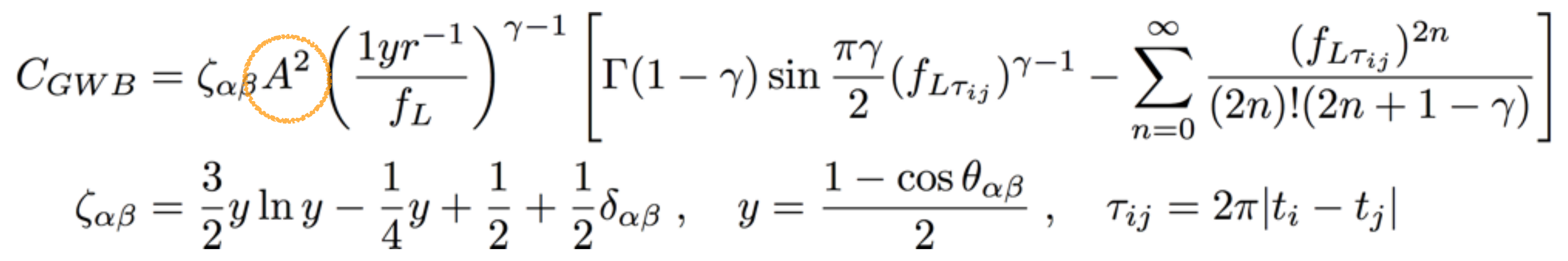
- Likelihood:

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}}$$

$$\times \exp\left(-\frac{1}{2}\left(\vec{\delta t}-\vec{h}\right)^T G(G^T C G)^{-1} G^T \left(\vec{\delta t}-\vec{h}\right)\right)$$

- Stochastic background shape: power-law
- Parametrization of the correlation matrix:



$$C_{GWB} = \zeta_{\alpha\beta} A^2 \left(\frac{1yr^{-1}}{f_L}\right)^{\gamma-1} \left[\Gamma(1-\gamma)\sin\frac{\pi\gamma}{2}(f_L\tau_{ij})^{\gamma-1} - \sum_{n=0}^{\infty}\frac{(f_L\tau_{ij})^{2n}}{(2n)!(2n+1-\gamma)}\right]$$

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$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}}$$

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DA stochastic background

- **Likelihood:**

$$p(\vec{\delta t}|\vec{\theta}) = \frac{1}{\sqrt{(2\pi)^{n-m} det(G^T C G)}} \times \exp\left(-\frac{1}{2}\left(\vec{\delta t} - \vec{h}\right)^T G (G^T C G)^{-1} G^T \left(\vec{\delta t} - \vec{h}\right)\right)$$

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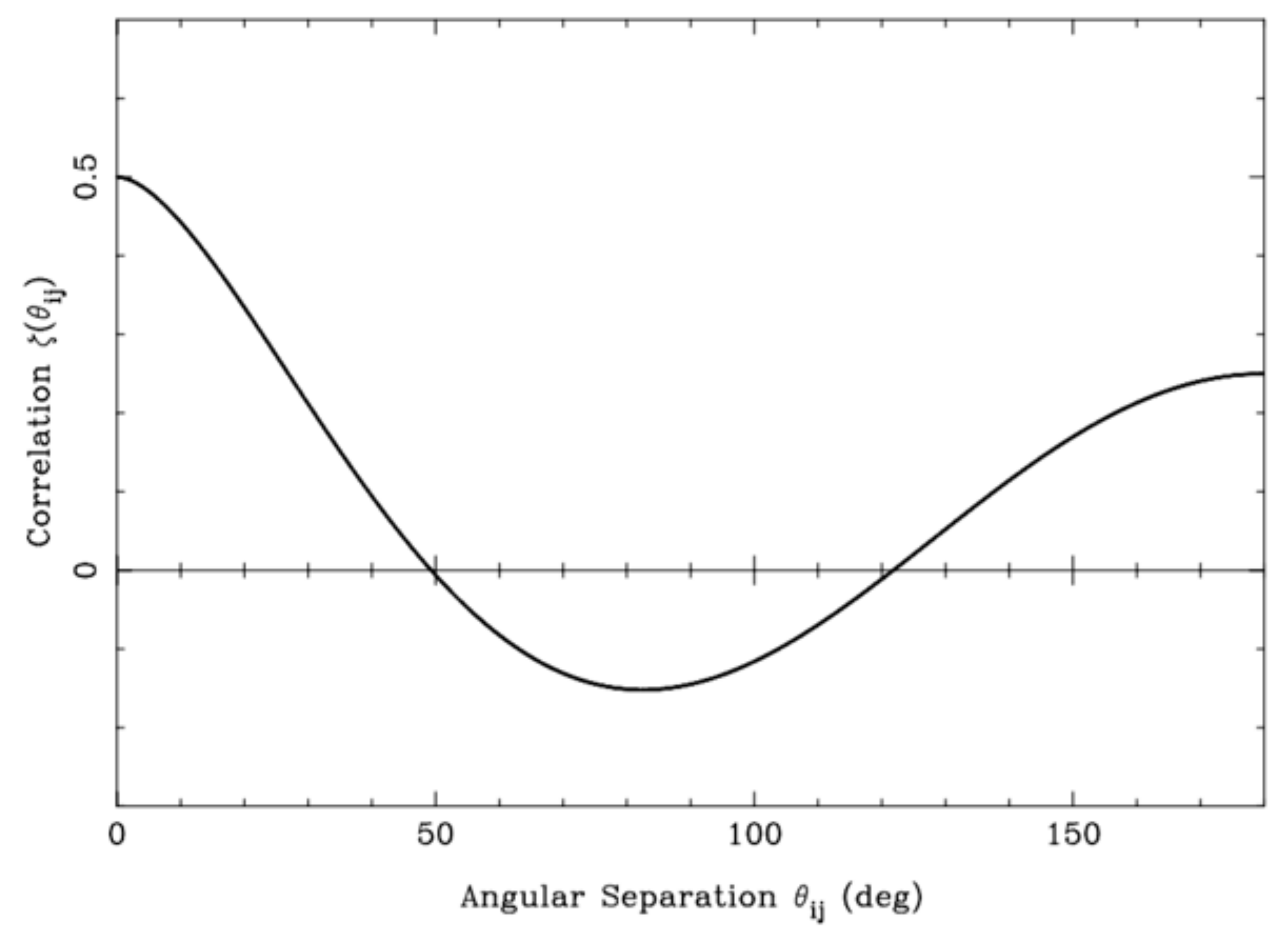
$$\boxed{\zeta_{\alpha\beta} = \frac{3}{2}y\ln y - \frac{1}{4}y + \frac{1}{2} + \frac{1}{2}\delta_{\alpha\beta}}, \quad y = \frac{1-\cos\theta_{\alpha\beta}}{2}, \quad \tau_{ij} = 2\pi|t_i - t_j|$$

Hellings Downs curve

Hellings Downs curve

- Overlap reduction function: expected correlation in pulsar timing residuals due to an isotropic stochastic GW background



Search for stochastic background

- Parameterization of C and inverting C is time consuming because its a matrix n x n, with n the total number of measurements.
- Several methods:
  - Bayesian / Frequentist
  - Fixed noise / varying noise
  - Fixed slope / varying slope
  - Various samplers

Search for stochastic background

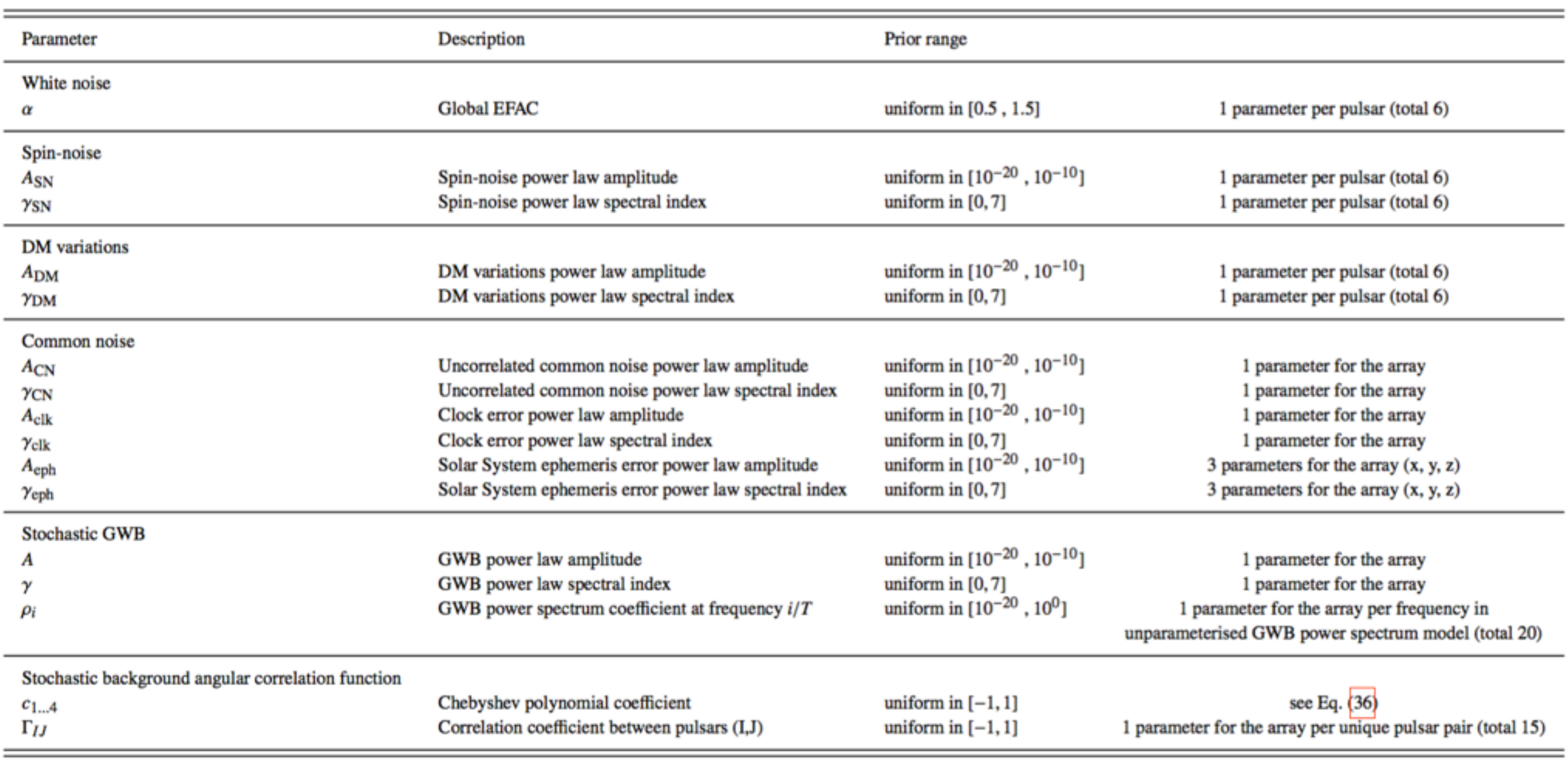
- Bayesian:
  - Use the likelihood previously define
  - Sampling algorithm:
    - MultiNest [EPTA]
    - parallel tempering MCMC [Ellis]
  - Inputs: priors
  - Results: posterior distribution for:
    - amplitude and slope of the background
    - noises parameters for individual pulsars
    - common noises

