METHODES D'APPROXIMATION

- Approximation post-newtonienne
- Problème des équations du mouvement et lagrangien de Fokker
- Régularisations du champ propre des particules ponctuelles
- Développement post-Minkowskien & équation de raccord
- Champ d'onde des binaires compactes spiralantes

Gauge-fixed Einstein-Hilbert action

We come back to the EH action without torsion.

$$S = \frac{1}{16\pi}\int d^4x \sqrt{-g}\, R \quad + S_m \qquad (G=c=1)$$

The Ricci scalar R is a function of the metric g, and space-time derivative ∂g and $\partial^2 g$. But the second-order derivatives appear in the form of a surface-term (already studied). We derive the Landau-Lifchitz form of the action, which depends only on g and ∂g (ignoring the surface term).

$$\sqrt{-g}\, R = \sqrt{-g}\, g^{\mu\nu}\left(\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\epsilon_{\mu\nu}\Gamma^\lambda_{\epsilon\lambda} - \Gamma^\epsilon_{\mu\lambda}\Gamma^\lambda_{\nu\epsilon}\right)$$

operate by parts and throw the pure divergence

$$= -\underbrace{\partial_\lambda(\sqrt{-g}\,g^{\mu\nu})}_{}\Gamma^\lambda_{\mu\nu} + \underbrace{\partial_\nu(\sqrt{-g}\,g^{\mu\nu})}_{}\Gamma^\lambda_{\mu\lambda} + \sqrt{-g}\left(\Gamma^\epsilon_{\mu\nu}\Gamma^\lambda_{\epsilon\lambda} - \Gamma^\epsilon_{\mu\lambda}\Gamma^\lambda_{\nu\epsilon}\right)$$

re-express $\partial_\lambda g^{\mu\nu}$ in terms of Christoffel symbols

Hence the Landau-Lifchitz form (which is not manifestly covariant, but in fact it's modulo a surface term)

$$\boxed{S = \frac{1}{16\pi}\int d^4x \sqrt{-g}\, g^{\mu\nu}\left(\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\lambda}\right) + S_m}$$

Posing $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ "gothic" metric (and its inverse $\mathfrak{g}_{\mu\nu} = \frac{1}{\sqrt{-g}}\, g_{\mu\nu}$) we have the completely explicit form

$$\boxed{S = -\frac{1}{64\pi}\int d^4x \Big[\Big(\mathfrak{g}_{\mu\rho}\mathfrak{g}_{\nu\sigma} - \frac{1}{2}\mathfrak{g}_{\mu\nu}\mathfrak{g}_{\rho\sigma}\Big)\mathfrak{g}^{\lambda\tau}\partial_\lambda \mathfrak{g}^{\mu\nu}\partial_\tau \mathfrak{g}^{\rho\sigma} - 2\,\mathfrak{g}_{\mu\nu}\partial_\rho \mathfrak{g}^{\mu\sigma}\partial_\sigma \mathfrak{g}^{\nu\rho}\Big] + S_m}$$

This action is generally covariant ("general diffeomorphism" invariance). In particular, in an infinitesimal coord. transformation

$$x'^\mu = x^\mu + \xi^\mu(x)$$

it is invariant when $g_{\mu\nu} \longrightarrow g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$ with $\mathcal{L}_\xi g_{\mu\nu} = -2\nabla_{(\mu}\epsilon_{\nu)}$

or equivalently $\mathfrak{g}^{\mu\nu} \longrightarrow \mathfrak{g}^{\mu\nu} + \mathcal{L}_\epsilon \mathfrak{g}^{\mu\nu}$ with (exercice)

$$\mathcal{L}_\xi \mathfrak{g}^{\mu\nu} = \sqrt{-g}\,\big(2\nabla^{(\mu}\xi^{\nu)} - g^{\mu\nu}\nabla_\rho \xi^\rho\big)$$

If we consider an expansion around a given background (for instance flat)

$$\mathfrak{g}^{\mu\nu}(x) = \mathfrak{g}_B^{\mu\nu}(x) + h^{\mu\nu}(x)$$

$$\mathfrak{g}'^{\mu\nu}(x) = \mathfrak{g}_B^{\mu\nu}(x) + h'^{\mu\nu}(x)$$

the diffeo invariance implies a <u>gauge invariance</u> for the potentials $h \longrightarrow h'$ with

$$\boxed{h'^{\mu\nu}(x) = h^{\mu\nu}(x) + \sqrt{-g_B}\,\big(2\nabla_B^{(\mu}\xi^{\nu)} - g_B^{\mu\nu}\nabla_{B\rho}\xi^\rho\big)}$$

cov. derivative w.r.t. the background

We now write the <u>gauge-fixed</u> action corresponding to harmonic (or de Donder) coordinates defined by

$$\boxed{\partial_\nu \mathfrak{g}^{\mu\nu} = 0}$$

or equivalently $\Gamma^\mu \equiv g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = 0$ since $\boxed{\sqrt{-g}\,\Gamma^\mu = -\partial_\nu \mathfrak{g}^{\mu\nu}}$

We simply have to add a term $\frac{-1}{2}\sqrt{-g}\,g_{\mu\nu}\Gamma^\mu\Gamma^\nu = -\frac{1}{2}\mathfrak{g}_{\mu\nu}\partial_\rho \mathfrak{g}^{\mu\rho}\partial_\sigma \mathfrak{g}^{\nu\sigma}$ to the LL action

$$S_{\substack{\text{gauge}\\\text{fixed}}} = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\left[g^{\mu\nu}\left(\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\lambda}\right) - \frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu\right] + S_m$$

$$\boxed{\begin{aligned} S_{g.f.} = -\frac{1}{64\pi}\int d^4x\Big[&\left(\mathfrak{g}_{\mu\rho}\mathfrak{g}_{\nu\sigma} - \frac{1}{2}\mathfrak{g}_{\mu\nu}\mathfrak{g}_{\rho\sigma}\right)\mathfrak{g}^{\lambda\sigma}\partial_\lambda \mathfrak{g}^{\mu\nu}\partial_\tau \mathfrak{g}^{\rho\sigma} \\ &\underbrace{-2\mathfrak{g}_{\mu\nu}\left(\partial_\rho \mathfrak{g}^{\mu\sigma}\partial_\sigma \mathfrak{g}^{\nu\rho} - \partial_\rho \mathfrak{g}^{\mu\rho}\partial_\sigma \mathfrak{g}^{\nu\sigma}\right)}_{\substack{\text{pure divergence at}\\\text{"quadratic order"}\\\text{in an expansion } \mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}}}\Big] + S_m \end{aligned}}$$

General form of the harmonic-gauge Einstein field eqs

$$\boxed{\mathfrak{g}^{\rho\sigma}\partial^2_{\rho\sigma}\mathfrak{g}^{\mu\nu} = 16\pi\,|g|\,T^{\mu\nu} + \underbrace{\Sigma^{\mu\nu}[\mathfrak{g},\partial\mathfrak{g}]}_{\substack{\text{(gravitational source}\\\text{term)}}}}$$

and the gauge condition $\partial_\nu \mathfrak{g}^{\mu\nu} = 0$ which is in fact implied by the latter equations when the EOM of matter are satisfied, $\nabla_\nu T^{\mu\nu} = 0$.

The field eqs in such form appear as wave-like equations, with an hyperbolic wave operator $\mathfrak{g}^{\rho\sigma}\partial_\rho\partial_\sigma$, with only first order derivatives $\partial\mathfrak{g}$ in the right-hand-side, and form a well defined hyperbolic system of eqs, with a well-posed Cauchy problem.

If we further assume that space-time is a small deviation from a flat Minkowski background

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \qquad \left(|h^{\mu\nu}| \ll 1\right)$$

we have (posing $\Box = \Box_\eta = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$ the flat D'Alembertian operator)

$$\boxed{\Box h^{\mu\nu} = 16\pi\, |g|\, T^{\mu\nu} + \Lambda^{\mu\nu}[h, \partial h, \partial^2 h]}$$

where $\Lambda^{\mu\nu} = \Sigma^{\mu\nu} - h^{\rho\sigma}\partial^2_{\rho\sigma}h^{\mu\nu}$. The gravitational source term is at least quadratic in h and is amenable to a perturbative expansion

$$\Lambda \sim \underbrace{h\,\partial^2 h + \partial h\,\partial h}_{\text{quadratic}} + \underbrace{h\,\partial h\,\partial h}_{\text{cubic}} + \underbrace{h\,h\,\partial h\,\partial h}_{\text{quartic}} + \dots$$

Posing

$$\boxed{\tau^{\mu\nu} = |g|\, T^{\mu\nu} + \frac{1}{16\pi}\Lambda^{\mu\nu}}$$

this is the pseudo-stress energy tensor of the matter fields and gravitational fields (in harmonic coordinates). The coord. condition is equivalent to the EOM of matter fields

$$\boxed{\partial_\nu h^{\mu\nu} = 0 \iff \partial_\nu \tau^{\mu\nu} = 0 \iff \nabla_\nu T^{\mu\nu} = 0}$$

The Fokker action

We consider the problem of motion of particles under their mutual gravitation. Thus this is a self-gravitating problem, where the dynamics is purely gravitational.

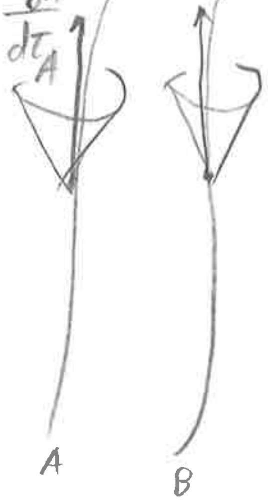
Thus we are not considering the motion of particles in a given background metric (as we did in a previous course) but the motion of particles in the metric generated by the particles.

The matter action will be

$$u_A^\mu = \frac{dy_A^\mu}{d\tau_A}$$

$$S_m = \sum_A -m_A \int_{-\infty}^{+\infty} d\tau_A$$

proper time $d\tau_A = \sqrt{-(g_{\mu\nu})_A\, dy_A^\mu\, dy_A^\nu}$



with stress-energy tensor

$$T^{\mu\nu} = \sum_A m_A \int_{-\infty}^{+\infty} d\tau_A\, u_A^\mu u_A^\nu \frac{\delta_{(4)}(x-y_A)}{\sqrt{-g}}$$

or (in 3-dimensional form)

$$\boxed{T^{\mu\nu} = \sum_A m_A \frac{v_A^\mu v_A^\nu}{\sqrt{-(g_{\rho\sigma})_A v_A^\rho v_A^\sigma}} \frac{\delta_{(3)}(\vec{x}-\vec{y}_A)}{\sqrt{-g}}}$$

where $y_A^\mu = (ct, \vec{y}_A(t))$ $\quad v_A^\mu = (c, \vec{v}_A(t))$ $\quad \vec{v}_A = \frac{d\vec{y}_A}{dt}$ coordinate velocity

This $T^{\mu\nu}$ is on the RHS of the Einstein field eqs and, solving them, we get a solution of the field eqs as an explicit functional of the positions, velocities, ... of particles.

$$\boxed{\bar{h}^{\mu\nu} = \bar{h}^{\mu\nu}\big(\vec{x}; \underbrace{\vec{y}_A(t), \vec{v}_A(t), \vec{a}_A(t), \dots}_{\substack{\text{dependence on particles} \\ \text{with in general presence of accelerations } \vec{a}_A, \\ \text{derivative of accelerations } \vec{b}_A, \dots}}\big)}$$

For instance the metric evaluated at position $\vec{y}_A(t)$ will be

$$(\bar{g}_{\mu\nu})_A = \bar{g}_{\mu\nu}\big(\vec{y}_A(t); \vec{y}_B(t), \vec{v}_B(t), \dots\big)$$

Such solution $\bar{h}^{\mu\nu}$ (or $\bar{g}_{\mu\nu}$) can be constructed order by order as a post-Newtonian (PN) expansion, see below.