Search for stochastic background

- Bayesian:

Lentati et al. EPTA (2015)

  - Prior on parameters



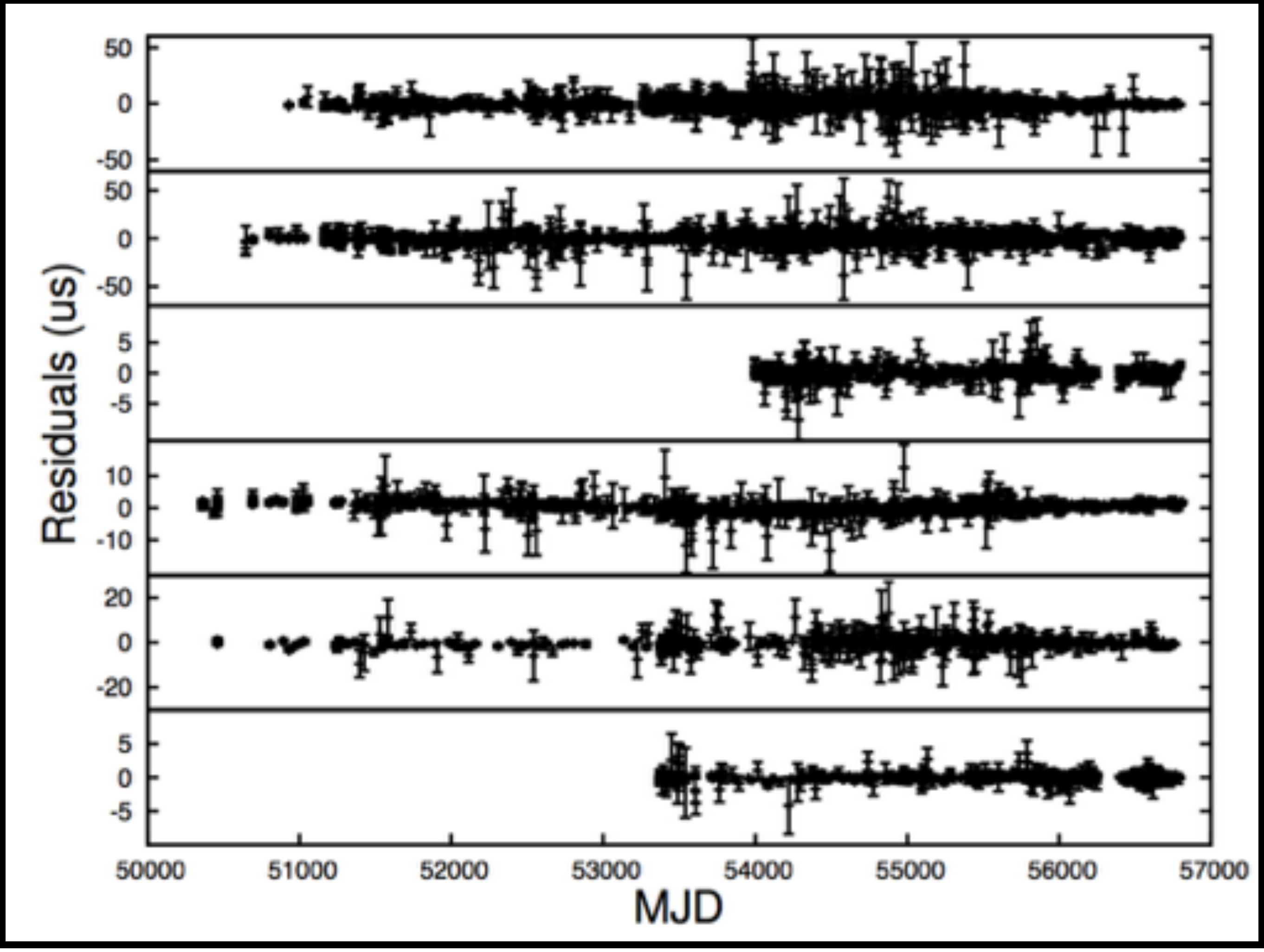
Parameter	Description	Prior range	
White noise			
α	Global EFAC	uniform in [0.5 , 1.5]	1 parameter per pulsar (total 6)
Spin-noise			
A_{SN}	Spin-noise power law amplitude	uniform in $[10^{-20}, 10^{-10}]$	1 parameter per pulsar (total 6)
γ_{SN}	Spin-noise power law spectral index	uniform in $[0, 7]$	1 parameter per pulsar (total 6)
DM variations			
A_{DM}	DM variations power law amplitude	uniform in $[10^{-20}, 10^{-10}]$	1 parameter per pulsar (total 6)
γ_{DM}	DM variations power law spectral index	uniform in $[0, 7]$	1 parameter per pulsar (total 6)
Common noise			
A_{CN}	Uncorrelated common noise power law amplitude	uniform in $[10^{-20}, 10^{-10}]$	1 parameter for the array
γ_{CN}	Uncorrelated common noise power law spectral index	uniform in $[0, 7]$	1 parameter for the array
A_{clk}	Clock error power law amplitude	uniform in $[10^{-20}, 10^{-10}]$	1 parameter for the array
γ_{clk}	Clock error power law spectral index	uniform in $[0, 7]$	1 parameter for the array
A_{eph}	Solar System ephemeris error power law amplitude	uniform in $[10^{-20}, 10^{-10}]$	3 parameters for the array (x, y, z)
γ_{eph}	Solar System ephemeris error power law spectral index	uniform in $[0, 7]$	3 parameters for the array (x, y, z)
Stochastic GWB			
A	GWB power law amplitude	uniform in $[10^{-20}, 10^{-10}]$	1 parameter for the array
γ	GWB power law spectral index	uniform in $[0, 7]$	1 parameter for the array
ρ_i	GWB power spectrum coefficient at frequency i/T	uniform in $[10^{-20}, 10^{0}]$	1 parameter for the array per frequency in unparameterised GWB power spectrum model (total 20)
Stochastic background angular correlation function			
$c_{1\ldots4}$	Chebyshev polynomial coefficient	uniform in $[-1, 1]$	see Eq. (36)
Γ_{IJ}	Correlation coefficient between pulsars (I,J)	uniform in $[-1, 1]$	1 parameter for the array per unique pulsar pair (total 15)

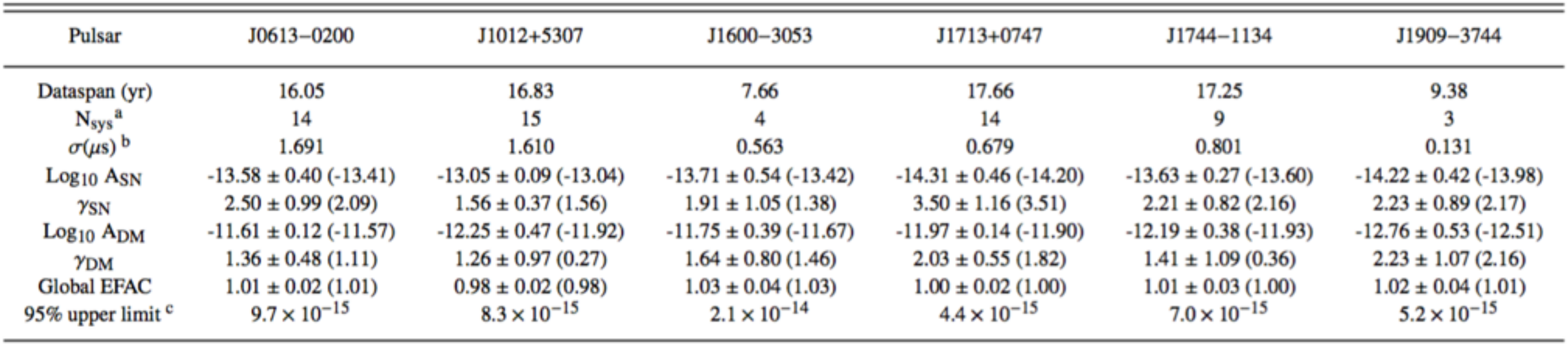
Search for stochastic background

- Example: EPTA

Lentati et al. EPTA (2015)

  - data



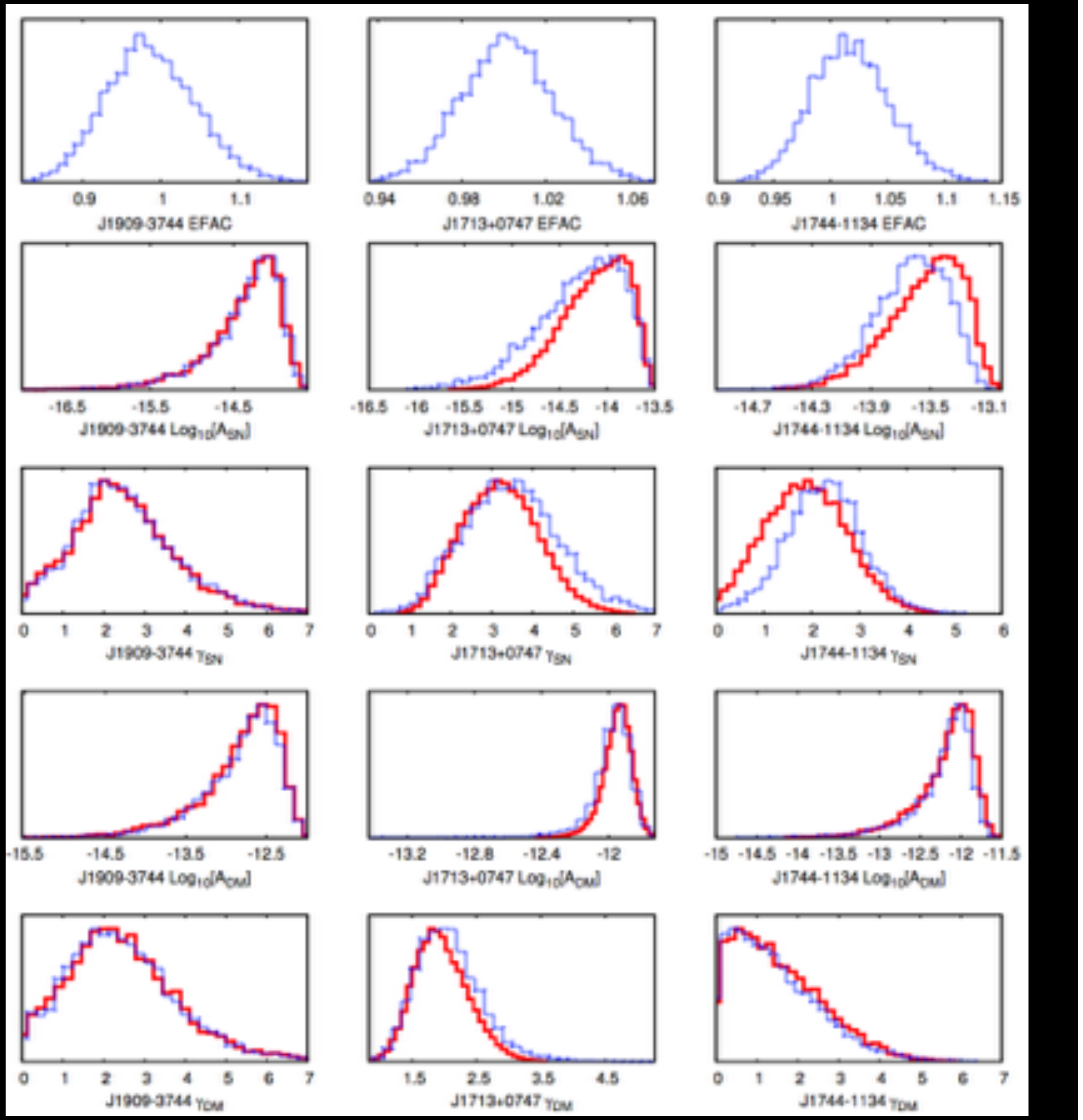


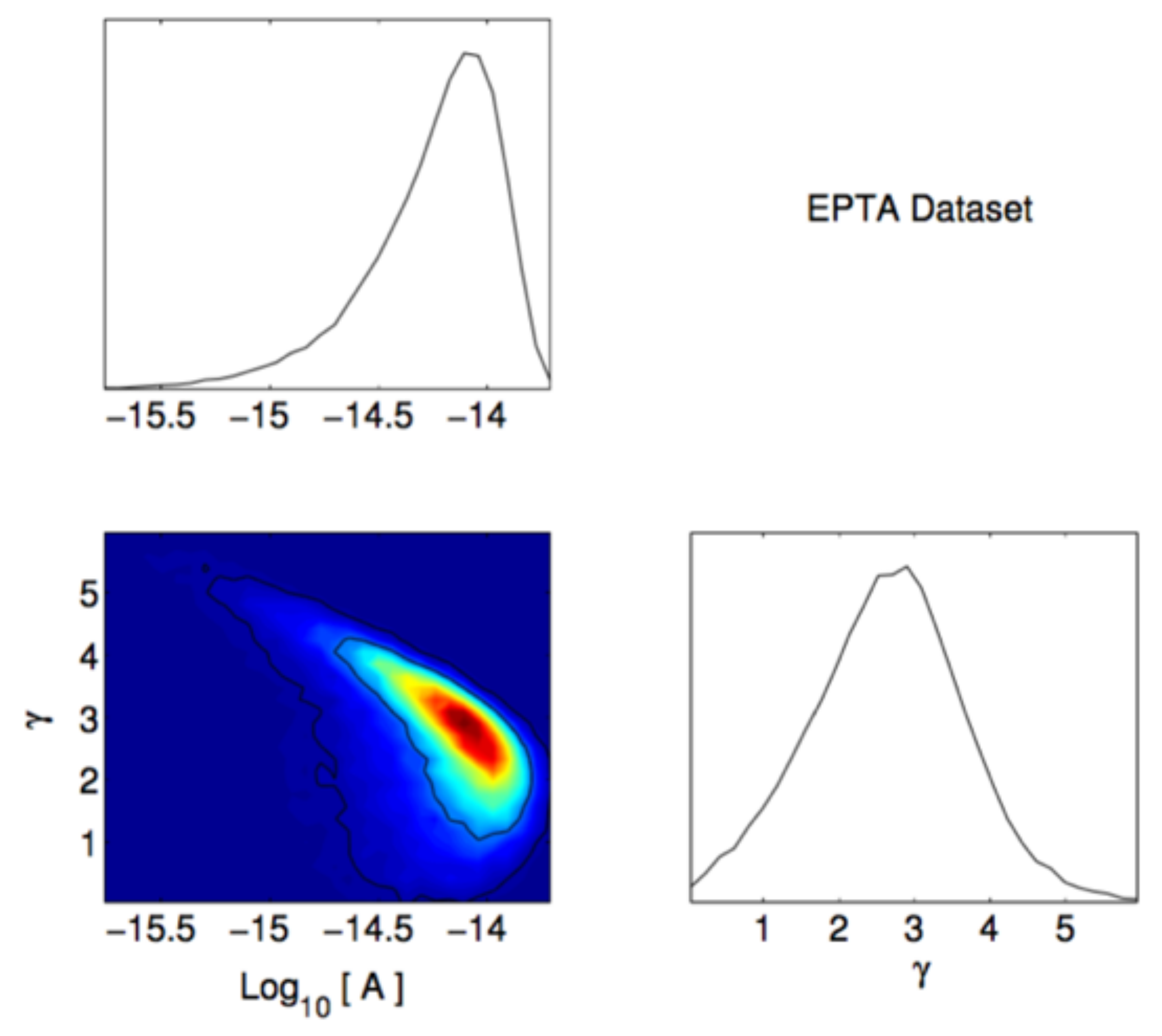
Pulsar	J0613−0200	J1012+5307	J1600−3053	J1713+0747	J1744−1134	J1909−3744
Dataspan (yr)	16.05	16.83	7.66	17.66	17.25	9.38
N_{sys} [a]	14	15	4	14	9	3
$\sigma(\mu s)$ [b]	1.691	1.610	0.563	0.679	0.801	0.131
$\mathrm{Log}_{10}\,A_{SN}$	-13.58 ± 0.40 (-13.41)	-13.05 ± 0.09 (-13.04)	-13.71 ± 0.54 (-13.42)	-14.31 ± 0.46 (-14.20)	-13.63 ± 0.27 (-13.60)	-14.22 ± 0.42 (-13.98)
γ_{SN}	2.50 ± 0.99 (2.09)	1.56 ± 0.37 (1.56)	1.91 ± 1.05 (1.38)	3.50 ± 1.16 (3.51)	2.21 ± 0.82 (2.16)	2.23 ± 0.89 (2.17)
$\mathrm{Log}_{10}\,A_{DM}$	-11.61 ± 0.12 (-11.57)	-12.25 ± 0.47 (-11.92)	-11.75 ± 0.39 (-11.67)	-11.97 ± 0.14 (-11.90)	-12.19 ± 0.38 (-11.93)	-12.76 ± 0.53 (-12.51)
γ_{DM}	1.36 ± 0.48 (1.11)	1.26 ± 0.97 (0.27)	1.64 ± 0.80 (1.46)	2.03 ± 0.55 (1.82)	1.41 ± 1.09 (0.36)	2.23 ± 1.07 (2.16)
Global EFAC	1.01 ± 0.02 (1.01)	0.98 ± 0.02 (0.98)	1.03 ± 0.04 (1.03)	1.00 ± 0.02 (1.00)	1.01 ± 0.03 (1.00)	1.02 ± 0.04 (1.01)
95% upper limit [c]	9.7×10^{-15}	8.3×10^{-15}	2.1×10^{-14}	4.4×10^{-15}	7.0×10^{-15}	5.2×10^{-15}

Search for stochastic background

- Example: EPTA

Lentati et al. EPTA (2015)

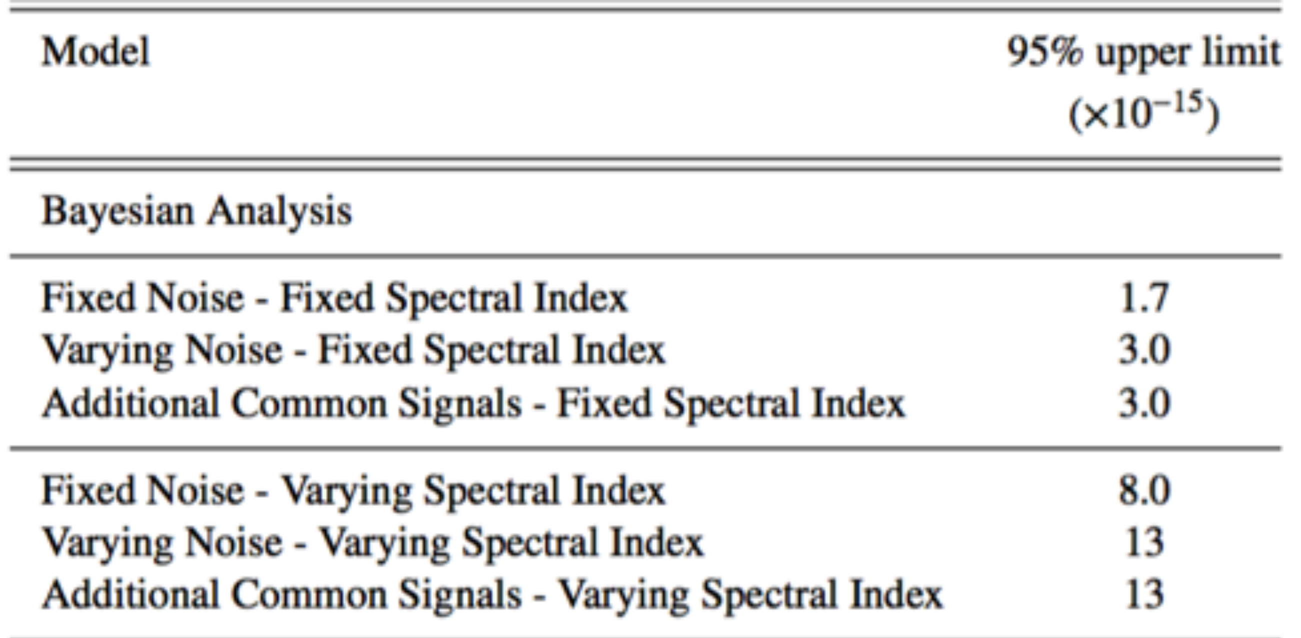




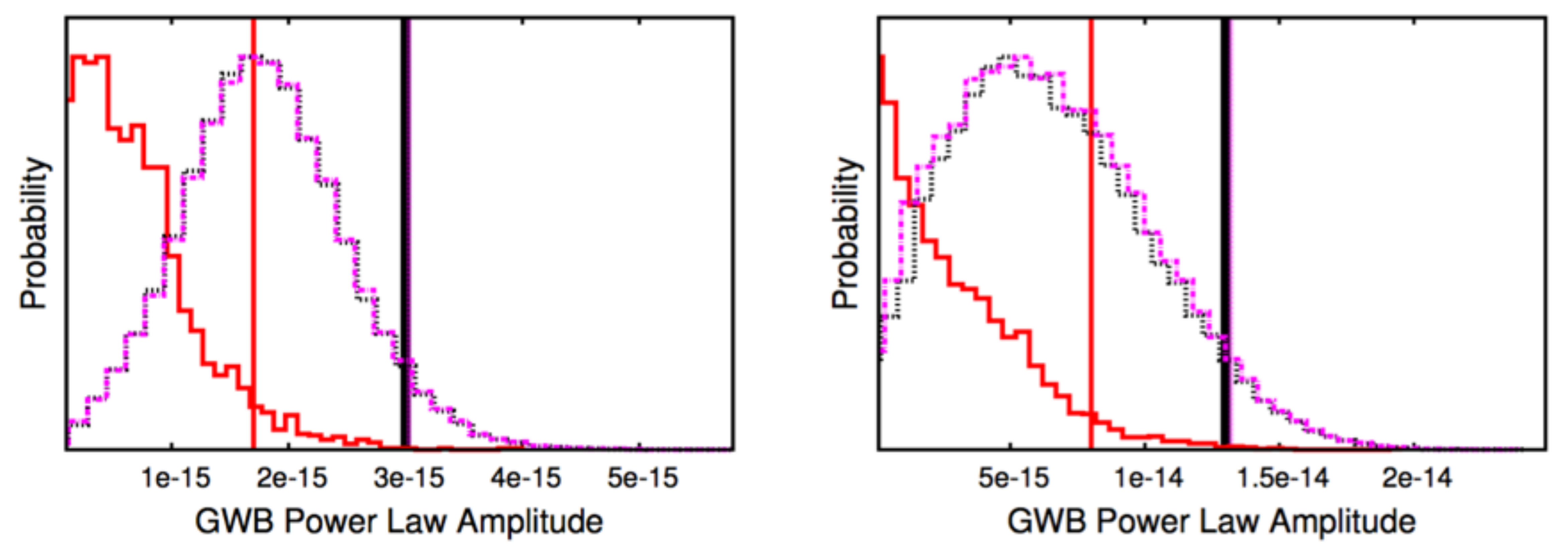
Search for stochastic background

- Example: EPTA

Lentati et al. EPTA (2015)