The Fokker action is obtained when we insert this explicit solution back into the (gauge-fixed) EH action thus

$$\boxed{\begin{aligned} S_{\text{Fokker}}[\vec{y}_A, \vec{v}_A, \vec{a}_A, \dots] = &\int d^4x\, \mathcal{L}_g\big[\bar{h}^{\mu\nu}(\vec{x}; \vec{y}_A(t), \vec{v}_A(t), \dots)\big] \\ &- \sum_B m_B \int dt \sqrt{-\bar{g}_{\mu\nu}(\vec{y}_B; \vec{y}_A, \vec{v}_A \dots)\, v_B^\mu v_B^\nu} \end{aligned}}$$

The E.O.M. of particles follow from varying the Fokker action wrt the particles.

$$\frac{\delta S_F}{\delta y_A^i} = \underbrace{\frac{\delta S}{\delta g_{\mu\nu}}[\bar{g}]}_{\substack{\text{functional variation} \\ \text{of the full EH action} \\ \text{w.r.t. metric}}} \frac{\delta \bar{g}_{\mu\nu}}{\delta y_A^i} + \underbrace{\frac{\delta S_m}{\delta y_A^i}[\bar{g}]}_{\substack{\text{variation of} \\ \text{matter action}}}$$

But $\frac{\delta S}{\delta g_{\mu\nu}}\left[\bar{g}(\vec{x};\vec{y}_A,\cdots)\right]=0$ because $\bar{g}$ is a solution of the field eqs

$$\boxed{\frac{\delta S_F}{\delta y^i_A}[\bar{g}] = \frac{\delta S_m}{\delta y^i_A}[\bar{g}] = 0}$$

which are thus the correct EOM of particles in the metric $\bar{g}_{\mu\nu}(\vec{x};\vec{y}_A\cdots)$ generated by the particles themselves.

Equations of motion in the PN approximation

Matter source (say N particles)

- isolated ($T^{\mu\nu}$ has a compact support)
- PN, i.e. "slowly moving", existence of a small parameter

$$\boxed{\varepsilon \sim \frac{v}{c} \ll 1}$$

- self-gravitating, internal motion is due to gravitational forces

$$\gamma \sim \frac{v^2}{a} \sim \frac{GM}{a^2}$$

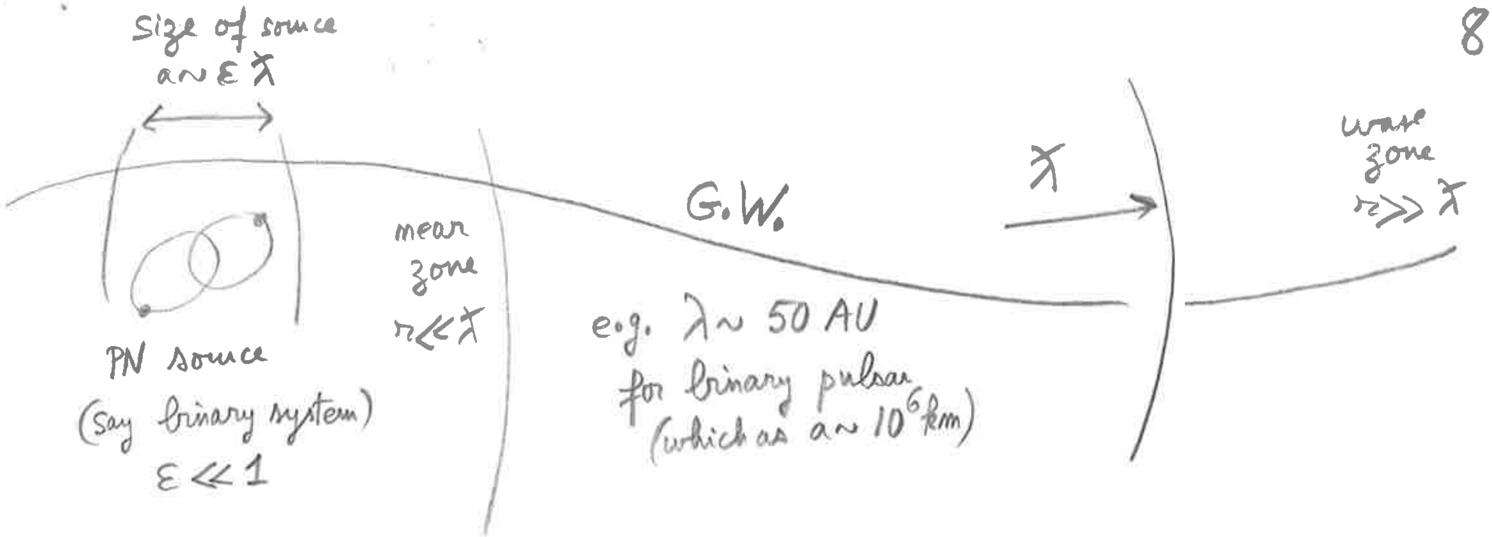
a = size of source

M = its mass

Period of motion $P \sim \frac{2\pi a}{v}$

Gravitational wave length $\lambda = cP$ $\quad \lambdabar = \frac{\lambda}{2\pi}$

$$\boxed{\frac{a}{\lambdabar} \sim \frac{v}{c} \sim \varepsilon}$$



The near zone $r \ll ƛ$ covers entirely the PN source.

In the near zone the PN expansion $\varepsilon \to 0$ is valid

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \underbrace{\Lambda^{\mu\nu}}_{\text{neglect to leading Newtonian order}}$$

$$T^{00} \sim \rho c^2 = \mathcal{O}(c^2)$$
$$T^{0i} \sim \rho c v^i = \mathcal{O}(c)$$
$$T^{ij} \sim \rho v^i v^j = \mathcal{O}(1)$$

to leading order

$$\Delta h^{00} = \frac{16\pi G}{c^2}\rho + \mathcal{O}\left(\frac{1}{c^4}\right)$$
$$\Delta h^{0i} = \mathcal{O}\left(\frac{1}{c^3}\right)$$
$$\Delta h^{ij} = \mathcal{O}\left(\frac{1}{c^4}\right)$$

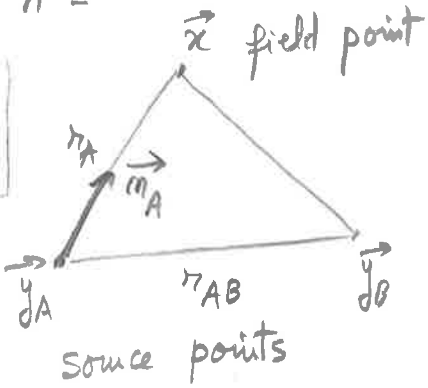
U usual Newtonian potential of source

$$\Delta U = -4\pi G\rho$$

$$h^{00} \simeq -\frac{4U}{c^2} \qquad h^{0i} \simeq 0 \qquad h^{ij} \simeq 0$$

For N point particles

$$\boxed{U = \sum_{A=1}^{N} \frac{G m_A}{r_A}}$$



$$r_A^i = x^i - y_A^i(t)$$

$$r_A = |\vec{r}_A| \qquad n_A^i = \frac{r_A^i}{r_A}$$

$$\mathcal{L}_g \approx -\frac{c^4}{64\pi G}\left[\partial_i h^{00}\partial_i h^{00} - \frac{1}{2}\partial_i h\,\partial_i h\right] + O\left(\frac{1}{c^2}\right)$$

$$h = -h^{00} + h^{ii} \approx -h^{00}$$

$$= -\frac{1}{8\pi G}\partial_i U\,\partial_i U + O\left(\frac{1}{c^2}\right)$$

$$g_{00} \approx -1 + \frac{2U}{c^2}$$

$$g_{ij} \approx \delta_{ij}$$

$$L_m = -\sum_A m_A c\sqrt{-(g_{\mu\nu})_A v_A^\mu v_A^\nu}$$

$$= \sum_A m_A\left(-c^2 + \frac{\vec{v}_A^{\,2}}{2} + U_A\right) + O\left(\frac{1}{c^2}\right)$$

Fokker Lagrangian at Newtonian order

$$L^F = -\frac{1}{8\pi G}\int d^3x\,\partial_i U\,\partial_i U + \sum_A m_A\left(-c^2 + \frac{\vec{v}_A^{\,2}}{2} + U_A\right) + O\left(\frac{1}{c^2}\right)$$

$$\int d^3x\,\partial_i U\,\partial_i U = -\int d^3x\,U\Delta U = 4\pi G\int d^3x\,U\sum m_A\,\delta_{(3)}(\vec{x}-\vec{y}_A)$$

$$= 4\pi G\sum_A m_A U_A$$

$$\boxed{L^F = \sum_A m_A\left(-c^2 + \frac{\vec{v}_A^{\,2}}{2} + \frac{U_A}{2}\right) + O\left(\frac{1}{c^2}\right)}$$

We need a <u>self-field regularization</u> to remove the infinite self-field of particles.

$$U_A = \left(\sum_B \frac{Gm_B}{r_B}\right)_A = \sum_{B\neq A}\frac{Gm_B}{r_{AB}}$$

$$\frac{1}{2}\sum_A m_A U_A = \frac{1}{2}\sum_{A=1}^{N} m_A \sum_{B\neq A}\frac{G m_B}{r_{AB}}$$

$$= \sum_{A=1}^{N}\sum_{B=A+1}^{N}\frac{G m_A m_B}{r_{AB}}$$

$$= \sum_{A<B}\frac{G m_A m_B}{r_{AB}}$$

$$\boxed{L^F = \sum_A m_A\left(-c^2+\frac{\vec{v}_A^{\,2}}{2}\right) + \sum_{A<B}\frac{G m_A m_B}{r_{AB}} + \mathcal{O}\left(\frac{1}{c^2}\right)}$$

and we recover the Lagrangian of N point masses in Newtonian dynamics. The Fokker Lagrangian of 2 point-masses is known at 4PN order, $\left(\frac{v}{c}\right)^8$ beyond Newtonian term.

For two particles the EOM, which derive from the Lagrangian for the conservative part, and on which we add some radiation-reaction terms (studied later) read

$$\frac{dv_1^i}{dt} = -\frac{G m_2}{r_{12}^2} n_{12}^i$$

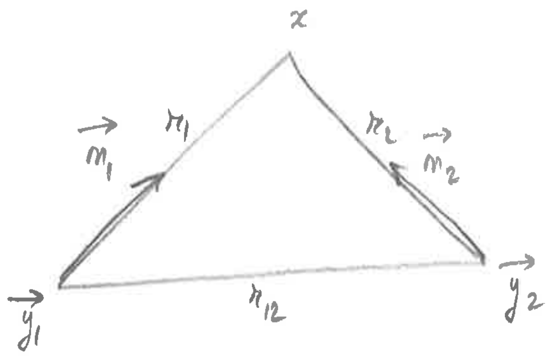
$$+ \frac{1}{c^2}\overbrace{\left\{\left(\frac{5G^2 m_1 m_2}{r_{12}^3} + 4\frac{G^2 m_2^2}{r_{12}^3} + \frac{G m_2}{r_{12}^1}\left(\frac{3}{2}(n_{12}v_2)^2 + \cdots\right)\right) n_{12}^i + \frac{G m_2}{r_{12}^2}\Big(4(n_{12}v_1) - 3(n_{12}v_2)\Big) v_{12}^i\right\}}^{\text{Lorentz-Droste-Einstein-Infeld-Hoffmann 1PN term}}$$

$$+ \frac{1}{c^4}\left\{\text{2PN term}\right\} + \frac{1}{c^5}\left\{\text{2.5PN }\underline{\text{radiation reaction term}}\right\}$$

$$\underbrace{+\frac{1}{c^6} + \frac{1}{c^7} + \frac{1}{c^8}}_{\text{also known}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

Problem of point-particles

Two particles (masses m_A)



Newtonian potential generated by the masses

$$\Delta U = -4\pi G\rho = -4\pi G\left[m_1\,\delta(\vec{x}-\vec{y}_1) + m_2\,\delta(\vec{x}-\vec{y}_2)\right]$$

Using $\Delta \frac{1}{r} = -4\pi\,\delta(\vec{x})$ $\qquad U(\vec{x}) = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}$

acceleration $\quad \frac{dv_1^i}{dt} = (\partial_i U)(\vec{y}_1) = \Big(\underbrace{-\frac{Gm_1}{r_1^2}\,n_1^i}_{\text{infinite self-force of the point particle}} - \frac{Gm_2}{r_2^2}\,n_2^i\Big)(\vec{y}_1)$

If $F(\vec{x})$ is singular at $\vec{y}_1$ and $\vec{y}_2$ (with power-like singular expansion around $\vec{y}_1, \vec{y}_2$) what are the meanings of

$F(\vec{y}_1)$?

$F(\vec{x})\,\delta(\vec{x}-\vec{y}_1)$?

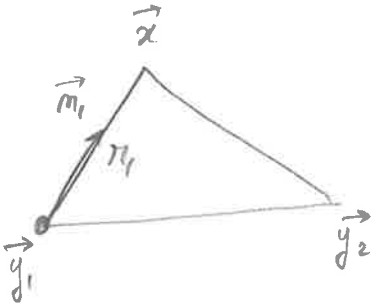
$\partial_i F$? $\quad$ (for instance $\partial_i\partial_j \frac{1}{r_1} = \frac{3n_1^i n_1^j - \delta^{ij}}{r_1^3} \underbrace{- \frac{4\pi}{3}\delta^{ij}\,\delta(\vec{x}-\vec{y}_1)}_{\text{distributional term}}$)

$\int d^3\vec{x}\, F(\vec{x})$?

Answer to all these problems needed to answer the problems of motion and radiation from <u>compact objects</u> modelled by point-particles in PN approximations.

Hadamard self-field regularization

$F(\vec{x})$ smooth except at $\vec{y}_1$ and $\vec{y}_2$.

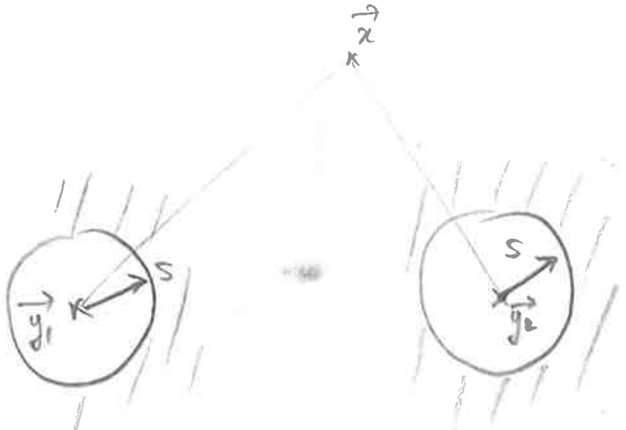


When $r_1 \longrightarrow 0$

$$F(\vec{x}) = \sum_{a_0 \leq a \leq N} r_1^a \underset{1}{f}_a(\vec{n}_1) + o(r_1^N) \qquad a_0, a \in \mathbb{Z}$$

Hadamard's "partie finie" (Pf)

$$\boxed{(F)_1 = \int \frac{d\Omega}{4\pi} \underset{1}{f}_0(\vec{n}_1)}$$



Two balls (radius s) excised

$$\boxed{\begin{aligned} \underset{s_1 s_2}{\mathrm{Pf}} \int d^3x\, F(\vec{x}) = \lim_{s \to 0} \Bigg\{ & \int_{\substack{r_1 > s \\ \text{and } r_2 > s}} d^3x\, F(\vec{x}) \\ & + \sum_{a+3<0} \frac{s^{a+3}}{a+3} \int d\Omega_1 \underset{1}{f}_a(\vec{n}_1) \\ & + \ln\left(\frac{s}{s_1}\right) \int d\Omega_1 \underset{1}{f}_{-3}(\vec{n}_1) + 1 \leftrightarrow 2 \Bigg\} \end{aligned}}$$

$s_1\ s_2$ two arbitrary UV cut-off scales.

$$\boxed{\underset{s_1, s_2}{\mathrm{Pf}} \int d^3x\, F = \underset{\alpha \to 0}{\mathrm{FP}} \ \underset{\beta \to 0}{\mathrm{FP}} \int d^3x \left(\frac{r_1}{s_1}\right)^{\alpha} \left(\frac{r_2}{s_2}\right)^{\beta} F}$$

using analytic continuation in α and $\beta \in \mathbb{C}$.

Hadamard's regularization is very convenient in practical calculactions but in higher PN orders is plagued with ambiguities. Basic reason is that

$$(FG)_1 \neq (F)_1 (G)_1 \quad \text{in general.}$$

Hence basic symmetries of GR such as diffeomorphism invariance are broken in high PN approximations (starting in fact at 3PN).

Dimensional self-field regularization (t'Hooft, Veltman)

We work in a space with d dimensions (so space-time has D=d+1 dimensions).

In D dimensions EFE take the same form

$$\boxed{R^{\mu\nu} - \underset{\substack{\uparrow \\ \text{coeff } \frac{1}{2} \text{ in any dim}}}{\frac{1}{2}} g^{\mu\nu} R = \frac{16\pi G}{c^4} T^{\mu\nu}}$$

but

$$\boxed{R^{\mu\nu} = \frac{8\pi G}{c^4}\left(T^{\mu\nu} - \underset{\substack{\uparrow \\ \text{coeff depends on d}}}{\frac{1}{d-1}} g^{\mu\nu} T\right)}$$

Here $\boxed{G = G_N \ell_0^{d-3}}$ where G_N is the usual Newton's constant and ℓ_0 is the characteristic length of dim.reg.

The dimension $d \in \mathbb{C}$ and we apply complex analytic regularization in d.

Volume element $\boxed{d^dx = r^{d-1}dr\,d\Omega_{d-1}}$

Volume of (d-1) dimensional sphere $\Omega_{d-1} = \int d\Omega_{d-1}$

To compute it we use the Gaussian integral

$$\int_{\mathbb{R}^d} d^dx\, e^{-r^2} = \left(\int_{-\infty}^{+\infty} dx\, e^{-x^2}\right)^d = \pi^{d/2}$$

$$= \int r^{d-1}dr\,d\Omega_{d-1}\, e^{-r^2} = \Omega_{d-1}\int_0^{+\infty} dr\, r^{d-1} e^{-r^2} = \frac{\Omega_{d-1}}{2}\Gamma\left(\frac{d}{2}\right)$$

$$\boxed{\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}}$$

$\Omega_2 = 4\pi$ $\Omega_1 = 2\pi$ and $\Omega_0 = 2$ (since sphere with 0 dimension is made of 2 points!)

Tools used in PN calculations

Green's function of the Laplace operator

$$\boxed{\begin{array}{ll} \Delta u = -4\pi\,\delta^{(d)}(\vec{x}) & \\ u = k\, r^{2-d} & k = \dfrac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-1}{2}}} \end{array}}$$

Riesz Euclidean kernels (generalize $\delta^{(d)}$ and u)

$$\boxed{\begin{array}{l} \delta_\alpha^{(d)}(\vec{x}) = K_\alpha\, r^{\alpha-d} \\[2em] \Delta\,\delta_{\alpha+2}^{(d)} = -\,\delta_\alpha^{(d)} \end{array}} \qquad K_\alpha = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2}\,\Gamma(d/2)}$$

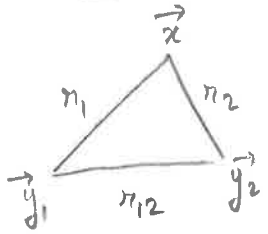
(So that $\delta^{(d)} = \delta_0^{(d)}$ and $u = 4\pi\,\delta_{(2)}^{(d)}$)

Permits to integrate term by term the PN series

Convolution relation $\boxed{\delta^{(d)}_\alpha * \delta^{(d)}_\beta = \delta^{(d)}_{\alpha+\beta}}$

which is an elegant formulation of Riesz's formula

$$\boxed{\int d^dx\, r_1^\alpha r_2^\beta = \pi^{d/2} \frac{\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\beta+d}{2}\right)\Gamma\left(-\frac{\alpha+\beta+d}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2d}{2}\right)}\, r_{12}^{\alpha+\beta+d}}$$



which is extensively used in PN calculations.

Difference between Had. reg. and Dim. reg.

iterating in PN form we have to solve Poisson equations

$$\Delta P = F$$

where F is a non-compact support function singular on $\vec{y}_1, \vec{y}_2$

Had. reg. is given by "partie finie"

$$\boxed{P(\vec{x}') = -\frac{1}{4\pi} \underset{s_1, s_2}{\mathrm{Pf}} \int \frac{d^3\vec{x}}{|\vec{x}-\vec{x}'|} F(\vec{x}')}$$

Dim. reg.

$$\boxed{P^{(d)}(\vec{x}') = -\frac{k}{4\pi} \int \frac{d^d\vec{x}}{|\vec{x}-\vec{x}'|^{d-2}} F^{(d)}(\vec{x}')}$$

We have to compute the values of these solutions at the location of singular point $\vec{y}_1$.

$(P)_1$ (in the sense of "partie finie") and $P^{(d)}(\vec{y}_1) = -\frac{k}{4\pi} \int \frac{d^d\vec{x}}{r_1^{d-2}} F^{(d)}(\vec{x})$

In d dim we have when $r_1 \longrightarrow 0$

$$\overset{(d)}{F}(\vec{x}) = \sum_{p,q} \underbrace{r_1^{\,p+q\varepsilon}}_{\substack{\text{complex powers}\\ \text{of } r_1 \text{ appear}}} \underset{1}{\overset{(\varepsilon)}{f}}_{p,q}(\vec{n}_1) + o(r_1^{\,N}) \qquad \text{where } \varepsilon = d-3$$

$$\mathcal{D}P(1) = P^{(d)}(\vec{y}_1) - (P)_1 \qquad \text{difference}$$

$$\boxed{\begin{aligned} \mathcal{D}P(1) = {} & -\frac{1}{\varepsilon(1+\varepsilon)} \sum_q \left(\frac{1}{q} + \varepsilon\left[\ln s_1 - 1\right]\right) \left< \underset{1}{\overset{(\varepsilon)}{f}}_{-2,q)}(\vec{n}_1)\right> \\ & -\frac{1}{\varepsilon(1+\varepsilon)} \sum_q \left(\frac{1}{q+1} + \varepsilon \ln s_2\right) \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r_1^{\,1+\varepsilon}}\right) \left< n_2^L \underset{2}{\overset{(\varepsilon)}{f}}_{-l-3,q}(\vec{n}_2)\right> \\ & + \mathcal{O}(\varepsilon) \end{aligned}}$$

(can be computed locally when $r_1 \to 0$, $r_2 \to 0$)

Apparition of poles $\propto \frac{1}{\varepsilon}$
when logarithmic divergences in Had. reg.

It can be shown that the poles can be <u>renormalized</u>

by a redefinition of the trajectories of the particles

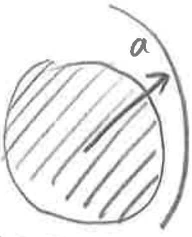
$$\left(y_A^i(t)\right)^{\text{bare}} \longrightarrow \left(y_A^i(t)\right)^{\text{dressed}} = \left(y_A^i\right)^{\text{bare}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) + \mathcal{O}(\varepsilon^0) + \dots$$

so that the final result is UV finite.