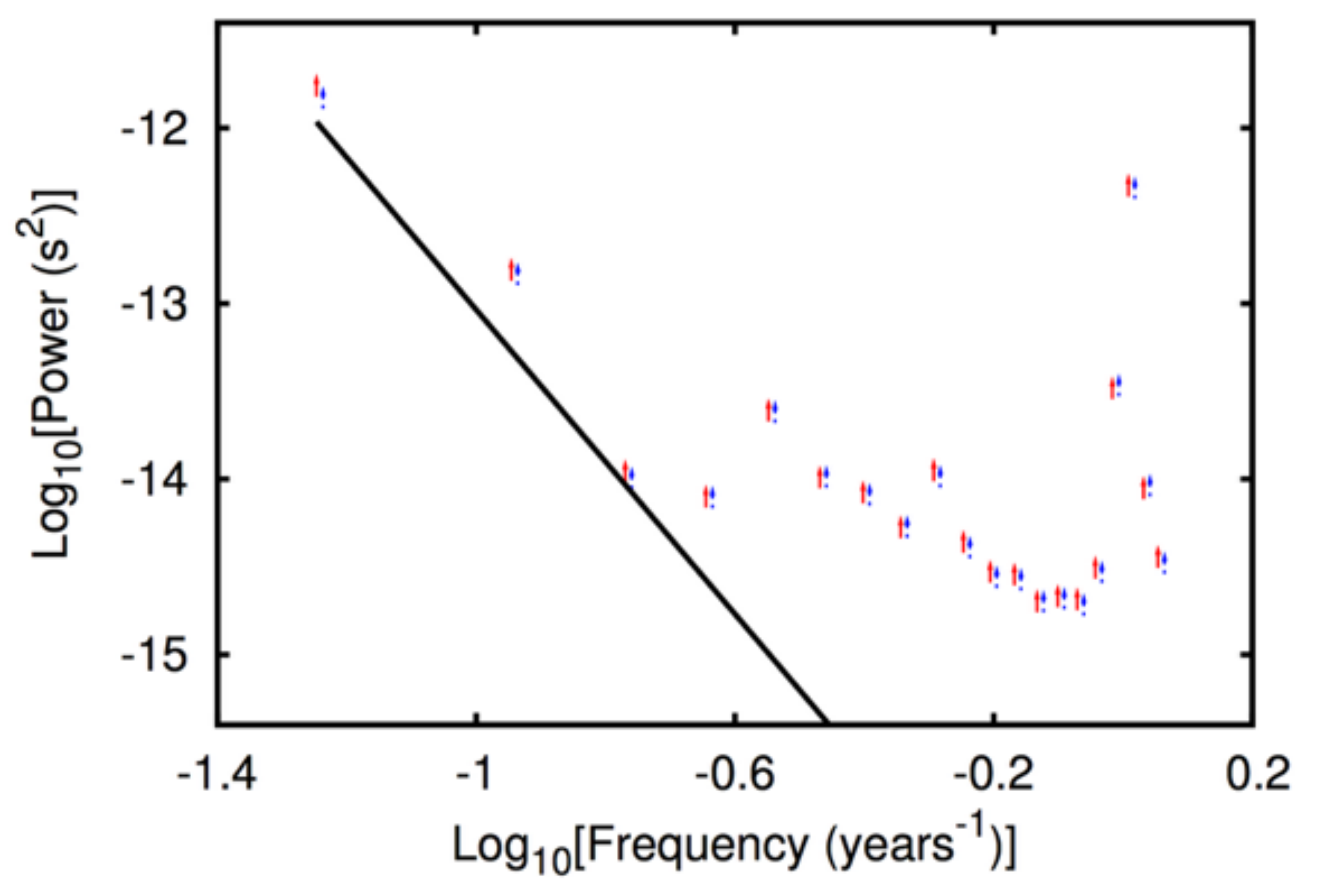
Model	95% upper limit ($\times 10^{-15}$)
Bayesian Analysis	
Fixed Noise - Fixed Spectral Index	1.7
Varying Noise - Fixed Spectral Index	3.0
Additional Common Signals - Fixed Spectral Index	3.0
Fixed Noise - Varying Spectral Index	8.0
Varying Noise - Varying Spectral Index	13
Additional Common Signals - Varying Spectral Index	13



Others bayesian methods

- Measure power in frequency bins [Lentati et al.]
- Unparameterised power spectrum analysis for a correlated Gravitational Wave Background

95% upper limits from an unparameterised power spectrum analysis for a correlated GWB (red points), and uncorrelated common red noise process (blue points) for the 6 pulsar



DA stochastic background

- Example of frequentist method used: *Optimal statistic*
  - weak signal maximum likelihood for GWB spectral amplitude (Anholm et al. 2009; Siemens et al. 2013; Chamberlin et al. 2014)
  - The statistic is:

$$\hat{A}^2 = \frac{\sum_{IJ} \delta\mathbf{t}_I^{\mathrm{T}} \mathbf{P}_I^{-1} \tilde{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \delta\mathbf{t}_J}{\sum_{IJ} \mathrm{tr}\left[\mathbf{P}_I^{-1} \tilde{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \tilde{\mathbf{S}}_{JI}\right]},$$

  - with:
    - autocovariance of the post-fit residuals $\mathbf{P}_I = \langle \delta\mathbf{t}_I \delta\mathbf{t}_I^{\mathrm{T}} \rangle$
    - signal term $A^2 \tilde{\mathbf{S}}_{IJ} = \langle \delta\mathbf{t}_I \delta\mathbf{t}_J^{\mathrm{T}} \rangle = \mathbf{S}_{IJ}$

DA stochastic background

- Example of frequentist method used: *Optimal statistic*
  - weak signal maximum likelihood for GWB spectral amplitude (Anholm et al. 2009; Siemens et al. 2013; Chamberlin et al. 2014)
  - The statistic is:

$$\hat{A}^2 = \frac{\sum_{IJ} \delta\mathbf{t}_I^{\mathrm{T}} \mathbf{P}_I^{-1} \tilde{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \delta\mathbf{t}_J}{\sum_{IJ} \mathrm{tr}\left[\mathbf{P}_I^{-1} \tilde{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \tilde{\mathbf{S}}_{JI}\right]},$$

  - with:
    - autocovariance of the post-fit residuals $\mathbf{P}_I = \langle \delta\mathbf{t}_I \delta\mathbf{t}_I^{\mathrm{T}} \rangle$
    - signal term $A^2 \tilde{\mathbf{S}}_{IJ} = \langle \delta\mathbf{t}_I \delta\mathbf{t}_J^{\mathrm{T}} \rangle = \mathbf{S}_{IJ}$

DA stochastic background

- Example of frequentist method used: *Optimal statistic*
  - SNR is: $$\rho = \frac{\hat{A}^2}{\sigma_0} = \frac{\sum_{IJ} \delta \mathbf{t}_I^{\mathrm{T}} \mathbf{P}_I^{-1} \tilde{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \delta \mathbf{t}_J}{\left(\sum_{IJ} \mathrm{tr}\left[\mathbf{P}_I^{-1} \tilde{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \tilde{\mathbf{S}}_{JI}\right]\right)^{1/2}}$$

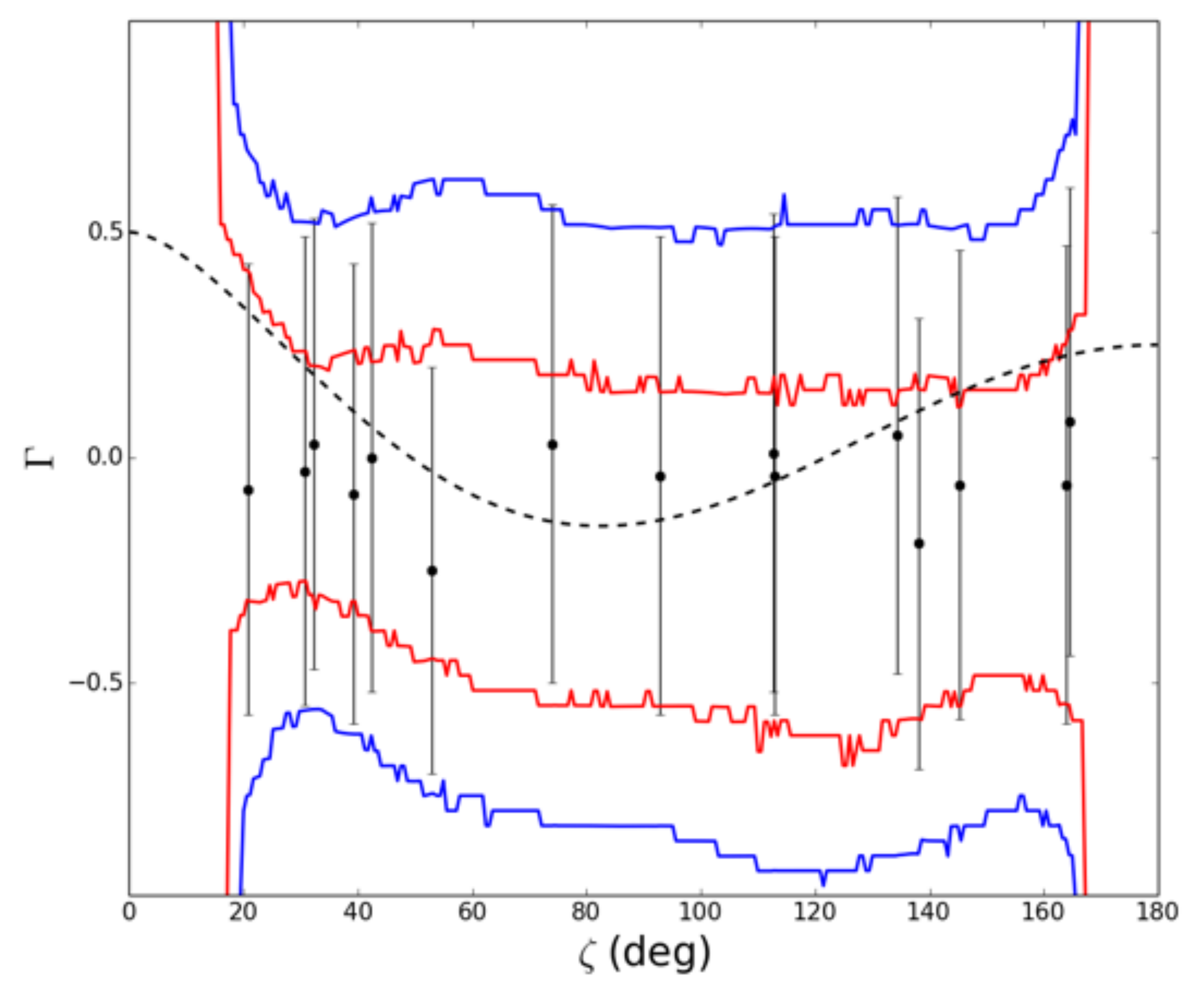
    measures how likely it is that we have found a cross-correlated signal in our data rather than an uncorrelated signal
  - Measurement of cross-power values $$\chi_{IJ} = \frac{\delta \mathbf{t}_I^{\mathrm{T}} \mathbf{P}_I^{-1} \hat{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \delta \mathbf{t}_J}{\mathrm{tr}\left[\mathbf{P}_I^{-1} \hat{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \hat{\mathbf{S}}_{JI}\right]}$$