Non linearity (or post-Minkowskian) expansion

In exterior region $(r>a)$

of order $\mathcal{O}(h_{ext}^2)$



isolated source (radius a)

$$\begin{cases} \Box h^{\mu\nu}_{ext} = \Lambda^{\mu\nu}(h_{ext}) \\ \partial_\nu h^{\mu\nu}_{ext} = 0 \end{cases}$$

harmonic coordinate condition

We solve these equations by means of <u>post-Minkowskian</u> (PM) or <u>non-linearity</u> expansion

$$\boxed{h^{\mu\nu}_{ext} = \sum_{n=1}^{+\infty} G^n h^{\mu\nu}_{(n)}}$$

G = Newton's constant (viewed here as a "bookeeping" parameter to label the successive PM orders)

Insert PM expansion into <u>vacuum</u> Einstein field eqs.

$$\Box\left(G h^{\mu\nu}_{(1)} + G^2 h^{\mu\nu}_{(2)} + \cdots\right) = G^2 \Lambda^{\mu\nu}_{(2)}(h_{(1)}) + G^3 \Lambda^{\mu\nu}_{(3)}(h_{(1)} h_{(2)}) + \cdots$$

$$\partial_\nu\left(\text{———————}\right) = 0$$

where $\Lambda_{(2)} \sim h_{(1)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(1)}$

$\Lambda_{(3)} \sim h_{(1)} \partial h_{(1)} \partial h_{(1)} + h_{(1)} \partial^2 h_{(2)} + h_{(2)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(2)}$

$\cdots$

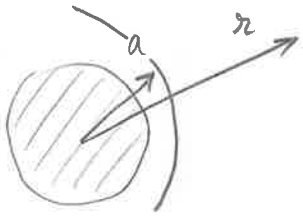
Hierarchy of PM equations equivalent to Einstein eqs.

$$\forall n \geqslant 1 \quad \boxed{\begin{aligned} \Box h^{\mu\nu}_{(n)} &= \Lambda^{\mu\nu}_{(n)}(h_{(1)} h_{(2)} \dots h_{(n-1)}) \\ \partial_\nu h^{\mu\nu}_{(n)} &= 0 \end{aligned}}$$

The source term $\Lambda_{(n)}$ is known from previous iterations

Linearized solution

Solve $\Box h_{(1)} = 0$ by means of multipole expansion (valid in exterior $r > a$)



"Monopolar" general solution

$$h^{\text{Mono.}}_{(1)}(\vec{x}, t) = \frac{R(t - r/c) + A(t + r/c)}{r}$$

Impose no incoming rad. cond.

$$0 = \lim_{\substack{t \to -\infty \\ t + \frac{r}{c} = \text{const}}} \left[\partial_r (r h_{(1)}) + \partial_t (r h_{(1)}) \right] = 2 A'(t + \tfrac{r}{c})$$ hence $A(u)$ is constant and can be included into definition of $R(t - \frac{r}{c})$.

$$h^{\text{Mono.}}_{(1)} = \frac{R(t - r/c)}{r}$$

$(i = 1, 2, 3)$

"Dipolar" solution is obtained by applying $\partial_i \equiv \dfrac{\partial}{\partial x^i}$

hence $h^{\text{Dip.}}_{(1)} = \partial_i\left(\frac{R_i(t-r/c)}{r}\right)$. General multipolar solution is obtained by applying l spatial derivatives

$$\boxed{h^{\mu\nu}_{(1)}(\vec{x},t) = \sum_{l=0}^{+\infty} \partial_L\left(\frac{R^{\mu\nu}_L(u)}{r}\right)} \quad \left(u \equiv t - \frac{r}{c}\right)$$

$L = i_1 i_2 \cdots i_l$ a multi-index with l spatial indices

$$\partial_L \equiv \partial_{i_1 i_2 \cdots i_l} = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}}$$

Without loss of generality we can assume that R_L is symmetric and trace-free (STF)

STF tensors

$$R_L = \hat{R}_L + \sum_{j \leq l-1} \underset{\substack{\uparrow \\ 0 \text{ or } 1 \\ \text{Levi-Civita} \\ \text{symbol}}}{\varepsilon} \underbrace{\delta\,\delta \ldots \delta}_{\substack{1 \text{ to } [\frac{l}{2}] \\ \text{Kronecker symbol}}} \hat{U}_J$$

where the $\hat{U}_J$'s are linear in the $\varepsilon\,\delta \ldots \delta\, R_K$'s.

For example:

$$\begin{cases} R_{ij} = \hat{R}_{ij} + \varepsilon_{ijk}\hat{U}_k + \delta_{ij}\hat{U} \\ \hat{U}_k = \frac{1}{2}\varepsilon_{kab}R_{ab} \\ \hat{U} = \frac{1}{3}R_{kk} \end{cases}$$

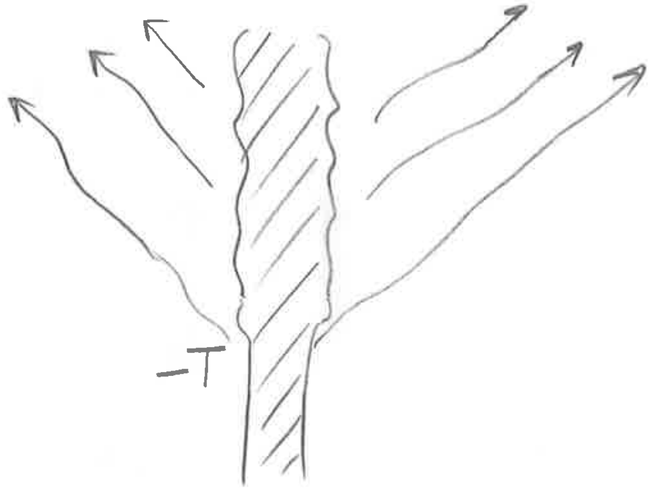
$\hat{R}_{ij} = \frac{R_{ij}+R_{ji}}{2} - \frac{1}{3}\delta_{ij}R_{kk}$ is the STF part of R_{ij}.

$$\partial_L\left(\frac{1}{r}R_L\right) = \partial_L\left(\frac{1}{r}\hat{R}_L\right) + \sum_{k\geqslant 1} \underset{\uparrow}{\Delta^k}\, \partial_{L-2K}\left(\frac{1}{r}\hat{U}_{L-2K}\right)$$

because of k Kronecker δs
(terms with one ε cancelled by symmetry of ∂_L)

$$\Delta^k \partial\left(\frac{1}{r}\hat{U}(u)\right) = \partial\left(\frac{1}{r}\frac{d^{2k}\hat{U}}{c^{2k}\,du^{2k}}(u)\right)$$ takes same structure

For simplicity assume that source emits GWs only from some finite instant $-T$ in the past (stationarity in the past)



$h^{\mu\nu}_{ext}(\vec{x})$ is independent of time when $t \leq -T$

(and even when
$$\underbrace{t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right)+\dots}_{\text{"light cone" in coordinates } (t,r)} \leq -T)$$

There are 10 independent functions $R^{\mu\nu}_L(u)$ (for each multi-index L) at this stage.

We impose now the harmonicity condition $\partial_\nu h^{\mu\nu}_{(1)} = 0$ which gives 4 differential relations between the R_L's. Hence we end up with 6 independent functions (6 types of "source" multipole moments).

Most general solution of $\Box h_{(1)} = 0 = \partial h_{(1)}$ is (Thorne 1980)

$$\boxed{h^{\mu\nu}_{(1)} = k^{\mu\nu}_{(1)} + \underbrace{\partial^\mu \varphi^\nu_{(1)} + \partial^\nu \varphi^\mu_{(1)} - \eta^{\mu\nu} \partial_\rho \varphi^\rho_{(1)}}_{\text{linearized gauge transformation}}}$$

where $k^{\mu\nu}_{(1)}$ depends on two sets of <u>STF multipole moments</u>

$$\boxed{\underset{\uparrow}{I}_L(u) \quad \text{and} \quad \underset{\uparrow}{J}_L(u)}$$

<u>mass-moment of order l</u> <u>current-moment of order l</u>

and $\varphi^\mu_{(1)}$ depends on four sets of moments (for its four components $\mu = 0,1,2,3$)

$$W_L(u) \quad X_L(u) \quad Y_L(u) \quad \text{and} \quad Z_L(u)$$

$$\boxed{\begin{aligned}
k^{00}_{(1)} &= -\frac{4}{c^2} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} I_L(u) \right) \\
k^{0i}_{(1)} &= \frac{4}{c^3} \sum_{l=1}^{+\infty} \frac{(-)^l}{l!} \left\{ \partial_{L-1} \left(\frac{1}{r} \dot{I}_{iL-1}(u) \right) + \frac{l}{l+1} \varepsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} J_{bL-1}(u) \right) \right\} \\
k^{ij}_{(1)} &= -\frac{4}{c^4} \sum_{l=2}^{+\infty} \frac{(-)^l}{l!} \left\{ \partial_{L-2} \left(\frac{1}{r} \ddot{I}_{ijL-2} \right) + \frac{2l}{l+1} \partial_{aL-2} \left(\frac{1}{r} \varepsilon_{ab(i} \dot{J}_{j)L-2} \right) \right\}
\end{aligned}}$$

Dots mean derivative w.r.t. time $u = t - r/c$

$I_L(u)$ and $J_L(u)$ are arbitrary functions of time u except for the conservation laws (directly issued from the harmonicity condition $\partial h_{(1)} = 0$)

$$M \equiv I = \text{const} \qquad \text{total mass}$$

$$X_i \equiv \frac{I_i}{I} = \text{const} \qquad \text{center-of-mass position}$$

$$P_i \equiv \dot{I}_i = 0 \qquad \text{linear momentum}$$

$$S_i \equiv J_i = \text{const} \qquad \text{angular momentum}$$

These conservation laws are exact (by definition of the moments) and refer to the total quantities associated with the source and and refer to the total quantities associated with the source including the contributions of GWs emitted by the source

They describe the initial state of the source before emisson of GWs.

In particular $M = I$ is the total ADM mass of source

Finally $h_{(1)}$ (and hence $h_{ext} = \sum G^n h_{(n)}$) will be described by

$$\boxed{\underbrace{I_L(u) \quad J_L(u)}_{\text{main moments}} \quad \underbrace{W_L(u) \ \dots \ Z_L(u)}_{\text{gauge moments}} = \text{six source multipole moments}}$$

main moments (encode all properties of source at linear order)

gauge moments (will play a role at non-linear order)

Non-linear vacuum solution

When $r \to 0$ $h_{(1)} \sim \partial\left(\frac{R(t-r)}{r}\right)$ diverges. This is because $h_{(1)}$ is valid only in the exterior $r > a$. Inserting $h_{(1)}$ into $\Lambda_{(2)}$ we get

$$\Lambda_{(2)} \sim \partial\left(\frac{R(t-r)}{r}\right)\partial\left(\frac{S(t-r)}{r}\right)$$

$$\sim \sum_{k\geq 2} \frac{\hat{n}_L}{r^k} F(t-r)$$

STF product of unit vectors n_i $\boxed{\hat{n}_L = n_{<i_1} \dots n_{i_\ell>}}$
is equivalent to spherical harmonics $Y_{\ell m}(\theta,\varphi)$

$$\begin{cases} \hat{n}_L(\theta,\varphi) = \sum_{m=-\ell}^{\ell} \alpha_L^m \, Y_{\ell m}(\theta,\varphi) \\ \alpha_L^m = \int d\Omega \, \hat{n}_L \, Y_{\ell m}^* \end{cases} \longleftarrow \text{constant STF tensor}$$

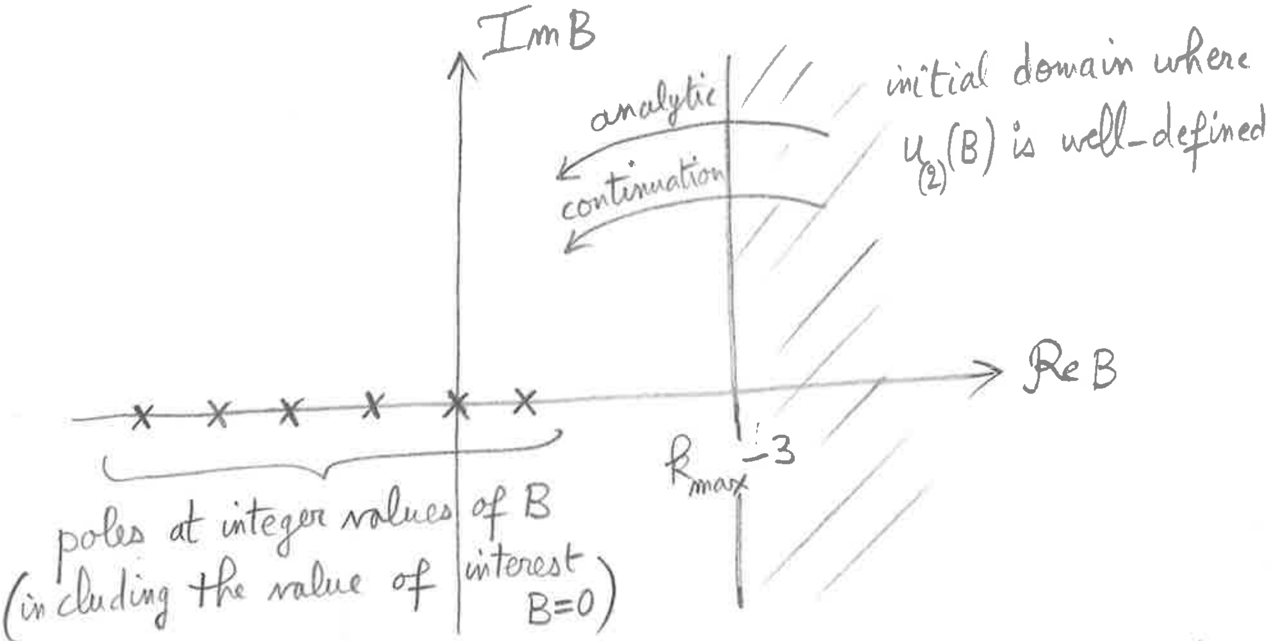
Because of divergence when $r \to 0$ one cannot apply the standard retarded integral.