    with $\mathbf{S}_{IJ} = A^2 \zeta_{IJ} \hat{\mathbf{S}}_{IJ}$ and error $\sigma_{0,IJ} = \left(\mathrm{tr}\left[\mathbf{P}_I^{-1} \hat{\mathbf{S}}_{IJ} \mathbf{P}_J^{-1} \hat{\mathbf{S}}_{JI}\right]\right)^{-1/2}$

    => high SNR limit: cross-power values = Hellings Downs curve

Results

- Results of optimal-statistic on EPTA data

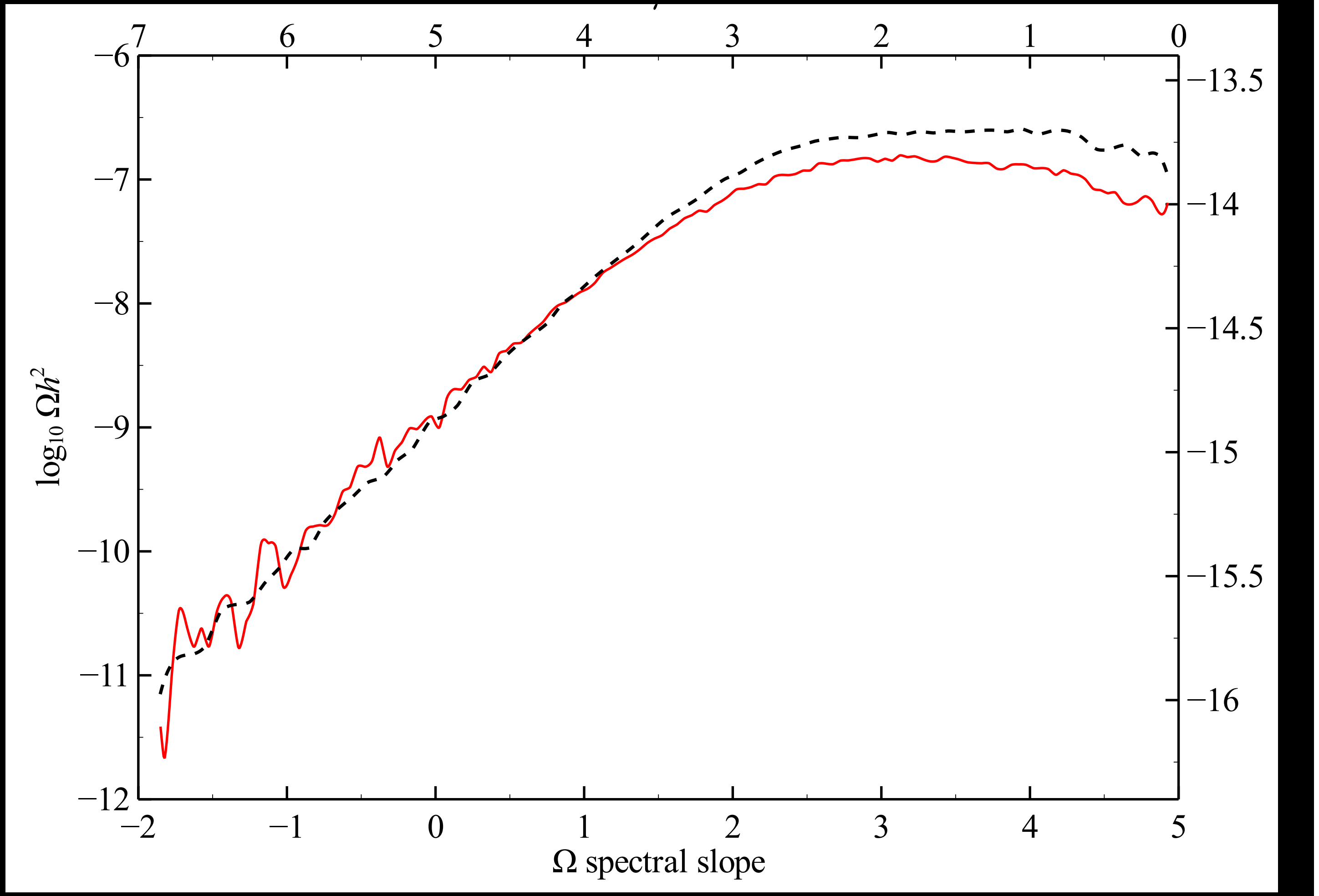


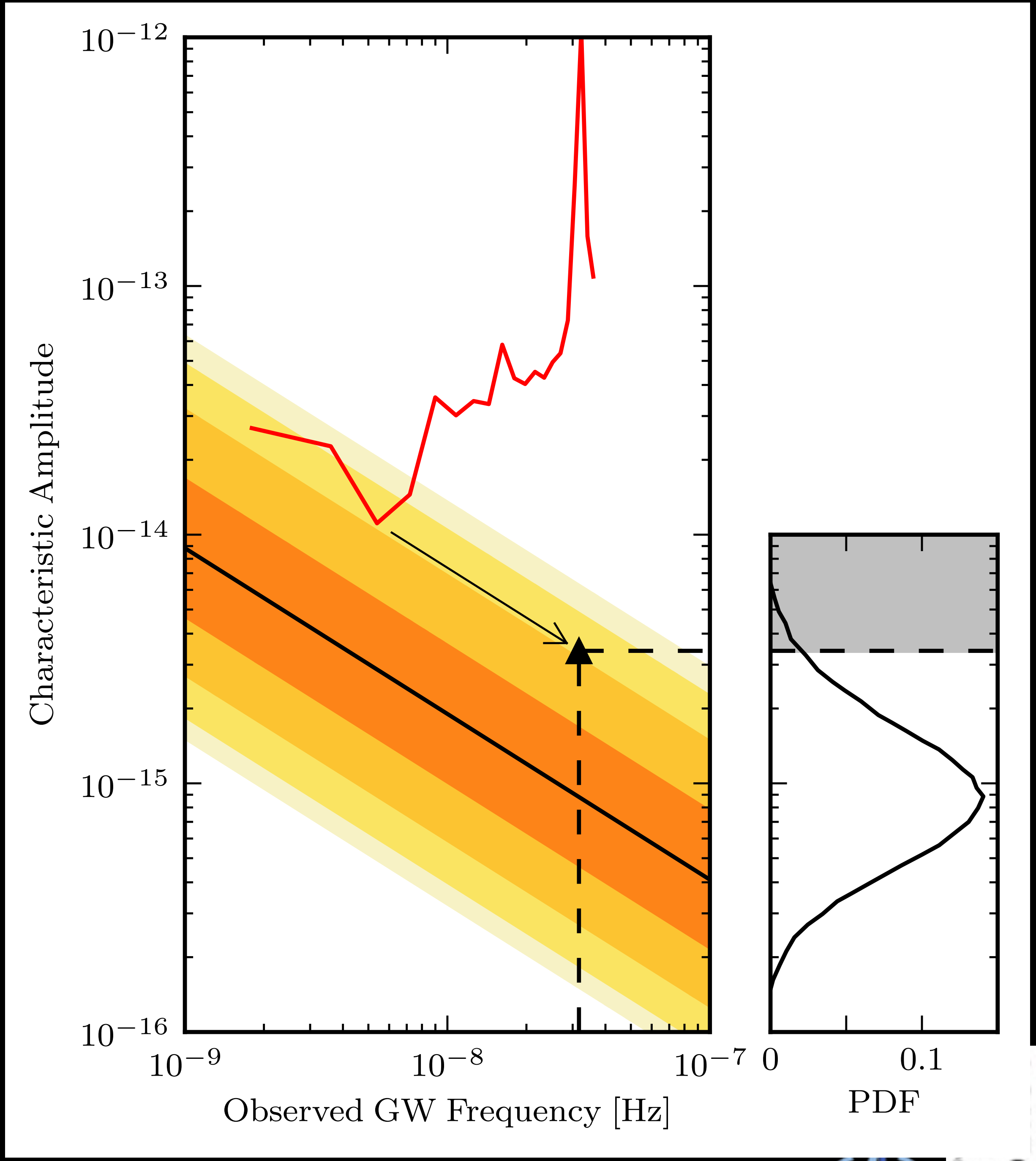
Lentati et al. EPTA (2015)

Upper limit result

- EPTA data

Lentati et al. EPTA (2015)

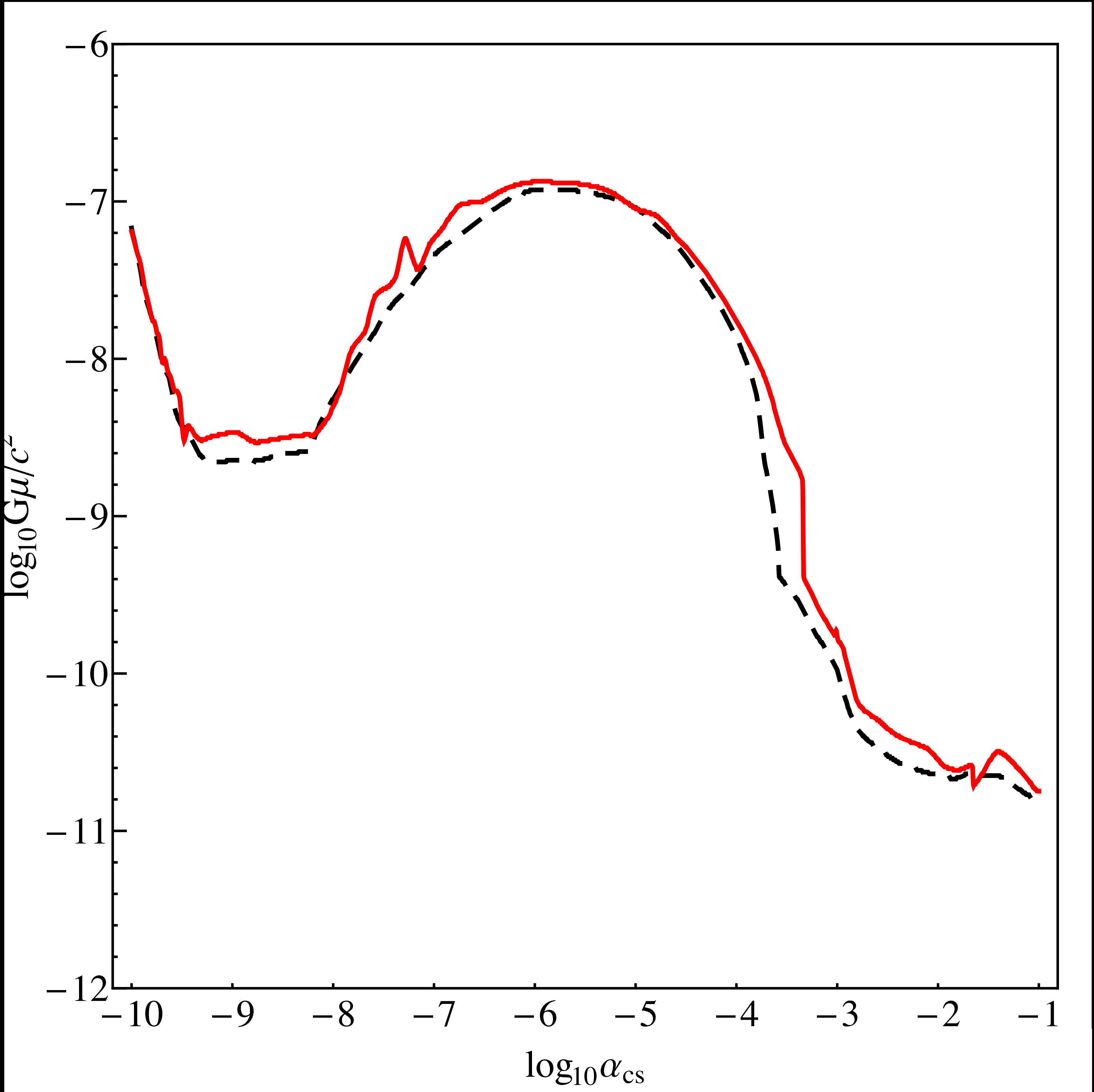




Upper limit on cosmic strings

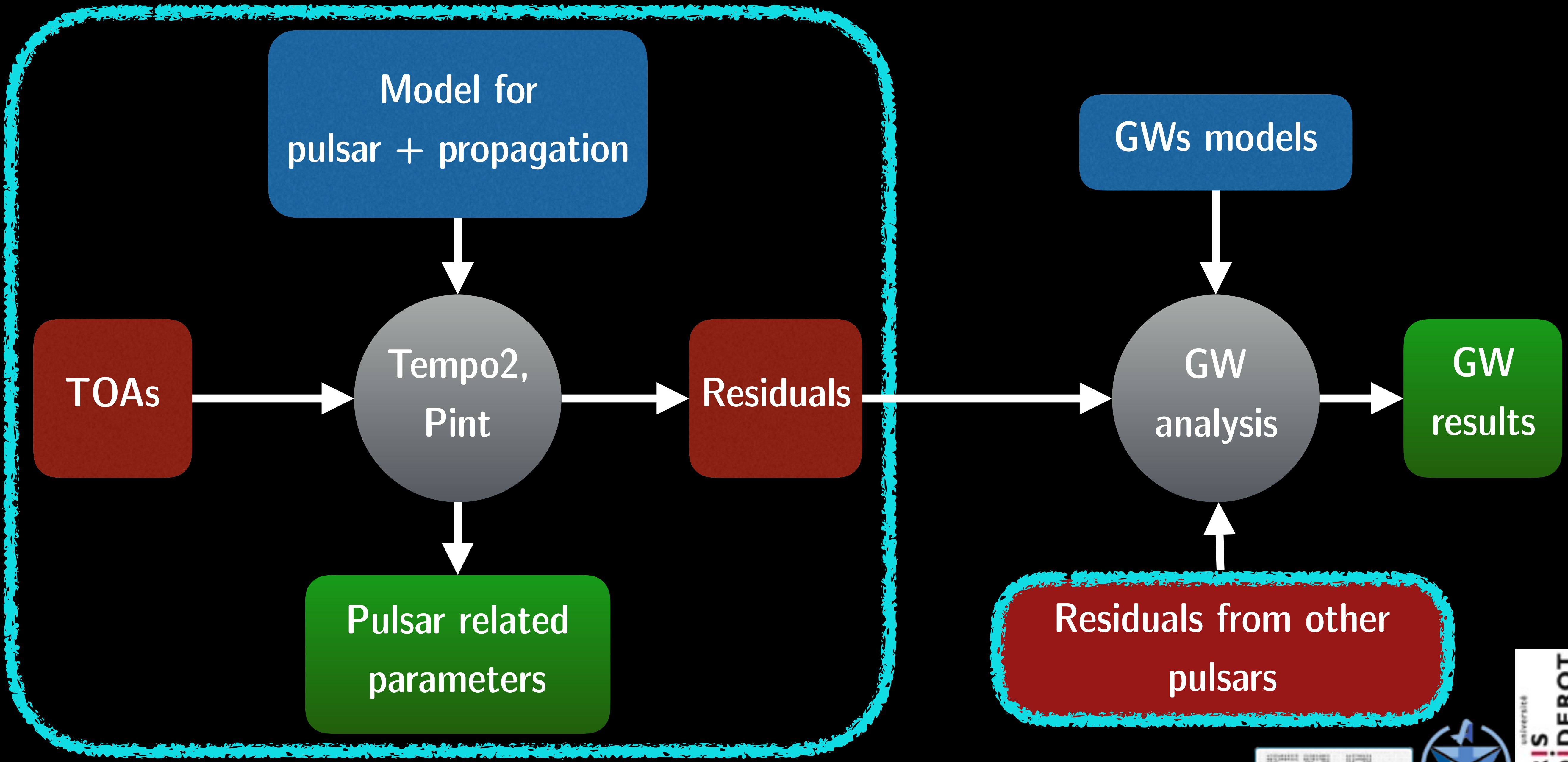
Lentati et al. EPTA (2015)

- Background from cosmic string network.
- Parameters
  - string tension $G\mu/c^2$,
  - α_{cs}: the birth-scale of loops relative to the horizon.
  - ...