If we assume $h_{(1)}$ is made of a finite set of moments, say $\ell \leq \ell_{max}$, there is a maximal order of divergencies in $\Lambda_{(2)}$, $k \leq k_{max}$. We can <u>regularize</u> $\Lambda_{(2)}$ by multiplying by some factor r^B (where $B \in \mathbb{C}$).

Next we define:

$$\boxed{u^{\mu\nu}_{(2)}(B) \equiv \Box^{-1}_{\mathrm{Ret}}\left[\left(\frac{r}{r_0}\right)^B \Lambda^{\mu\nu}_{(2)}\right]}$$

The retarded integral is convergent when $\Re e\, B > k_{max} - 3$



$$u_{(2)}(B) = \sum_{p=p_0}^{+\infty} \lambda_p B^p$$

Laurent expansion
when $B \to 0$
$(p \in \mathbb{Z})$

Applying $\Box$ we get $\left(\frac{r}{r_0}\right)^B \Lambda_{(2)} = \sum (\Box \lambda_p) B^p$

$$p_0 \leq p \leq -1 \Rightarrow \Box \lambda_p = 0$$

$$p \geqslant 0 \Rightarrow \Box \lambda_p = \frac{(\ln(r/r_0))^p}{p!} \Lambda_{(2)}$$

In particular when $p=0$ we obtain a solution of the eq. we want $(\Box u_{(2)} = \Lambda_{(2)})$. Pose $u^{\mu\nu}_{(2)} \equiv \lambda^{\mu\nu}_0$

$$\boxed{u^{\mu\nu}_{(2)} = \underset{B\to 0}{\text{Finite Part}}\ \Box^{-1}_{Ret}\left[r^{B} \Lambda^{\mu\nu}_{(2)}\right]} \qquad (r_0 = 1)$$

Thus $\Box u_{(2)} = \Lambda_{(2)}$ is satisfied and $u_{(2)}$ has the same structure $\sim \sum \frac{\hat{n}_L}{r^k} G(t-r)$ as $\Lambda_{(2)}$ but $\partial_\nu u^{\mu\nu}_{(2)} \neq 0$ in general.

$$\boxed{w^{\mu}_{(2)} \equiv \partial_\nu u^{\mu\nu}_{(2)} = \underset{B\to 0}{FP}\ \Box^{-1}_{Ret}\left[B\, n_i\, r^{B-1} \Lambda^{\mu i}_{(2)}\right]}$$

↑ computed from the fact that $\partial_\nu \Lambda^{\mu\nu}_{(2)} = 0$

Because of factor B (coming from $\partial_i r^B = B r^{B-1} n_i$) $w^\mu_{(2)}$ is non zero when the integral develops a pole $\propto \frac{1}{B}$. The structure of the pole is that of a source-free (retarded) solution of d'Alembert's eq.

$$\boxed{w^{\mu}_{(2)} = \sum_{l=0}^{\infty} \partial_L \left(\frac{S^{\mu}_{L}(u)}{r} \right)}$$

Indeed $\Box w_{(2)} = \underset{B\to 0}{FP}\left(B n r^{B-1} \Lambda\right) = 0$. From that structure one can construct "algorithmically"

$$v^{\mu\nu}_{(2)} = \mathcal{H}^{\mu\nu}\left(w_{(2)}\right)$$

↑ an algorithm which gives a unique $v^{\mu\nu}_{(2)}$ (source-free solution) starting from any $w^\mu_{(2)}$

such that (at once) $\Box v_{(2)} = 0$ and $\partial v_{(2)} = - u_{(2)}$

$$\boxed{v^{\mu\nu}_{(2)} = \sum_{l=0}^{\infty} \partial_L \left(\frac{T^{\mu\nu}_L(u)}{r} \right)}$$

where the $T^{\mu\nu}_L$'s are given in terms of the S^{μ}_L's by the algorithm $\mathcal{H}$. Solution is thus

$$\boxed{h^{\mu\nu}_{(2)} = u^{\mu\nu}_{(2)} + v^{\mu\nu}_{(2)}}$$

Same method applies by induction to any n (Blanchet & Damour 1986)

$$\boxed{\begin{aligned} u^{\mu\nu}_{(n)} &= \underset{B\to 0}{\text{Finite Part}}\ \Box^{-1}_{\text{Ret}} \left[\left(\frac{r}{r_0}\right)^B \Lambda_{(n)}\left(h_{(1)} \cdots h_{(n-1)}\right) \right] \\ v^{\mu\nu}_{(n)} &= \mathcal{H}^{\mu\nu}\left(\partial u_{(n)}\right) \\ h^{\mu\nu}_{(n)} &= u^{\mu\nu}_{(n)} + v^{\mu\nu}_{(n)} \end{aligned}}$$

To $h_{(n)}$ one can still add a homogeneous solution (such that $\Box h^{\text{Hom}}_{(n)} = 0 = \partial h^{\text{Hom}}_{(n)}$) but $h^{\text{Hom}}_{(n)}$ is necessarily of the form $h_{(1)}$[some moments]. Hence

$$h^{gen}_{(n)} = h_{(n)}[I_L \cdots Z_L] + \underbrace{h_{(1)}[\delta I_L \cdots \delta Z_L]}_{\substack{\text{can be re-absorbed into} \\ h_{(1)}[I_L \cdots Z_L] \text{ by posing}}}$$

$$\begin{cases} I_L^{new} = I_L + G^{n-1}\,\delta I_L \\ \quad\vdots \\ Z_L^{new} = Z_L + G^{n-1}\,\delta Z_L \end{cases}$$

Hence the previous construction represents the <u>most general</u> solution of Einstein's field eqs. outside the source

Resulting metric $$\boxed{g^{ext}_{\mu\nu}\big(x;\ \underbrace{I_L\ J_L\ \overbrace{W_L\ X_L\ Y_L\ Z_L}^{\text{4 gauge moments}}}_{\text{6 source moments}}\big)}$$

One can define by coord. transformation $x \longrightarrow x'$ a "canonical" metric which depends only on 2 moments $M_L\ S_L$.

Thus $$\boxed{g^{can}_{\mu\nu}\big(x';\ \underbrace{M_L\ S_L}_{\text{2 canonical moments}}\big)}$$

is <u>isometric</u> to $g^{ext}_{\mu\nu}$ i.e. $g^{can}_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g^{ext}_{\rho\sigma}(x)$ where

$$\boxed{x'^\mu = x^\mu + G\, \underbrace{\varphi^\mu_{(1)}(x;\ W_L X_L Y_L Z_L)}_{\substack{\text{gauge vector in the} \\ \text{general linear solution}}} + \underset{\substack{\uparrow \\ \text{crucial} \\ \text{non-linear} \\ \text{corrections}}}{O(G^2)}}$$

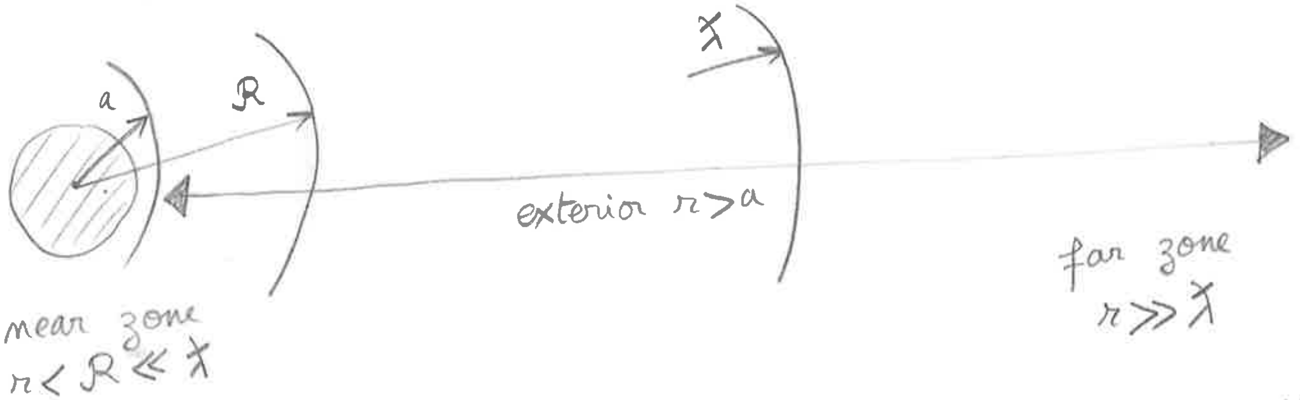
Hence any isolated system can be described by <u>2 sets of moments</u>

$$\underset{\text{mass-type}}{M_L(u)} \qquad \underset{\text{current-type}}{S_L(u)}$$

$$\boxed{\begin{aligned} M_L &= I_L + \mathcal{O}(G) \\ S_L &= J_L + \mathcal{O}(G) \end{aligned}}$$

complicated non-linear functionals of $I_L \; J_L \; X_L \ldots Z_L$

General structure of the solution



The solution $h_{ext} = \sum G^n h_{(n)}$ is physically valid in the exterior $r>a$ but is defined for any $r>0$. When $r\to 0$

$$\boxed{h_{(n)} = \sum_{p\leq N} \hat{n}_L(\theta,\varphi)\, r^p (\ln r)^q F(t) + \mathcal{O}(r^N)}$$

(proved by induction on n in the construction of $h_{(n)}$).
Note appearance of powers of $\ln r$ with $q\leq n-2$.

Since $r \to 0$ means $\frac{r}{c} \to 0$ or $c \to \infty$ we have the general structure of the post-Newtonian (PN) expansion

$$\boxed{h_{(m)}(c) = \sum_{p \leq N} \frac{(\ln c)^q}{c^p} + O\left(\frac{1}{c^N}\right)}$$

When $r \to \infty$ (wave zone) we find also a "poly-logarithmic" structure

$$h_{(m)} = \sum_{k \leq N} \hat{n}_L \frac{(\ln r)^q}{r^k} G(u) + o\left(\frac{1}{r^N}\right)$$

where $u = t - r/c$

(expansion at $\mathcal{J}^+$)

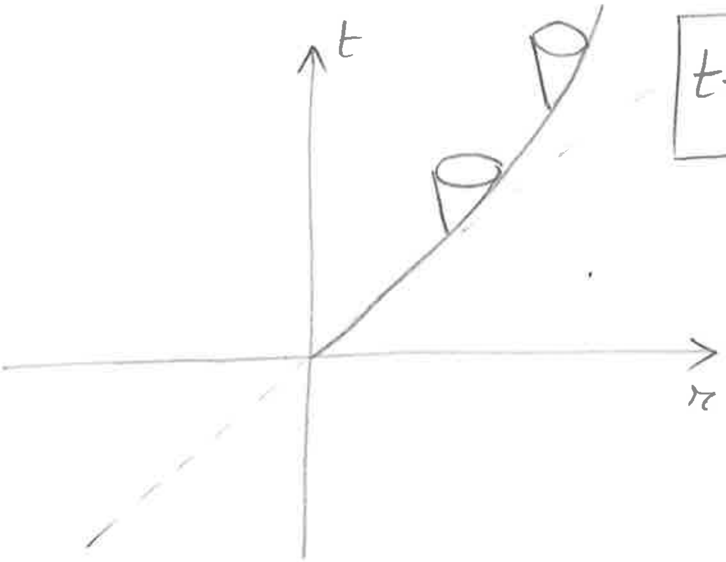
The logs here come from the well-known deviation of light rays in harmonic coordinates.

Schwarzschild in harmonic coord.

$$ds^2 = -\left(\frac{r-M}{r+M}\right)dt^2 + \left(\frac{r+M}{r-M}\right)dr^2 + (r+M)^2 d\Omega^2$$

For an <u>outgoing radial</u> ($\theta = \text{const}$ $\varphi = \text{const}$) photon

$$dt = \frac{r+M}{r-M} dr \quad \Rightarrow \quad t = r + 2M \ln\left(\frac{r-M}{\text{const}}\right)$$



$$\boxed{t = \frac{r}{c} + \frac{2GM}{c^3} \ln\left(\frac{r}{r_0}\right) + O(G^2)}$$

We shall see that all these logs (in the FZ) can be removed by a coord. transformation

The matching equation

We have contructed the exterior field (physically valid when $r>a$) of any isolated source

$$h_{\text{ext}} = \sum_{n=1}^{+\infty} G^n h_{(n)} [\underbrace{I_L\ J_L\ W_L \dots Z_L}_{\text{source moments (for the moment arbitrary)}}]$$

We suppose that h_{ext} comes from the multipole expansion of h defined everywhere inside and outside the source (for any r)

$$\boxed{h_{\text{ext}} = \mathcal{M}(h)}$$

operation of taking the multipole expansion

Note that $\mathcal{M}(h)$ is defined of any $r>0$ but agrees with the "true" field h only when $r>a$

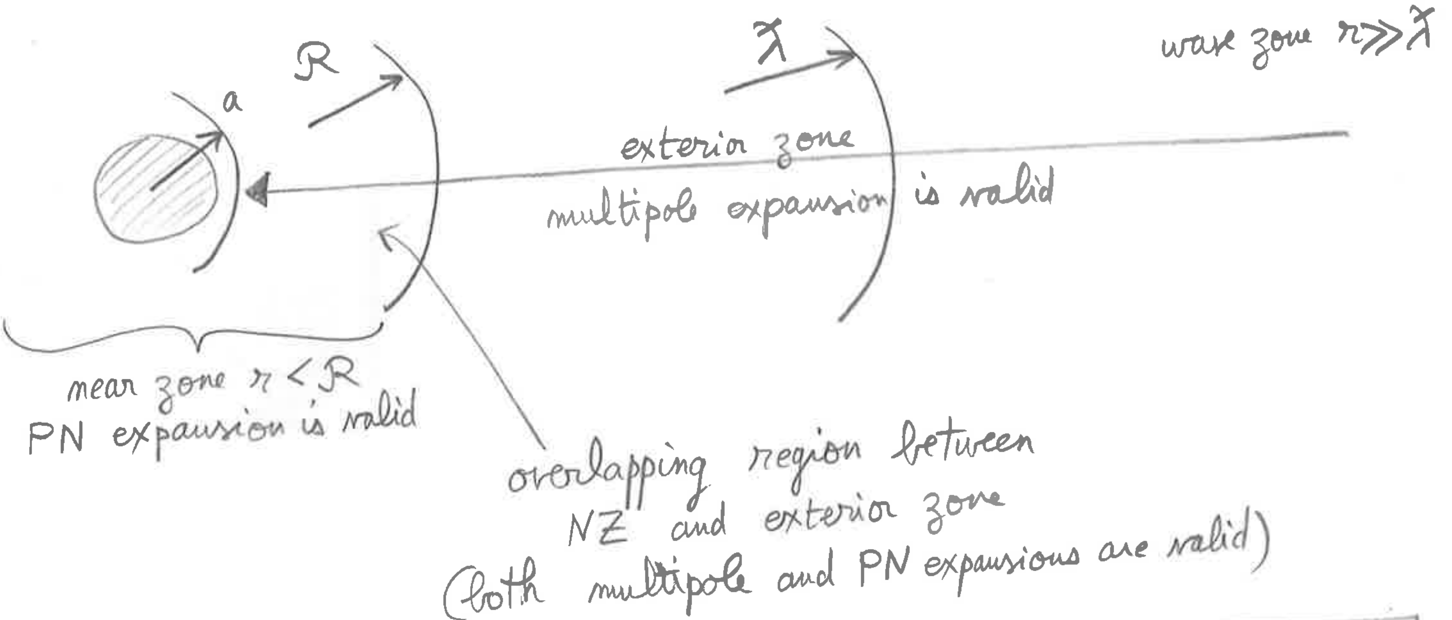
$$\boxed{r>a \quad \Longrightarrow \quad \mathcal{M}(h) = h \quad \text{(numerically)}}$$

But when $r \longrightarrow 0$ $\mathcal{M}(h)$ diverges while h is a perfectly smooth solution of Einstein field eqs. inside the matter (of the extended source).

Suppose the source is post-Newtonian (existence of the PN parameter $\varepsilon = \frac{v}{c} \ll 1$). We know that the near zone $r < \mathcal{R}$ where $\mathcal{R} \ll \lambdabar$ encloses totally the PN source ($\mathcal{R} > a$).

In the NZ the field h can be expanded as a PN expansion ($\bar{h} = \sum c^{-p} (\ln c)^q$)

$$\boxed{r < \mathcal{R} \quad \Longrightarrow \quad h = \bar{h} \quad \text{(numerically)}}$$



$$\boxed{a < r < \mathcal{R} \quad \Longrightarrow \quad \mathcal{M}(h) = \bar{h} \quad \text{(numerically)}}$$

The matching equation follows from transforming the latter numerical equality in a functional identity (valid $\forall (\vec{x}, t)$ in $\mathbb{R}^3_* \times \mathbb{R}$) between two formal asymptotic series

Matching equation: $\boxed{\overline{\mathcal{M}(h)} \equiv \mathcal{M}(\bar{h})}$

NZ expansion $\left(\frac{r}{c} \to 0\right)$
of each multipolar coeff.
of $\mathcal{M}(h)$

multipole expansion of
each PN coefficient of $\bar{h}$

We assume (as part of our fundamental assumptions) that the matching eq. is correct (in the sense of formal series)

$$\boxed{\underset{\frac{r}{c} \to 0}{\text{NZ expansion}} \begin{pmatrix} \text{multipolar} \\ \text{expansion} \\ \frac{a}{r} \to 0 \end{pmatrix} \equiv \underset{r \to \infty}{\text{FZ expansion}} \begin{pmatrix} \text{PN series} \\ c \to \infty \end{pmatrix}}$$

The NZ expansion $\frac{r}{c} \to 0$ is "equivalent" to the PN expansion
$c \to +\infty$ for fixed r

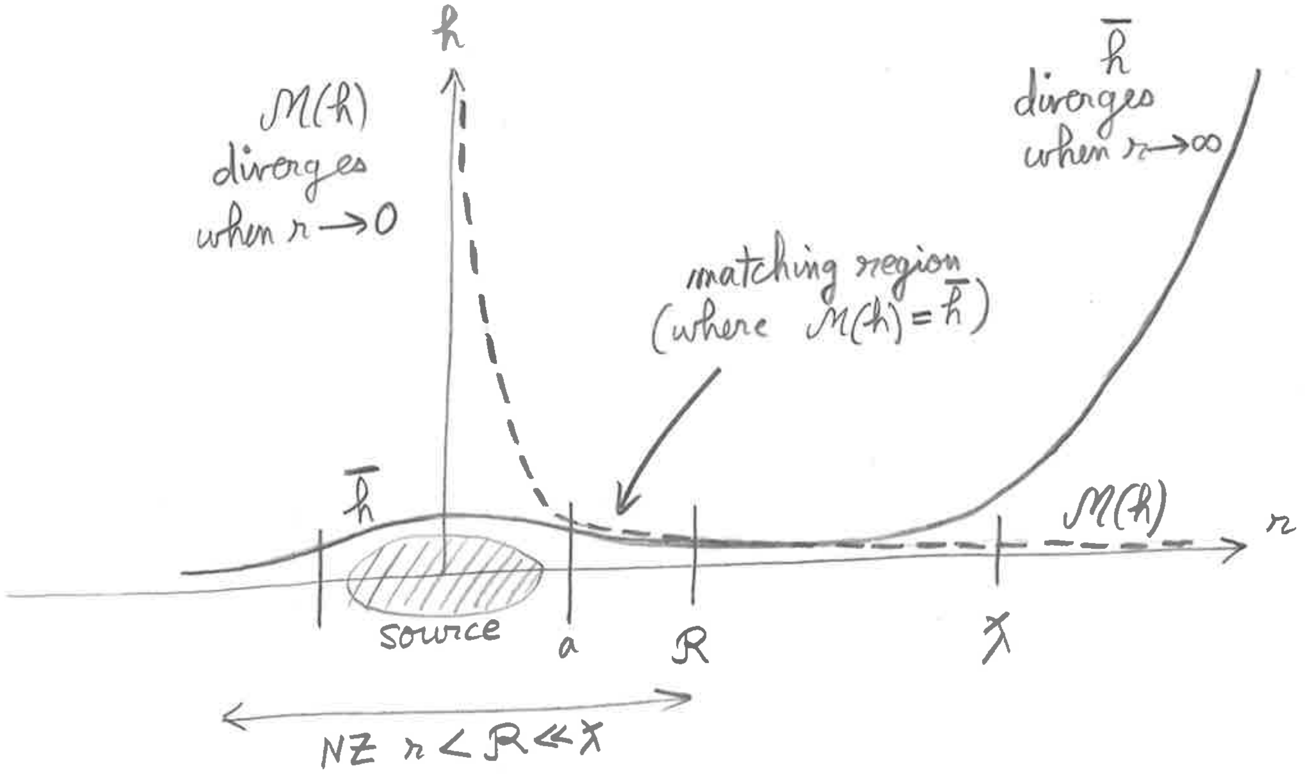
The multipole expansion $\frac{a}{r} \to 0$ is "equivalent" to the
FZ expansion $r \to +\infty$ for a given source (fixed a)

The matching equation says basically the NZ and multipole expansions can be commuted.

Thus there is a common structure for the formal NZ and FZ expansions

$$\overline{\mathcal{M}(h)} \equiv \underbrace{\sum \hat{n}_L r^p (\ln r)^q F(t)}_{\text{can be interpreted either as}} \equiv \mathcal{M}(\bar{h})$$

- NZ singular expansion when $r \to 0$
- FZ ——————————— $r \to \infty$



Expression of the multipole moments

h is the sol. of Einstein eqs (in harmonic coord. $\partial h = 0$) valid everywhere inside and outside the source

$$h = \frac{16\pi G}{c^4} \Box^{-1}_{\text{Ret}} \tau \qquad \begin{pmatrix}\text{suppress}\\ \text{indices } \mu\nu\end{pmatrix}$$

where $\tau = |g|\, T + \underbrace{\frac{c^4}{16\pi G}\Lambda}_{\substack{\text{gravitational source-term}\\ \text{(non-linearities in } h)}}$

Define $$\boxed{\Delta \equiv h - \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda)}$$

where $\mathcal{M}(\Lambda) = \Lambda[\mathcal{M}(\Lambda)] = \Lambda_{\mathrm{ext}}$ and FP is the finite part when $B \to 0$ (plays a crucial role because Λ_{ext} diverges when $r \to 0$)

$$\Delta = \underbrace{\frac{16\pi G}{c^4} \Box^{-1}_{\mathrm{Ret}} \tau}_{\substack{\text{no FP here} \\ \text{since } \tau \text{ is regular } (C^\infty)}} - \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda)$$

However we can add FP on the first term (do not change the value because it converges). Using also $\mathcal{M}(T) = 0$ since T has a compact support

$$\Delta = \frac{16\pi G}{c^4} \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \left[\tau - \mathcal{M}(\tau)\right]$$