Global analysis

- Separated approach …


Model for pulsar + propagation

TOAs

Tempo2, Pint

Residuals

Pulsar related parameters

GWs models

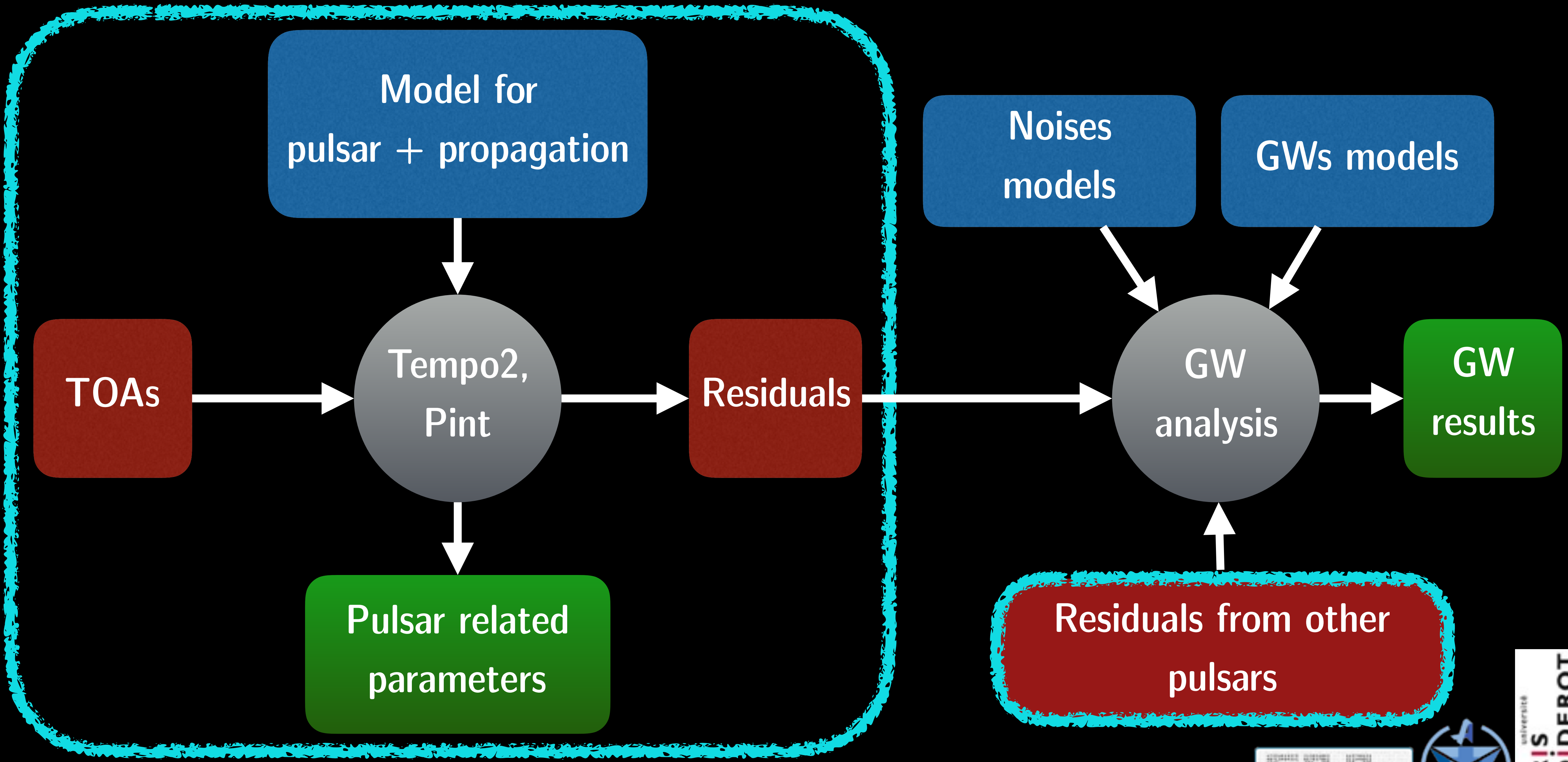
GW analysis

GW results

Residuals from other pulsars

Global analysis

- Separated approach including noises



Model for pulsar + propagation

TOAs

Tempo2, Pint

Residuals

Pulsar related parameters

Noises models

GWs models

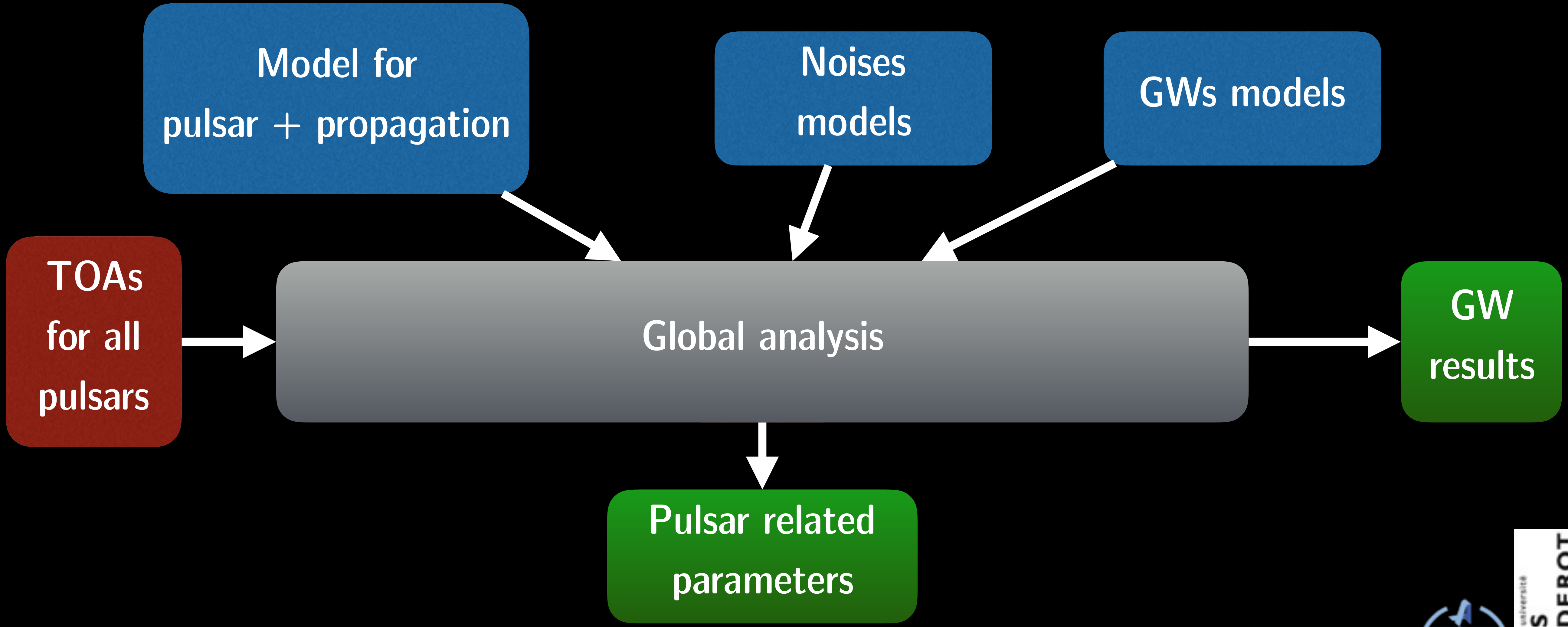
GW analysis

GW results

Residuals from other pulsars

Global analysis

- Global analysis including:
  - Pulsar + propagation parameters
  - Noises
  - GWs: continuous wave sources + backgrounds



Model for pulsar + propagation

Noises models

GWs models

TOAs for all pulsars

Global analysis

GW results

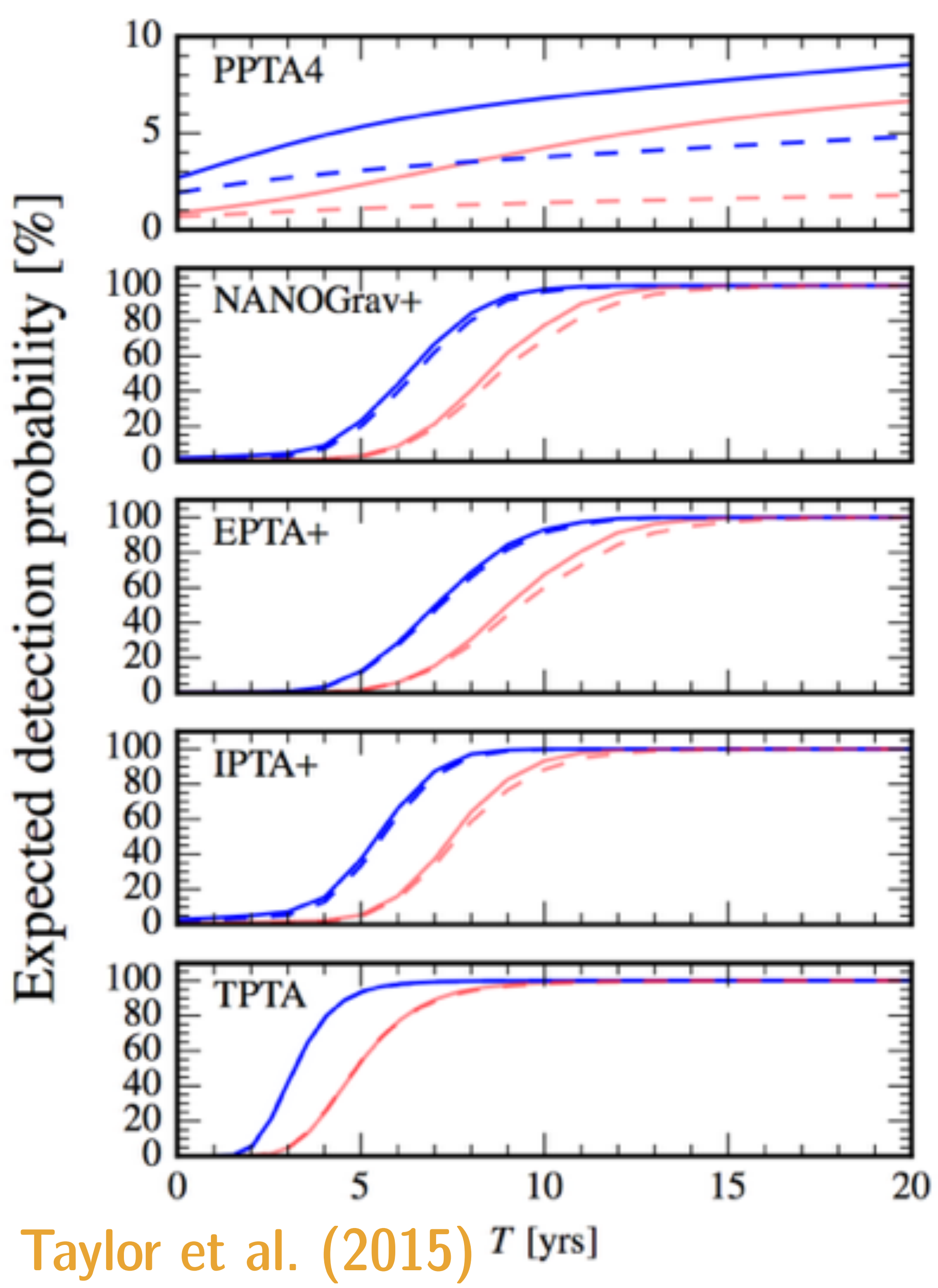
Pulsar related parameters

Global analysis

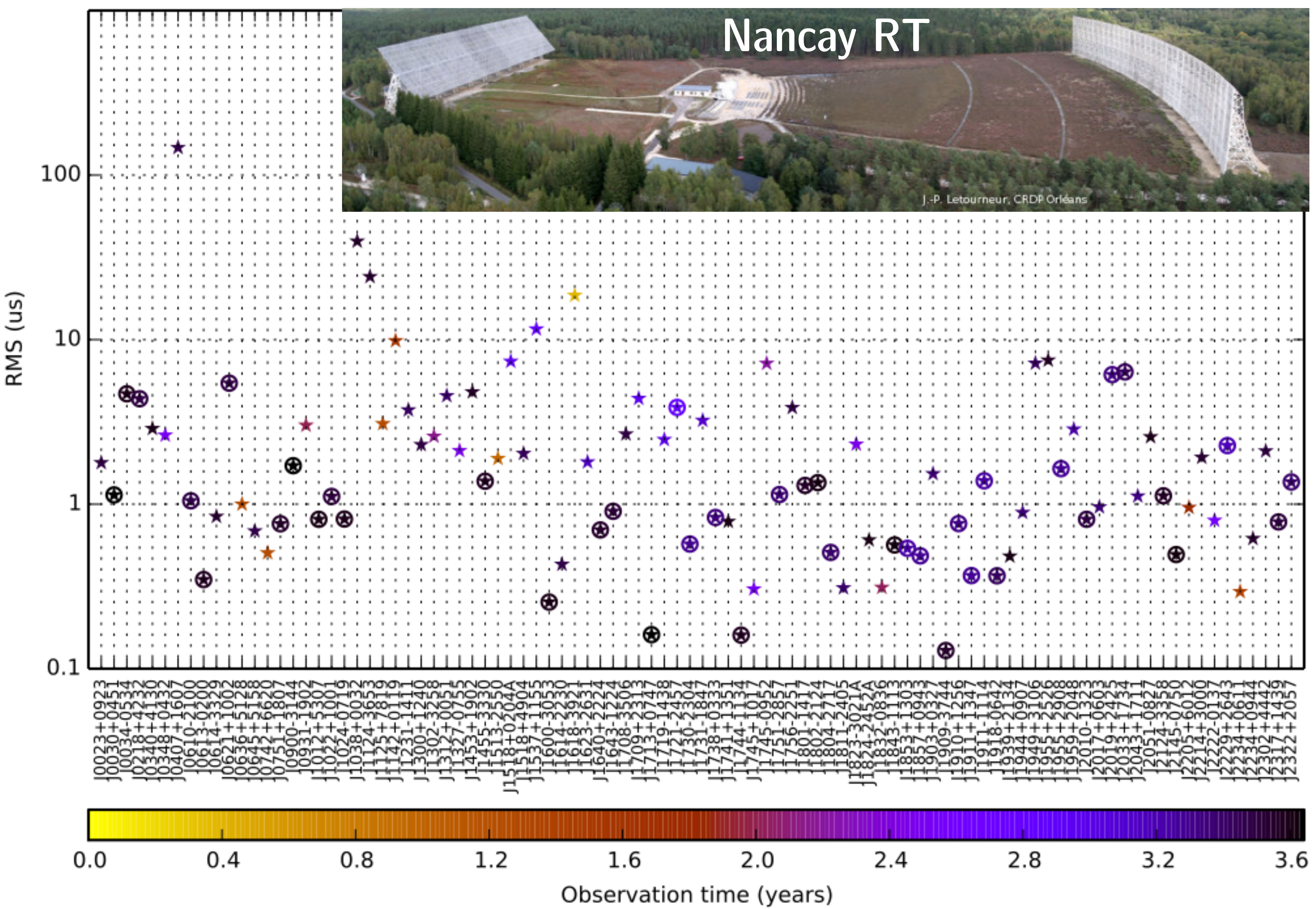
- Global analysis including:
  - Pulsar + propagation parameters
  - Noises
  - GWs: continuous wave sources + backgrounds
- Could also include the pulse template matching
- Work in progress