Hence Δ appears as the retarded integral of a source with compact support. Indeed

$$\tau = \mathcal{M}(\tau) \quad \text{when } r > a$$

$$\boxed{\mathcal{M}(\Delta) = -\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r} \mathcal{H}_L(u)\right)}$$

This is standard expression of multipolar expansion outside a compact-support source. Here the moments are

$$\mathcal{H}_L = \mathrm{FP} \int d^3x \, x_L \underbrace{\left[\tau - \mathcal{M}(\tau) \right]}$$

since this has compact support ($r < a$, inside the NZ) we can replace by the NZ or PN expansion

$$\mathcal{H}_L = \mathrm{FP} \int d^3x \, x_L \left[\overline{\tau} - \overline{\mathcal{M}(\tau)} \right]$$

But we know the structure $\overline{\mathcal{M}(\tau)} = \sum \hat{n}_L r^p (\ln r)^q F(t)$ which is sufficient to prove that the second term is <u>zero</u> by analytic continuation

$$\mathrm{FP} \int d^3x \, x_L \, \overline{\mathcal{M}(\tau)} = \sum \mathrm{FP} \int d^3x \, x_L \, \hat{n}_Q \, r^p (\ln r)^q$$

$$= \sum \underset{B \to 0}{\text{Finite Part}} \int dr \, r^{B+S} (\ln r)^p$$

integrate over angles

$$= \sum \underset{B \to 0}{\mathrm{FP}} \left(\frac{d}{dB} \right)^p \int_0^{+\infty} dr \, r^{B+S}$$

$$\int_0^{+\infty} dr \, r^{B+S} = \underbrace{\int_0^{\mathcal{R}} dr \, r^{B+S}}_{\substack{\text{computed} \\ \text{when } \Re B > -S-1}} + \underbrace{\int_{\mathcal{R}}^{+\infty} dr \, r^{B+S}}_{\substack{\text{computed} \\ \text{when } \Re B < -S-1}}$$

$$= \underbrace{\frac{\mathcal{R}^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}} \qquad = \underbrace{- \frac{\mathcal{R}^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}}$$

$$\boxed{\text{Analytic Continuation} \qquad \int_0^{+\infty} dr\, r^{B+s} (\ln r)^p = 0 \qquad \forall B \in \mathbb{C}}$$

The general multipole expansion outside the domain of a PN isolated source reads (Blanchet 1995, 1998)

$$\boxed{\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} \mathcal{H}_L(u) \right)}$$

where

$$\boxed{\mathcal{H}_L(u) = \mathrm{FP} \int d^3x\; x_L\, \underset{\uparrow}{\overline{\tau}}(\vec{x}, u)}$$

PN expansion crucial here
(this is where the formalism applies only to PN sources)

Same result but in STF guise

$$\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} \mathcal{F}_L(u) \right)$$

where

$$\mathcal{F}_L(u) = \mathrm{FP} \int d^3x\; \hat{x}_L \int_{-1}^{1} dz\, \delta_l(z)\, \overline{\tau}(\vec{x}, u + z|\vec{x}|/c)$$

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}\, l!} (1-z^2)^l \quad \text{such that} \quad \int_{-1}^{1} dz\, \delta_l(z) = 1$$

$$\lim_{l \to +\infty} \delta_l(z) = \delta(z)$$

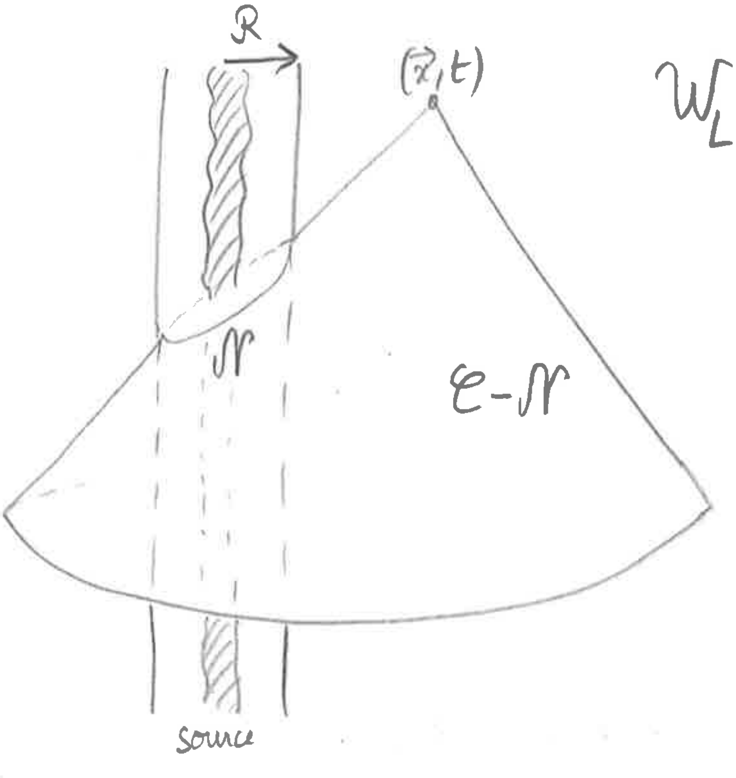
Practical way to implement the STF multipole expansion is to use the PN series

$$\boxed{\int_{-1}^{1} dz\, \delta_\ell(z)\, \overline{\tau}\left(\vec{x}, u + z|\vec{x}|/c\right) = \sum_{k=0}^{+\infty} \underset{\substack{\uparrow \\ \frac{(2\ell+1)!!}{(2k)!!\,(2\ell+2k+1)!!}}}{\alpha_k^\ell} \left(\frac{|\vec{x}|}{c}\frac{\partial}{\partial u}\right)^{2k} \overline{\tau}(\vec{x},u)}$$

There is an alternative formalism for writing the general multipole expansion (Will & Wiseman 1996)

$$\mathcal{M}(h) = \underbrace{\Box^{-1}_{\mathrm{Ret}}\, \mathcal{M}(\Lambda)\Big|_{\mathcal{C}-\mathcal{N}}}_{\text{the retarded integral excludes the NZ of source}} - \frac{4G}{c^4}\sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!}\, \partial_L\left(\frac{1}{r}\mathcal{W}_L(t-r)\right)$$

where



$$\mathcal{W}_L(u) = \underbrace{\int_{r<\mathcal{R}} d^3x\; x_L\, \overline{\tau}(\vec{x},u)}_{\text{volume integral limited to the NZ of the source } (\mathcal{N})}$$

The two formalisms are equivalent

Next we identify $h_{ext} = \mathcal{M}(h)$ which means

$$G\, h_{(1)}[I_L J_L W_L \cdots Z_L] + G^2 h_{(2)} + \cdots + G^m h_{(m)} + \cdots$$

$$= \underbrace{-\frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L\left(\frac{1}{r} \mathcal{F}_L(u)\right)}_{\substack{\text{has the form of the linear} \\ \text{metric } Gh_{(1)} \text{ where the } \mathcal{F}_L\text{'s} \\ \text{are "equivalent" to } I_L \cdots Z_L}} + \underbrace{FP\, \Box^{-1}_{Ret}\, \mathcal{M}(\Lambda)}_{\substack{\text{represents the} \\ \text{non-linear corrections} \\ G^2 h_{(2)} + \cdots + G^m h_{(m)} + \cdots}}$$

Note that for the identification to work the $\mathcal{F}_L$'s in the right-hand-side should be considered as of <u>zero-th order in G</u>

Then we obtain $I_L \cdots Z_L$ in terms of the components of $\mathcal{F}_L^{\mu\nu}$ and hence of the source's pseudo-tensor $\overline{\tau}^{\mu\nu}$.

Decompose the $\mathcal{F}_L^{\mu\nu}$'s into ten irreducible STF tensors $R_L\ \overset{(+)}{T}_{L+1} \cdots \overset{(-2)}{U}_{L-2}\ V_L$

$$\begin{cases} \mathcal{F}_L^{00} = R_L \\ \mathcal{F}_L^{0i} = \overset{(+)}{T}_{iL} + \varepsilon_{ai<i_l} \overset{(0)}{T}_{L-1>a} + \delta_{i<i_l} \overset{(-)}{T}_{L-1>} \\ \mathcal{F}_L^{ij} = \overset{(+2)}{U}_{ijL} + \underset{L}{\mathrm{STF}}\, \underset{ij}{\mathrm{STF}} \Big[\varepsilon_{aii_l} \overset{(+1)}{U}_{ajL-1} + \delta_{ii_l} \overset{(0)}{U}_{jL-1} \\ \qquad + \delta_{ii_l} \varepsilon_{aji_{l-1}} \overset{(-1)}{U}_{aL-2} + \delta_{ii_l} \delta_{ji_{l-1}} \overset{(-2)}{U}_{L-2} \Big] + \delta_{ij} V_L \end{cases}$$

The final result is

$$\boxed{\begin{aligned} I_L &= \mathrm{FP}\int d^3x \int_{-1}^{1} dz \left\{ \delta_\ell(z)\,\hat{x}_L\,\Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\,\delta_{\ell+1}\,\hat{x}_{iL}\,\Sigma^{(1)}_i \right. \\ &\qquad \left. + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\,\delta_{\ell+2}\,\hat{x}_{ijL}\,\Sigma^{(2)}_{ij} \right\}(\vec{x}, u+z|x|/c) \\[2ex] J_L &= \mathrm{FP}\int d^3x \int_{-1}^{1} dz\ \varepsilon_{ab\langle i_\ell} \left\{ \delta_\ell\,\hat{x}_{L-1\rangle a}\,\Sigma_b \right. \\ &\qquad \left. - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\,\delta_{\ell+1}\,\hat{x}_{L-1\rangle ac}\,\Sigma^{(1)}_{bc} \right\}(\vec{x}, u+z|\vec{x}|/c) \end{aligned}}$$

where
$$\begin{cases} \Sigma = \dfrac{\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2} \\ \Sigma_i = \dfrac{\overline{\tau}^{0i}}{c} \\ \Sigma_{ij} = \overline{\tau}^{ij} \end{cases}$$

There are similar expressions for $W_L \cdots Z_L$

These expressions give the source moments of any isolated PN source, up to any PN order (formally).

PN expansion in the near-zone

Consider the PN expansion of the field in the NZ ($r<\mathcal{R}$)

$$\boxed{\overline{h}(\vec{x},t,c) = \sum_{p=2}^{+\infty} \frac{1}{c^p}\,\overline{h}_p(\vec{x},t,\ln c)}$$

formal PN series (appearance of $\ln c$'s at 4PN order)

Note: $\overline{h}_p$ denotes the PN coefficient of $\frac{1}{c^p}$ while $h_{(m)}$ denotes the PM coefficient of G^m

To compute iteratively the $\bar{h}_m$'s we meet two problems

① Problem of NZ limitation

$\bar{h}$ is valid only in NZ
(and diverges in the FZ, when $r \to \infty$)

How to incorporate into the PN series the information about boundary conditions at infinity (notably the no-incoming radiation condition which is imposed at $\mathcal{J}^-$)?

② Problem of divergencies

$$\Delta \bar{h}_p = \begin{pmatrix} \text{source term} \\ \text{with } \underline{\text{non-compact}} \\ \text{support} \\ \text{which blows up when } r \to +\infty \end{pmatrix}$$

Then the usual Poisson integral is divergent

$$\boxed{\bar{h}_p = \underbrace{\int \frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\left(\text{source term}\right)}_{\text{diverges at the bound } |\vec{x}'|=+\infty \text{ (for high } p)}}$$

Problem ① will be solved by matching: $\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})$

Problem ② will be solved by finding a suitable solution of the Poisson equation (different from the Poisson integral)

Insert $\bar{h} = \sum \frac{1}{c^p} \bar{h}_p$ into $\begin{cases} \Box \bar{h} = \frac{16\pi G}{c^4} \bar{\tau} \\ \partial \bar{h} = 0 \end{cases}$

Hierarchy of PN equations $(\forall n \geqslant 2)$

$$\boxed{\begin{aligned} \Delta \bar{h}^{\mu\nu}_p &= 16\pi G \, \bar{\tau}^{\mu\nu}_{p-4} + \partial_t^2 \bar{h}^{\mu\nu}_{p-2} \\ \partial_\nu \bar{h}^{\mu\nu}_p &= 0 \end{aligned}}$$

At any given p the right-hand-side is known from previous iteration (using recursive treatment).

Construct first a particular solution of these equations using the generalized Poisson integral (Poujade & Blanchet 2002)

$$\mathrm{FP}\Delta^{-1}\left[\bar{\tau}_p\right] \equiv \underset{B \to 0}{\text{Finite Part}} \underbrace{\frac{-1}{4\pi} \int \frac{d^3\vec{x}' \, |\vec{x}'|^B}{|\vec{x} - \vec{x}'|} \bar{\tau}_p(\vec{x}', t)}_{\substack{\text{defined by analytic} \\ \text{continuation}}}$$

Then we add the general homogeneous solution of Laplace's equation which is regular in the source $(r \to 0)$

$$\Delta\left[\underset{\substack{\uparrow \\ \text{solution} \\ \text{regular} \\ \text{when } r \to 0}}{a \, \hat{x}_L} + \underset{\substack{\uparrow \\ \text{solution} \\ \text{regular} \\ \text{when } r \to \infty}}{b \, \hat{\partial}_L \frac{1}{r}} \right] = 0$$

Most general solution is

$$\boxed{\bar{h}_p^{\mu\nu} = \underbrace{FP\,\Delta^{-1}\left\{16\pi G\,\bar{\tau}_{p-4}^{\mu\nu} + \partial_t^2\,\bar{h}_{p-2}^{\mu\nu}\right\}}_{\substack{\text{particular solution}\\ \text{(well-defined thanks to}\\ \text{the Finite Part)}}} + \underbrace{\sum_{l=0}^{+\infty} B_{pL}^{\mu\nu}(t)\,\hat{x}_L}_{\substack{\text{homogeneous solution}\\ \text{(unknown for the}\\ \text{moment)}}}}$$

To compute the homogenous solution we require that it matches the external field in the sense

$$\mathcal{M}\left(\sum \frac{1}{c^p}\,\bar{h}_p^{\mu\nu}\right) = \overline{\mathcal{M}(h)} = \overline{\sum G^n h_{(n)}}$$

where $\mathcal{M}(h) = h_{ext} = \sum G^n h_{(n)}$. This fixes uniquely the homogeneous solution which is associated with radiation reaction forces inside the source, appropriate to an isolated system emitting GWs but not receiving GWs from $\mathcal{J}^-$.

Summing up $\bar{h} = \sum \frac{1}{c^p}\bar{h}_p$ we get

$$\boxed{\bar{h}^{\mu\nu} = \underbrace{\frac{16\pi G}{c^4}\left\{\sum_{k=0}^{\infty}\left(\frac{\partial}{c\,\partial t}\right)^{2k} FP\,\Delta^{-k-1}\,\bar{\tau}^{\mu\nu}\right\}}_{\substack{\text{particular solution}\\ \text{of d'Alembert eq.}\\ \text{denoted } FP\,\Box^{-1}\bar{\tau}^{\mu\nu}}} - \underbrace{\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{A_L^{\mu\nu}(t-r) - A_L^{\mu\nu}(t+r)}{2r}\right\}}_{\substack{\text{homogeneous solution}\\ \text{of d'Alembert eq.}\\ \text{which is regular when } r\to 0}}}$$

It's an anti-symmetric wave (retarded)−(advanced)

Result of the matching is (Poujade & Blanchet 2002)

$$\mathcal{A}_L^{\mu\nu}(u) = \mathcal{F}_L^{\mu\nu}(u) + \mathcal{R}_L^{\mu\nu}(u)$$

where $\mathcal{F}_L^{\mu\nu}$ is the source's multipole moment $\begin{pmatrix}\text{computed}\\ \text{previously}\end{pmatrix}$

$$\mathcal{F}_L^{\mu\nu}(u) = \mathrm{FP}\int d^3x\, \hat{x}_L \int_{-1}^{1} dz\, \delta_\ell(z)\, \underset{\substack{\uparrow\\ \underline{\text{PN expansion of } \tau}}}{\overline{\tau}^{\mu\nu}}(\vec{x}, u+z|\vec{x}|/c)$$

and where $\mathcal{R}_L^{\mu\nu}(u)$ is a new type of moment which turns out to parametrize <u>non-linear radiation reaction effects</u> in the source (Blanchet 1993)

$$\boxed{\mathcal{R}_L^{\mu\nu}(u) = \mathrm{FP}\int d^3x\, \hat{x}_L \int_{1}^{+\infty} dz\, \gamma_\ell(z)\, \underset{\substack{\uparrow\\ \underline{\text{multipole expansion of } \tau}}}{\mathcal{M}(\tau^{\mu\nu})}(\vec{x}, u-z|\vec{x}|/c)}$$

where $\gamma_\ell(z) = -2\,\delta_\ell(z)$ satisfies (by analytic continuation in ℓ)

$$\int_1^{+\infty} dz\, \gamma_\ell(z) = 1 \qquad \gamma_\ell(z) = (-)^{\ell+1}\frac{(2\ell+1)!!}{2^\ell \ell!}(z^2-1)^\ell$$

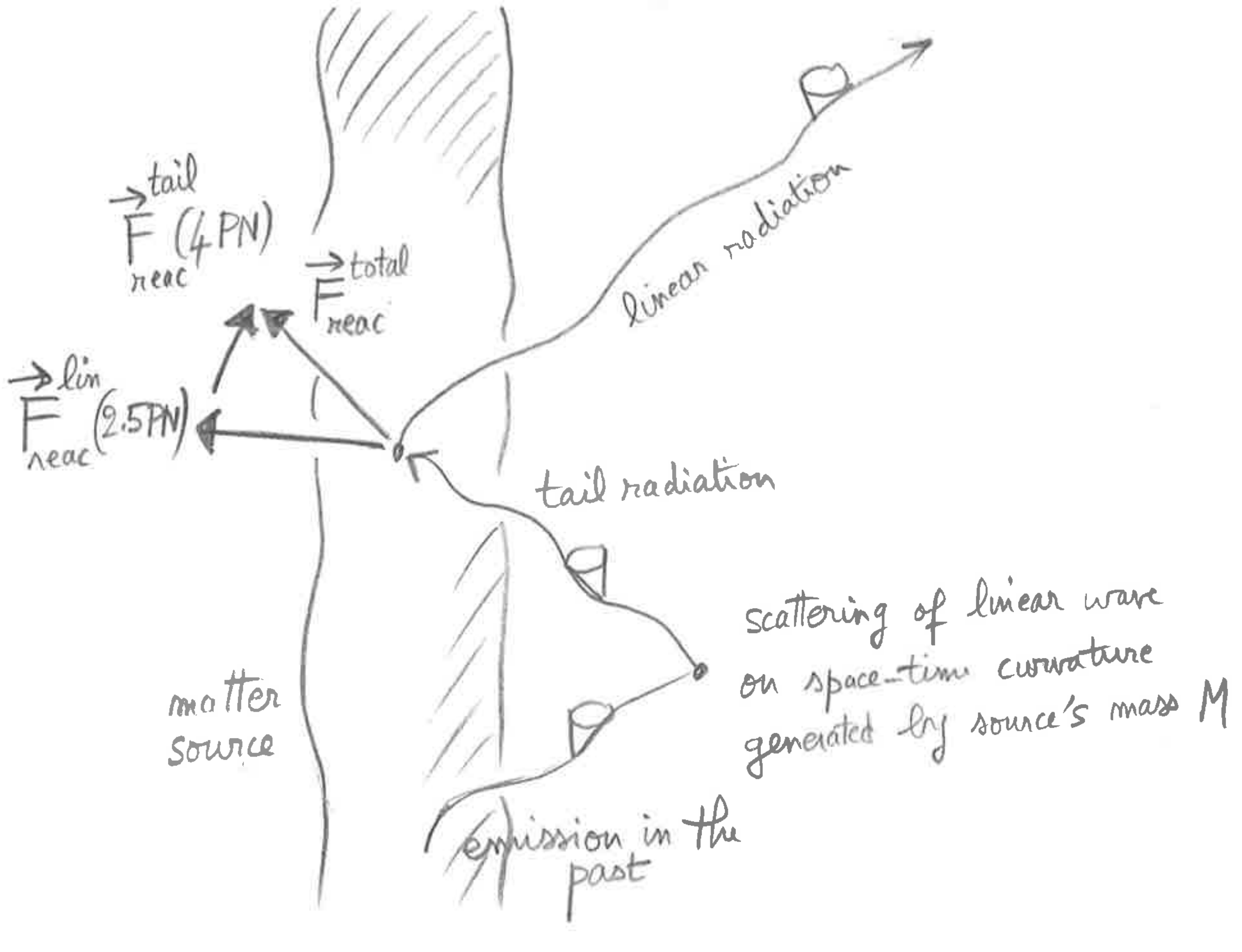
This comes from

$$0 = \underbrace{\int_{-\infty}^{+\infty} dz\, \delta_\ell(z)}_{\text{by analytic continuation in } \ell\in\mathbb{C}} = 2\int_1^{\infty} dz\, \delta_\ell(z) + \int_{-1}^{1} dz\, \delta_\ell(z) = -\int_1^{\infty} dz\, \gamma_\ell(z) + 1$$

Note that the PN expansion in the NZ ($r<\mathcal{R}$) depends on the multipole exp. $\mathcal{M}(\tau^{\mu\nu})$ and therefore on the properties of the field in the FZ ($r \gg \lambda\!\!\!^{-}$).

Indeed the PN exp. includes the radiation reaction terms appropriate to an isolated system, satisfying the correct boundary conditions at infinity (notably $\mathscr{I}^-$).

$$\mathcal{A}_L^{\mu\nu} = \underbrace{\mathcal{F}_L^{\mu\nu}}_{\substack{\text{describes "linear"} \\ \text{radiation reaction terms} \\ \text{and } \underline{\text{starts at 2.5PN}}}} + \underbrace{\mathcal{R}_L^{\mu\nu}}_{\substack{\text{describes "non-linear"} \\ \text{effects (tails) in the} \\ \text{radiation reaction and} \\ \underline{\text{starts at 4PN}}}}$$



The linear rad. reac. (parametrized by $\mathcal{F}_L^{\mu\nu}$) can be recombined with the particular solution

$$\mathrm{FP}\,\mathcal{I}^{-1}\overline{\tau}^{\mu\nu} = \sum_{k=0}^{+\infty}\left(\frac{\partial}{c\partial t}\right)^{2k}\mathrm{FP}\,\Delta^{-k-1}\overline{\tau}^{\mu\nu}$$

to give simply the retarded integral

$$\mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\overline{\tau}^{\mu\nu} = -\frac{1}{4\pi}\underbrace{\sum_{p=0}^{+\infty}\frac{(-)^p}{p!}\left(\frac{\partial}{c\partial t}\right)^{p}\mathrm{FP}\int d^3x'\,|x-x'|^{p-1}\,\overline{\tau}^{\mu\nu}(\vec{x}',t)}_{\substack{\text{formal expansion } c\to+\infty \\ \text{of the retardation } t-\frac{1}{c}|\vec{x}-\vec{x}'| \\ \text{(well-defined thanks to the FP)}}}$$

The sol. $\mathrm{FP}\,\mathcal{I}^{-1}$ corresponds to the even-parity part $p=2k$.
The odd-parity $p=2k+1$ is exactly given by the terms with $\mathcal{F}_L^{\mu\nu}$

Final result is thus (Blanchet, Faye & Nissanke 2005)

$$\boxed{\overline{h}^{\mu\nu} = \underbrace{\frac{16\pi G}{c^4}\,\mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\overline{\tau}^{\mu\nu}}_{\substack{\text{corresponds to the old} \\ \text{way of performing the} \\ \text{PN expansion (Anderson \& DeCanio} \\ \text{1975)}}} - \underbrace{\frac{4G}{c^4}\sum_{l=0}^{+\infty}\partial_L\left\{\frac{\mathcal{R}_L^{\mu\nu}(t-r)-\mathcal{R}_L^{\mu\nu}(t+r)}{2r}\right\}}_{\text{starts at 4PN}}}$$