Future

- More data:
  - Continue to observe the pulsar
  - Group all data in IPTA: EPTA, NANOGrav, PPTA
  - Use more pulsars
  - New instruments: SKA and it's precursor



Taylor et al. (2015)


Nancay RT
J.-P. Letourneur, CRDP Orléans
RMS (us)
100
10
1
0.1
J0023+0923
J0030+0451
J0034-0534
J0218+4232
J0340+4130
J0348+0432
J0407+1607
J0610-2100
J0613-0200
J0614-3329
J0621+1002
J0636+5128
J0645+5158
J0742+6620
J0751+1807
J0900-3144
J0931-1902
J1012+5307
J1022+1001
J1024-0719
J1038+0032
J1124-3653
J1125+7819
J1142+0119
J1231-1411
J1300+1240
J1302-3258
J1312+0051
J1327-0755
J1453+1902
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J1518+4904
J1537+1155
J1600-3053
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J1618-3921
J1623-2631
J1640+2224
J1643-1224
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J1730-2304
J1731-1847
J1738+0333
J1741+1351
J1744-1134
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J1745-0952
J1751-2857
J1756-2251
J1801-1417
J1802-2124
J1804-2717
J1811-2405
J1823-3021A
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J1857+0943
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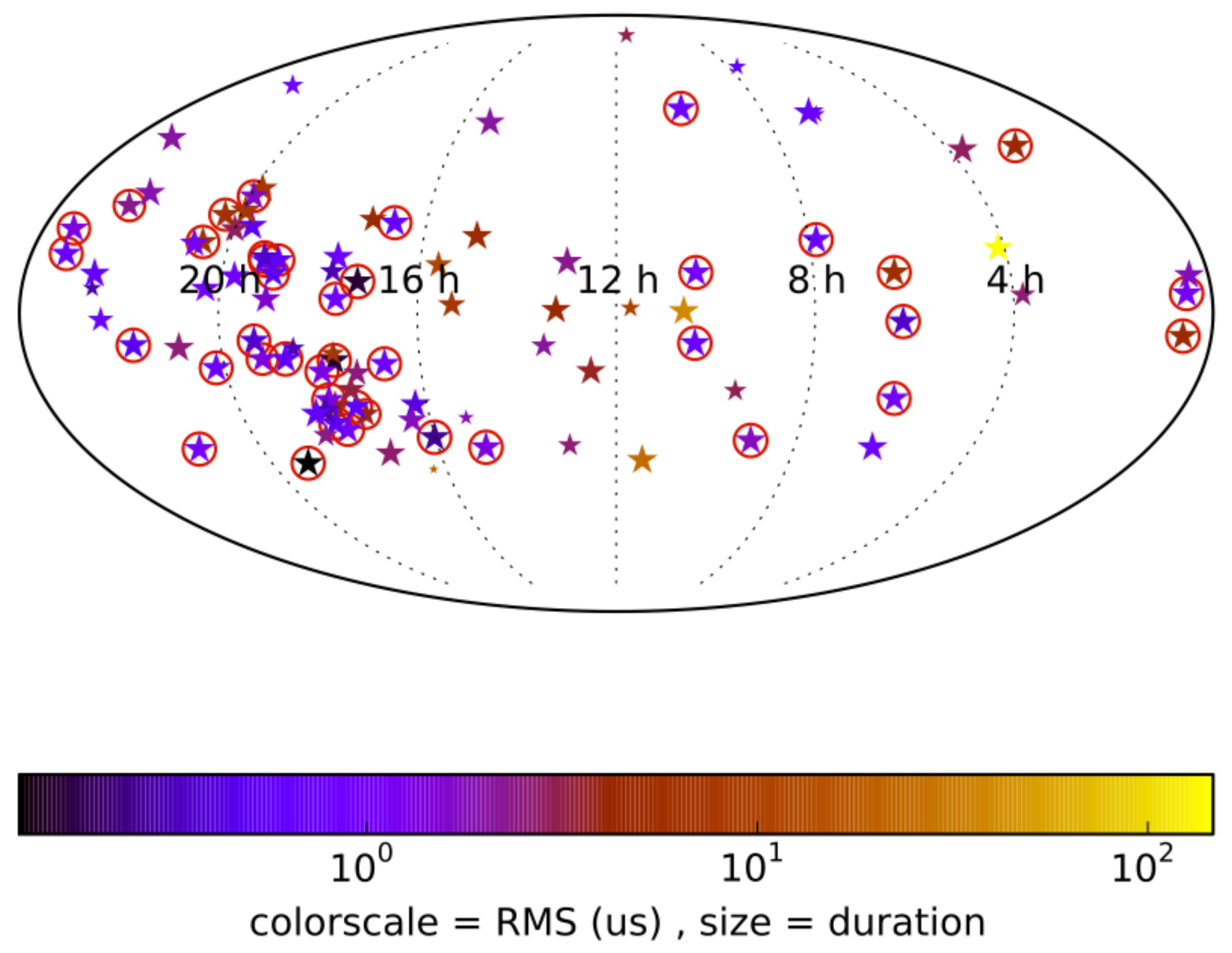
J1910+1256
J1911+1347
J1911-1114
J1918-0642
J1939+2134
J1944+0907
J1949+3106
J1955+2526
J1955+2908
J1959+2048
J2010-1323
J2017+0603
J2019+2425
J2033+1734
J2043+1711
J2051-0827
J2124-3358
J2145-0750
J2205+6012
J2214+3000
J2222-0137
J2229+2643
J2234+0611
J2234+0944
J2302+4442
J2317+1439
J2322+2057
0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
Observation time (years)

Future

- M



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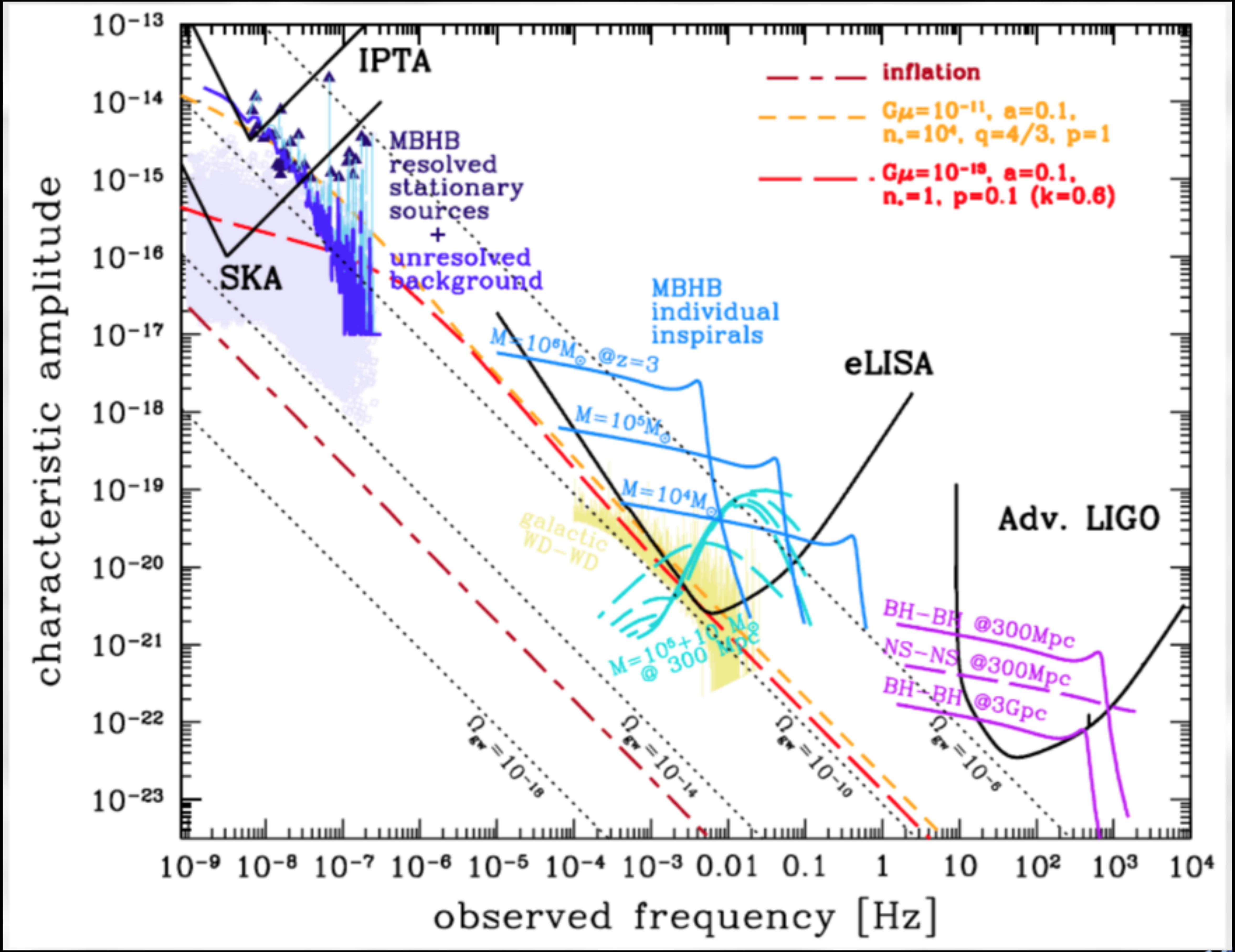

IPTA
SKA
MBHB resolved stationary sources + unresolved background
inflation
Gμ=10⁻¹¹, a=0.1, n.=10⁴, q=4/3, p=1
Gμ=10⁻¹³, a=0.1, n.=1, p=0.1 (k=0.6)
MBHB individual inspirals
M=10⁶M⊙ @z=3
M=10⁵M⊙
M=10⁴M⊙
eLISA
galactic WD-WD
M=10⁵+10 M⊙ @ 300 Mpc
Adv. LIGO
BH-BH @300Mpc
NS-NS @300Mpc
BH-BH @3Gpc
Ω_gw = 10⁻¹⁸
Ω_gw = 10⁻¹⁴
Ω_gw = 10⁻¹⁰
Ω_gw = 10⁻⁶
characteristic amplitude
10⁻¹³
10⁻¹⁴
10⁻¹⁵
10⁻¹⁶
10⁻¹⁷
10⁻¹⁸
10⁻¹⁹
10⁻²⁰
10⁻²¹
10⁻²²
10⁻²³
10⁻⁹
10⁻⁸
10⁻⁷
10⁻⁶
10⁻⁵
10⁻⁴
10⁻³
0.01
0.1
1
10
10²
10³
10⁴
observed frequency [Hz]

Thank you